3-Step $y-$*function* HYBRID METHODS FOR DIRECT NUMERICAL INTEGRATION OF SECOND ORDER IVPs IN ODEs

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Abstract: *This article is concerned with implicit $y-$function hybrid numerical methods for direct integration solution of general second-order differential equations. The approach is based on interpolation of the basis function at both grid and off-grid points and collocation of its associated differential system at all grid points using power series as the basis function to the solution of the problem. The methods developed are continuous, consistent, and symmetric and the main predictor of the same order of accuracy with the methods was also developed to evaluate the implicit scheme. Comparisons of results of the derived methods with existing methods of higher order of accuracy show that the proposed method is better than the existing methods.*

Key words: Linear multistep, Hybrid, Predictor-Corrector, Continuous, $y-$function.

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1 Introduction

In this work, a three step $y-$function algorithm is developed to directly implement a general second order differential equation of the form

$$y'' = f(t, y, y'), \quad y(t_0) = y_0, \quad y'(t_0) = y_1. \tag{1}$$

Literature has shown that many empirical problems can be modeled into problem (1). Though the conventional method for solving (1) is by reducing it to system of first order ordinary differential equations, attempt is hereby made to solve (1) directly to avoid the drawbacks in the reduction methods [Onumanyi, Awoyemi, Jator and Sirisena (1994); Awoyemi (2005); Waeleh, Majid, Ismail and Suleiman (2012); Jator (2010); Majid and Suleiman (2006); Adesanya, Anake and Oghonyon (2009); Yusuph and Onumanyi (2005)]. Waeleh et al (2012) developed a code based on 2-point Block methods for solving higher order IVPs of ODEs directly. Majid (2004) in Majid, Azumin and Suleiman (2009) developed the two-point block method for solving first and second order ODEs using variable stepsize. Moreso, Majid and Suleiman (2006), have introduced a direct integration implicit variable steps method for solving higher order systems of ODEs. Jator (2010) solve second order IVPs directly using the application of a self starting multistep method. Onumanyi et al (1994), Kayode (2005); Anake, Awoyemi, Adesanya and Famewo (2012). These authors have solve problem (1) directly but the location of the hybrids are at $f-$function which made the qualities of these methods not desirable as they have low order of accuracy and less efficient. To make these methods desirable and more efficient, there is need to introduce the hybrid points at $y-$function [Kayode (2011), Kayode and Adeyeye (2011), Kayode and obarhua (2013)]. The aim of this paper is to extend the work in Kayode and Obarhua (2013) proposing 3-step implicit $y-$function hybrid methods for direct numerical integration of initial value problems (IVPs) of ordinary differential equations to address these observed limitations. This we intend for efficiency and economically.

2 Derivation of the Method

Let consider the approximate solution to problem (1) to be a partial sum of a power series of the form

$$y(x) = \sum_{j=0}^{2(k+1)} a_j x^j. \tag{2}$$

Taking the second derivative of (2) and using this in (1) yields

$$\sum_{j=2}^{2(k+1)} j(j-1)a_j x^{j-2} = f(x, y, y'). \tag{3}$$

Equations (2) and (3) are respectively interpolated and collocated at selected grid and off-grid points x_{n+i} as $i = 0, r, 1, 2, v$ and x_{n+c} as $c = 0, 1, 2, 3$ where $r \in (0, 1)$ when the stepnumber $k = 3$, $0 < r < 1, 2 < v < 3$, giving rise to a system of $c + i$ equations written as matrix equation

$$\mathbf{Ax} = \mathbf{b}$$

as

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\ 1 & x_{n+r} & x_{n+r}^2 & x_{n+r}^3 & x_{n+r}^4 & x_{n+r}^5 & x_{n+r}^6 & x_{n+r}^7 & x_{n+r}^8 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 \\ 1 & x_{n+v} & x_{n+v}^2 & x_{n+v}^3 & x_{n+v}^4 & x_{n+v}^5 & x_{n+v}^6 & x_{n+v}^7 & x_{n+v}^8 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 & 56x_{n+1}^6 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 12x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 & 56x_{n+2}^6 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 12x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 & 56x_{n+3}^6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+r} \\ y_{n+1} \\ y_{n+2} \\ y_{n+v} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix}. \tag{4}$$

