A characterization of ruled real hypersurfaces in non-flat complex space forms

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Abstract

On a real hypersurface M in a non-flat complex space form we have the Levi-Civita and generalized Tanaka-Webster connections. For any nonnull constant k and any vector field X tangent to M an operator on M, $F_X^{(k)}$, related to both connections, is defined and is called k-th Cho operator. If X belongs to the maximal holomorphic distribution $\mathbb{D}$ on M, the corresponding operator does not depend on k and is denoted by F_X and is called Cho operator. The aim of the present paper is to classify real hypersurfaces in non-flat space forms such that $F_X S = S F_X$, where S denotes the Ricci tensor of M and a further condition is satisfied.

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1 Introduction.

A *complex space form* is an n-dimensional Kähler manifold of constant holomorphic sectional curvature c. A complete and simply connected complex space form is analytically isometric to a complex projective space $\mathbb{C}P^n$ if $c > 0$, a complex Euclidean space $\mathbb{C}^n$ if $c = 0$, or a complex hyperbolic space $\mathbb{C}H^n$ if $c < 0$. In case of $\mathbb{C}P^n$ c is considered 4 and in case of $\mathbb{C}H^n$ c is equal to -4. Furthermore, the complex projective and complex hyperbolic spaces are called non-flat complex space forms and the symbol $M_n(c), n \geq 2$, is used to denote them when it is not necessary to distinguish them.

Let M be a *connected real hypersurface* of $M_n(c)$ without boundary. Let ∇ be the Levi-Civita connection on M and J the complex structure of $M_n(c)$. Take a locally defined unit normal vector field N on M and denote by $\xi = -JN$. This is a tangent vector field to M called the *structure vector field* on M . If it is an eigenvector of the shape operator A of M the real hypersurface is called *Hopf hypersurface* and the corresponding eigenvalue is $\alpha = g(A\xi, \xi)$. Moreover, on M there exists an almost contact metric structure (ϕ, ξ, η, g) induced by the Kählerian structure of $M_n(c)$, where ϕ is the tangent component of J and η is an one-form given by $\eta(X) = g(X, \xi)$ for any X tangent to M.

The classification of homogeneous real hypersurfaces in $\mathbb{C}P^n$ was obtained by Takagi (see [7], [22], [23], [24]). His classification contains 6 types of real hypersurfaces. Among them we find type (A_1) real hypersurfaces that are geodesic hyperspheres of radius r, $0 < r < \frac{\pi}{2}$, type (A_2) real hypersurfaces that are tubes of radius r, $0 < r < \frac{\pi}{2}$, over totally geodesic complex projective spaces $\mathbb{C}P^k$, $0 < k < n-1$ (both types of real hypersurfaces are called type (A) real hypersurfaces) and type (B) real hypersurfaces that are tubes of radius r, $0 < r < \frac{\pi}{4}$, over the complex quadric. All of them are Hopf ones with constant principal curvatures. In case of $\mathbb{C}H^n$, the study of real hypersurfaces with constant principal curvatures, was started by Montiel in [14] and completed by Berndt in [1]. They are divided into two types: type (A) real hypersurfaces which are either a horosphere in $\mathbb{C}H^n$, or a geodesic hypersphere or a tube over a totally geodesic complex hyperbolic hyperplane $\mathbb{C}H^{n-1}$, or a tube over a totally geodesic $\mathbb{C}H^k$ $(1 \leq k \leq n-2)$ and type (B) real hypersurfaces which are tubes of radius $r > 0$ over totally real hyperbolic space $\mathbb{R}H^n$. All of them are homogeneous and Hopf.

Ruled real hypersurfaces are another important class of real hypersurfaces in $M_n(c)$. They can be described as follows: consider a regular curve γ in $M_n(c)$ with tangent vector field X. Then at each point of γ there is a unique hyperplane of $M_n(c)$ cutting γ in a way to be orthogonal to both X and JX. The union of all these hyperplanes is a ruled hypersurface. Equivalently, for ruled hypersurfaces in $M_n(c)$ we have that the maximal holomorphic distribution $\mathbb{D}$ of M at any point, which consists of all the vectors orthogonal to ξ, is integrable and it has as integrable manifold $M_{n-1}(c)$, i.e $g(A\mathbb{D}, \mathbb{D}) = 0$. For examples of ruled real hypersurfaces see [8] or [11].

The *Jacobi operator* R_X with respect to a unit vector field X is defined by $R_X = R(., X)X$, where R is the curvature tensor field on M. Then we see that R_X is a self-adjoint endomorphism of the tangent space. It is related to Jacobi vector fields, which are solutions of the second-order differential equation (the Jacobi equation) $\nabla_{\dot\gamma}(\nabla_{\dot\gamma}Y) + R(Y, \dot\gamma)\dot\gamma = 0$ along a geodesic γ in M. The Jacobi operator with respect to the structure vector field ξ, R_ξ, is called the *structure Jacobi operator* on M.

In the problem of classifying real hypersurfaces in non-flat complex space forms the *Ricci tensor, S,* of them plays an important role. Let R denote the Riemannian curvature tensor of M then the Ricci tensor is defined by

$$SX = \sum_{i=1}^{2n-1} R_{E_i}(X) = \sum_{i=1}^{2n-1} R(X, E_i)E_i,$$

where $\{E_i\}_{i=1,\dots,2n-1}$ is an orthonormal basis of TM, for any X tangent to M.

It is well known, [5], that $M_n(c)$, $n \geq 3$, does not admit real hypersurfaces M

whose Ricci tensor is parallel (that is, $\nabla_X S = 0$, for any vector field X tangent to M). Moreover, in [6] Kim extended the result of non-existence of real hypersurfaces with parallel Ricci tensor in case of three dimensional real hypersurfaces. Therefore it is natural to investigate real hypersurfaces that satisfy weaker conditions than the parallelism of S. In [9] Kimura and Maeda provided the classification of Hopf hypersurfaces in non-flat complex space forms with constant mean curvature and ξ-parallel Ricci tensor. Furthermore, Maeda in [12], classified Hopf real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$, such that $A\xi = 2cot(2r)\xi$ and the focal map ϕ_r has constant rank on M, satisfying $\nabla_\xi S = 0$, obtaining particular cases of the homogeneous real hypersurfaces in Takagi's list and two kinds of non-homogeneous hypersurfaces. In [21] Suh classified Hopf real hypersurfaces in $M_n(c)$, $n \geq 2$, whose Ricci tensor is η-parallel, that is, $g((\nabla_X S)Y, Z) = 0$, for any $X, Y, Z \in \mathbb{D}$, obtaining real hypersurfaces either of type (A) or of type (B). More details on the study of Ricci tensor of real hypersurfaces in non-flat complex spaces forms are included in Section 6 of [16].

The *Tanaka-Webster connection* is the canonical affine connection defined on a non-degenerate, pseudo-Hermitian CR-manifold (see [25], [27]). As a generalization of this connection, in [26] Tanno defined the *generalized Tanaka-Webster connection* for contact metric manifolds by

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y. \tag{1.1}$$

Using the naturally extended affine connection of Tanno's generalized Tanaka-Webster connection, Cho defined the generalized Tanaka-Webster connection $\hat{\nabla}^{(k)}$ for a real hypersurface M in $M_n(c)$ given by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{1.2}$$

for any X, Y tangent to M where k is a nonnull real number (see [3], [4]). Then the following relations hold

$$\hat{\nabla}^{(k)}\eta = 0, \quad \hat{\nabla}^{(k)}\xi = 0, \quad \hat{\nabla}^{(k)}g = 0, \quad \