Solving (4) for a_j's and substituting their results into (2) to obtain

$$y_k(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \{\tau_1(x) y_{n+r} + \tau_2(x) y_{n+v}\} + h^2 \sum_{j=0}^{k} \beta_j(x) f_{n+j}. \tag{5}$$

$$\begin{aligned} y_{n+3} = {} & \frac{1}{T_0}\alpha_0 y_n + \frac{1}{T_1}\tau_1 y_{n+r} - \frac{1}{T_2}\alpha_1 y_{n+1} + \frac{1}{T_3}\alpha_2 y_{n+2} + \frac{1}{T_4}\tau_2 y_{n+v} \\ & + \frac{h^2}{6T_5}\big(- \beta_0 f_n + 3\beta_1 f_{n+1} - 3\beta_2 f_{n+2} - \beta_3 f_{n+3}\big), \end{aligned} \tag{6}$$

and its first derivative is

$$\begin{aligned} y'_{n+3} = {} & \frac{1}{T'_0}\alpha'_0 y_n + \frac{1}{T'_1}\tau'_1 y_{n+r} - \frac{1}{T'_2}\alpha'_1 y_{n+1} + \frac{1}{T'_3}\alpha'_2 y_{n+2} + \frac{1}{T'_4}\tau'_2 y_{n+v} \\ & + \frac{h^2}{6T'_5}\big(- \beta'_0 f_n + 3\beta'_1 f_{n+1} - 3\beta'_2 f_{n+2} - \beta'_3 f_{n+3}\big), \end{aligned} \tag{7}$$

where

$$\alpha_0 = 2(r-3)(3-v)\left\{\begin{array}{l} -408(r^4v + rv^4) + 783(rv^3 + r^3v) + 144(r^3v^4 + r^4v^3) \\ -1662(r^2v + rv^2) - 261(r^4v^2 + r^2v^4) - 18r^4v^4 + 56730r^3v^3 \\ -1984r^2v^2 + 665rv + 252 + 1590(r^3v^2 + r^2s^3) + 63(r^4 + v^4) \\ -37(r^3 + v^3) + 413(r^2 + v^2) + 44202(r + v) \end{array}\right\}$$

$$\tau_1 = 6(3-v)(-63v^4 + 378v^3 - 413v^2 - 462v - 252)$$

$$\alpha_1 = -3(r-3)(3-v)\left\{\begin{array}{l} -864(r+v) - 851r^3v^3 + 262046r^2v^2 + 1116rv \\ +2106(r^3v + rv^3) + 1763(r^3v^2 + r^2v^3) - 828(r^3 + v^3) \\ +1836(r^2 + v^2) + 12096 - 3993(r^4v + rv^4) + 702(r^2v + rv^2) \\ -237(r^4v^2 + r^2v^4) + 120(r^3v^4 + r^4v^3) - 18r^4v^4 \\ +108(r^4 + v^4) \end{array}\right\}$$

$$\alpha_2 = 6(r-3)(3-v)\left\{\begin{array}{l} -1188(r+v) - 563r^3v^3 - 2292r^2v^2 - 315rv - 453(r^3v + rv^3) \\ +964(r^3v^2 + r^2v^3) - 360(r^3 + v^3) + 1268(r^2 + v^2) \\ +36(r^4v + rv^4) + 1332(r^2v + rv^2) - 129(r^4v^2 + r^2v^4) \\ +15648(r^3v^4 + r^4v^3) - 18r^4v^4 + 27(r^4 + v^4) \end{array}\right\}$$

$$\tau_2 = 6(r-3)(-63r^4 + 378r^3 - 413r^2 - 462r - 252)$$

$$\beta_0 = -2(r-3)(3-v)\left\{\begin{array}{l} 4920(rv^2+r^2v) - 2612r^2v^2 - 8005rv - 60468(r^2+v^2) \\ +732(r+v) - 747(r^3v+rv^3) + 360(r^3v^2+r^2v^3) \\ +108(r^3+v^3) - 45r^3v^3 + 3780 \end{array}\right\}$$

$$\beta_1 = 6(r-3)(3-v)\left\{\begin{array}{l} 4290(rv^2+r^2v) + 2556(r^2+v^2) - 4176(r+v) - 2801r^2v^2 \\ -5100rv - 726(r^3v+rv^3) + 395(r^3v^2+r^2v^3) \\ -11769(r^3+v^3) - 45r^3v^3 \end{array}\right\}$$

$$\beta_2 = -6(r-3)(3-v)\left\{\begin{array}{l} 2010(rv^2+r^2v) + 1224(r^2+v^2) - 1494(r+v) - 664r^2v^2 \\ -1515rv - 429(r^3v+rv^3) - 10(r^3v^2+r^2v^3) - 234(r^3+v^3) \\ +45r^3v^3 \end{array}\right\}$$

$$\beta_3 = -6(r-3)(3-v)\left\{\begin{array}{l} 750(rv^2+r^2v) + 468(r^2+v^2) - 528(r+v) - 6568r^2v^2 \\ +3388rv - 198(r^3v+rv^3) - 45(r^3v^2+r^2v^3) \\ -108(r^3+v^3) + 45r^3v^3 \end{array}\right\}$$

$$T_0 = 2rv\left\{\begin{array}{l} -2976(r+v) + 3930(rv^2+r^2v) - 1926(r^3v+rv^3) + 1059(r^2v^3+r^3v^2) \\ +3556(r^2+v^2) - 908rv - 4361r^2v^2 + 143r^3v^3 + 270(r^4v+rv^4) \\ -63(r^4v^2+r^2v^4) - 72(r^4v^3+r^3v^4) - 1404(r^3+v^3) + 180(r^4+v^4) \\ +18r^4v^4 \end{array}\right\}$$

$$T_1 = r(r-1)(r-2)(r-v)\left\{\begin{array}{l} -2976(r+v) + 3930(rv^2+r^2v) - 1926(r^3v+rv^3) \\ +3556(r^2+v^2) - 908rv - 4361r^2v^2 + 143r^3v^3 \\ +270(r^4v+rv^4) - 63(r^4v^2+r^2v^4) - 72(r^4v^3+r^3v^4) \\ +180(r^4+v^4) + 18r^4v^4 + 1059(r^2v^3+r^3v^2) \\ -1404(r^3+v^3) \end{array}\right\}$$

$$T_2 = (r-1)(v-1)\left\{\begin{array}{l} -2976(r+v) + 3930(rv^2+r^2v) - 1926(r^3v+rv^3) \\ +3556(r^2+v^2) - 908rv - 4361r^2v^2 + 143r^3v^3 + 270(r^4v+rv^4) \\ -63(r^4v^2+r^2v^4) - 72(r^4v^3+r^3v^4) + 180(r^4+v^4) + 18r^4v^4 \\ +1059(r^2v^3+r^3v^2) - 1404(r^3+v^3) \end{array}\right\}$$

$$
T_3 = 2(r-2)(v-2)\left\{\begin{array}{l}
-2976(r+v)+3930(rv^2+r^2v)-1926(r^3v+rv^3)\\
+3556(r^2+v^2)-908rv-4361r^2v^2+143r^3v^3+270(r^4v+rv^4)\\
-63(r^4v^2+r^2v^4)-72(r^4v^3+r^3v^4)+180(r^4+v^4)+18r^4v^4\\
+1059(r^2v^3+r^3v^2)-1404(r^3+v^3)
\end{array}\right\}
$$

$$
T_4 = v(r-v)(v-1)(v-2)\left\{\begin{array}{l}
-2976(r+v)+3930(rv^2+r^2v)-1926(r^3v+rv^3)\\
+3556(r^2+v^2)-908rv-4361r^2v^2+143r^3v^3\\
+270(r^4v+rv^4)-63(r^4v^2+r^2v^4)-72(r^4v^3+r^3v^4)\\
+180(r^4+v^4)+18r^4v^4+1059(r^2v^3+r^3v^2)\\
-1404(r^3+v^3)
\end{array}\right\}
$$

$$
T_5 = \left\{\begin{array}{l}
-2976(r+v)+3930(rv^2+r^2v)-1926(r^3v+rv^3)-1404(r^3+v^3)\\
+1059(r^2v^3+r^3v^2)+3556(r^2+v^2)-908rv-4361r^2v^2+143r^3v^3\\
+270(r^4v+rv^4)-63(r^4v^2+r^2v^4)-72(r^4v^3+r^3v^4)+180(r^4+v^4)+18r^4v^4
\end{array}\right\}. \tag{8}
$$

and

$$
\alpha_0' = 3\left\{\begin{array}{l}
19764(rv^4+r^4v)-73224(rv^3+r^3v)+103842(rv^2+r^2v)+1229823rv\\
+37071r^2v^2+125874r^3v^3-256716r^4v^4-14445(r^4+v^4)+2557737(r^3+v^3)\\
-24003(r^2+v^2)+223398(r+v)-47223(r^4v^3+r^3v^4)+32454(r^4v^2+r^2v^4)\\
-875761(r^3v^2-r^2v^3)-1863(r^5v+rv^5)+50058(r^5v^4+r^4v^5)\\
-41958(r^5v^3+r^3v^5)-3645(r^5v^2+r^2v^5)+1701(r^5+v^5)-9396r^5v^5\\
+6651288
\end{array}\right\}
$$

$$
\tau_1' = 3(1341v^5-11637v^4+1031098v^3-18051v^2-17658v-11772)
$$

$$
\alpha_1' = -3\left\{\begin{array}{l}
-147744rv+36288(rv^2+r^2v)-63563(r^4v^3+r^3v^4)-293112(r^3v^2+r^2v^3)\\
+86304(r^2v^4+r^4v^2)+129600(r^3+v^3)-186624(r^2+v^2)+1710720(r+v)\\
+17960r^4v^4+223200r^3v^3+365976r^2v^2-8739(r^5v^2+r^2v^5)\\
-1794(r^5v^4+r^4v^5)+180r^5v^5+6348(r^5v^3+r^3v^5)+1782(r^5v+rv^5)\\
-1090476(r^4v+rv^4)+28728(r^3v+rv^3)+3564(r^5+v^5)-36072(r^4+v^4)
\end{array}\right\}
$$

$$\alpha_2' = 3\left\{\begin{array}{l} -829359rv + 55566(rv^2 + r^2v) + 981693(r^4v^3 + r^3v^4) - 1809(r^3v^2 + r^2v^3) \\ -2659857(r^2v^4 + r^4v^2) + 1189161(r^3 + v^3) - 1133595(r^2 + v^2) + 304236(r + v) \\ -178032r^3v^3 + 767205r^2v^2 + 36531(r^5v^2 + r^2v^5) + 68850(r^5v^4 + r^4v^5) \\ -142074(r^5v^3 + r^3v^5) + 56943(r^5v + rv^5) - 396846(r^4v + rv^4) \\ +735372(r^3v + rv^3) + 56943(r^5 + v^5) - 453789(r^4 + v^4) - 494532r^4v^4 \\ -9396r^5v^5 \end{array}\right\}$$

$$\tau_2' = 3(-1341r^5 + 11637r^4 - 1031098r^3 + 18051r^2 + 17658r + 11772)$$

$$\beta_0' = -\left\{\begin{array}{l} -1313109(rv^2 + r^2v) + 433593(rv^3 + r^3v) - 52157277rv + 1171547r^2v^2 \\ -363687(r^3v^2 + r^2v^3) - 65880(r^3 + v^3) + 104544(r^2 + v^2) + 217944(r + v) \\ -529740 + 1143r^4v^4 - 11277(r^4v^3 + r^3v^4) + 38241(r^4v^2 + r^2v^4) \\ +8748(r^4 + v^4) - 47223(r^4v + rv^4) + 109507r^3v^3 \end{array}\right\}$$

$$\beta_1' = -3\left\{\begin{array}{l} -715662(rv^2 + r^2v) + 273564(rv^3 + r^3v) + 338580rv + 1013871r^2v^2 \\ -354426(r^3v^2 + r^2v^3) + 196560(r^3 + v^3) - 608148(r^2 + v^2) + 631152(r + v) \\ +292959r^4v^4 - 12886(r^4v^3 + r^3v^4) + 39273(r^4v^2 + r^2v^4) - 21276(r^4 + v^4) \\ -31914(r^4v + rv^4) + 119476r^3v^3 \end{array}\right\}$$

$$\beta_2' = -3\left\{\begin{array}{l} -211383(rv^2 + r^2v) + 94311(rv^3 + r^3v) + 58725rv + 241629r^2v^2 \\ -69069(r^3v^2 + r^2v^3) + 68310(r^3 + v^3) - 186462(r^2 + v^2) + 163458(r + v) \\ -279r^4v^4 + 371(r^4v^3 + r^3v^4) + 6117(r^4v^2 + r^2v^4) - 8154(r^4 + v^4) \\ -12231(r^4v + rv^4) + 9389r^3v^3 \end{array}\right\}$$

$$\beta_3' = \left\{\begin{array}{l} 6119334(rv^2 + r^2v) + 90072(rv^3 + r^3v) + 39420rv + 197893r^2v^2 \\ -45768(r^3v^2 + r^2v^3) + 65880(r^3 + v^3) - 163404(r^2 + v^2) + 135215(r + v) \\ -1143r^4v^4 + 4572(r^4v^3 + r^3v^4) + 1989(r^4v^2 + r^2v^4) - 8748(r^4 + v^4) \\ +70872(r^4v + rv^4) + 264470r^3v^3 \end{array}\right\}. \tag{9}$$

To test the accuracy of (6), we take an example by making $r = \frac{1}{2}$ and $v = \frac{5}{2}$, to have

$$y_{n+3} = -\frac{55}{3}y_{n+2} + \frac{32}{3}y_{n+\frac{5}{2}} + \frac{55}{3}y_{n+1} - \frac{32}{3}y_{n+\frac{1}{2}} + y_n + \frac{h^2}{36}(f_{n+3} - 63f_{n+2} + 63f_{n+1} - f_n). \tag{10}$$

The order p and the principal error constant c_{p+2} of (10) are $p = 7$ and $c_{p+2} = -0.000029624$ respectively and its first derivative is

$$y'_{n+3} = \frac{1}{h}\left(-\frac{8567059}{156366}y_{n+2} + \frac{11212304}{390915}y_{n+\frac{5}{2}} + \frac{4276442}{78183}y_{n+1} - \frac{2474704}{78183}y_{n+\frac{1}{2}} + \frac{37989}{12410}y_n\right) + \frac{h}{9381960}(1726769f_{n+3} - 52604847f_{n+2} + 47501847f_{n+1} - 819569f_n). \tag{11}$$

The order p and the principal error constant c_{p+2} of (11) are $p = 7$ and $c_{p+2} = -0.0035276$ respectively.

3 Implementation of the Method

To implement the derived method to solve problem (1) of the discrete scheme (10) obtained from (6) requires the generation of some starting values. This is obtained in Predictor-Corrector mode of the same order of accuracy. The following symmetric explicit predictor scheme and its derivative of the same order with the corrector scheme are obtained using the same procedure in section 2 y_{n+3} and y'_{n+3}.

$$y_{n+3} = \left(-\frac{14422}{359}y_n + \frac{25584}{359}y_{n\frac{1}{2}} - \frac{5915}{359}y_{n+1} - \frac{12840}{359}y_{n+2} + \frac{7952}{359}y_{n+\frac{5}{2}}\right) + \frac{h^2}{4308}\left(3033f_n + 39712f_{n+\frac{1}{2}} + 5526f_{n+1} - 21811f_{n+2}\right). \tag{12}$$

$$y'_{n+3} = \frac{1}{h}\left(-\frac{3389333}{12565}y_n + \frac{1284680}{12565}y_{n+\frac{1}{2}} - \frac{1326461}{7539}y_{n+1} - \frac{61136}{359}y_{n+2} + \frac{3949384}{37695}y_{n+\frac{5}{2}}\right) + \frac{h}{904680}\left(4307697f_n + 55256608f_{n\frac{1}{2}} + 1779534f_{n+1} - 24931099f_{n+2}\right). \tag{13}$$

The principal error constants of (12) and (13) are $c_{p+2} = 0.0021109$ and $C_{p+2} = 0.0014857$ respectively. The schemes (12) and (13) above have the same order $p = 7$. Other explicit schemes were also generated to evaluate other starting values and Taylor's series was used to evaluate the values for y_{n+r}

$$y_{n+r} = y_n + (rh)y'_n + \frac{(rh)^2}{2!}f_n + \frac{(rh)^3}{3!}\left\{\frac{\partial f_n}{\partial x_n} + y'_n\frac{\partial f_n}{\partial y_n} + f_n\frac{\partial f_n}{\partial y'_n}\right\} + O(h^4) \tag{14}$$

and

$$y'_{n+r} = y'_n + (rh)f_n + \frac{(rh)^2}{2!}\left\{\frac{\partial f_n}{\partial x_n} + y'_n\frac{\partial f_n}{\partial y_n} + f_n\frac{\partial f_n}{\partial y'_n}\right\} + O(h^4). \tag{15}$$

3.1 Numerical Examples

The method is applied to solve the following linear and non-linear second order initial value problems of ordinary differential equations directly without reduction to system of first order equations.

Problem 1:

$$y'' = x(y')^2, \quad y(0) = 1, \quad y'(0) = \frac{1}{2}, \quad h = \frac{1}{100}.$$

The Exact Solution:

$$y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right).$$

In this example, the results of our methods of order 7 are compared with the method of (Kayode & Awoyemi, 2005) a five step which is of order 8. This can be seen in table 1 at some selected points.

Table 1: Results and absolute errors $|y_e xact - y_c omputed|$ for problem 1

x	y_{exact}	$y_{computed}$	Errors in Kayode & Awoyemi (2005)	Errors in new scheme (10)
0.1	1.050041729278	1.050041729281	0.1708719055e-09	2.312595e-12
0.2	1.100335347731	1.100335347742	0.6836010114e-08	1.088329e-11
0.3	1.151140435936	1.151140435961	0.1555757709e-07	2.430833e-11
0.4	1.202732554054	1.202732554094	0.2880198295e-07	4.018186e-11
0.5	1.255412811883	1.255412811937	0.4802328029e-07	5.422818e-11
0.6	1.309519604203	1.309519604262	0.7628531256e-07	5.901679e-11
0.7	1.365443754271	1.365443754313	0.1157914170e-06	4.161738e-11
0.8	1.423648930194	1.423648930173	0.1727046080e-06	2.077827e-11
0.9	1.484700278594	1.484700278425	0.2561456831e-06	1.692806e-10
1.0	1.549306144334	1.549306143854	0.3815695118e-06	4.802496e-10

Problem 2:

$$y_1'' = -y_2 + \cos x, \quad y_1(0) = -1, \quad y_1'(0) = -1.$$

$$y_2'' = y_1 + \sin x, \quad y_2(0) = 1, \quad y_2'(0) = 0.$$

The Exact Solution:

$$y_1(x) = -\cos x - \sin x.$$

$$y_2(x) = \cos x. \qquad \text{(Majid et al (2009))}$$

In this example, the results of the new method (10) of order $p = 7$ are compared with those of Majid et al (2009) and Adeyeye (2012).

Table 2: Results and absolute errors $|y_e xact - y_c omputed|$ for problem 2

TOL	Majid et al (2009)				MTD	TS	Adeyeye (2012)		New Method (10)	
	MTD	TS	MAXE	t_c			MAXE	t_c	MAXE	t_c
10^{-2}	2P4SDIR	33	2.73003E-2	710	3-STEP	33	2.993106E-10	144	1.961956E-10	122
10^{-4}	2P4SDIR	42	1.72828E-3	837	3-STEP	42	6.394885E-14	285	1.598721E-14	131
10^{-6}	2P4SDIR	69	6.87609E-6	1182	3-STEP	69	3.030909E-14	276	1.443290E-14	198
10^{-8}	2P4SDIR	84	9.64221E-7	1552	3-STEP	84	3.208545E-13	337	2.430278E-13	312
10^{-10}	2P4SDIR	160	2.04449E-9	2485	3-STEP	160	1.035838E-13	612	1.310063E-14	581

Problem 3:

$$y'' = -y, \quad y(0) = 1, \quad y'(0) = 1, \quad h = 0.1.$$

The Exact Solution:

$$y(x) = \cos x + \sin x$$

. In this example, the optimal errors of the method (10) are compared with the optimal errors of Ehigie et al, (2010). The results are as shown in Table 3a and Graph 3b below:

Table 3a: Results and absolute optimal errors for problem 3

x	y_{exact}	$y_{computed}$	Optimal errors in Ehigie et al (2010)	Optimal errors in New Method (10)
0.3	1.250856695787	1.250856675130	1.26e-05	2.07e-08
0.4	1.310479336312	1.310479337927	1.66e-05	1.62e-09
0.5	1.357008100495	1.357008130874	2.05e-05	3.04e-08
0.6	1.389978088305	1.389978152572	2.41e-05	6.43e-08
0.7	1.409059874522	1.409059976326	2.75e-05	1.02e-07
0.8	1.414062800247	1.414062941683	3.07e-05	1.41e-07
0.9	1.404936877898	1.404937059457	3.35e-05	1.82e-07
1.0	1.381773290676	1.381773511219	3.60e-05	2.21e-07

Problem 4:

$$y'' = \lambda(1 - y^2)y' - y, \quad y(0) = 2, \quad y'(0) = 0, \quad h = 0.1 \text{ when } \lambda = 1.$$

This example is solved using the new methods of order 7. This can be seen in Table 4a and the graph 4b.

Table 4a: Numerical solution for problem 4

x	$y_{computed}$	x	$y_{computed}$	x	$y_{computed}$	x	$y_{computed}$
0.3	1.9106729679	0.4	1.8421219814	0.5	1.7551651171	0.6	1.6506712174
0.7	1.5296843493	0.8	1.3934133719	0.9	1.2432198584	1.0	1.0806044914
1.1	0.9071920686	1.2	0.7247152685	1.3	0.5349973381		

Conclusion: In this paper, the efficiency and low error term was established by extending earlier results of Kayode and Obarhua (2013), the performance of the continuous $y-$function hybrid methods developed have significantly improved by introducing a step higher. The methods were derived by interpolation and collocation procedure using power series as the basis function. The main predictor, which is of the same order with the method (10), was derived to implement the method. The new hybrid methods are continuous, consistent, symmetric and of higher order of accuracy than earlier ones in Kayode and Obarhua (2013). These methods were compared with some existing methods. The results show that the accuracy of the new methods is better than the existing methods.

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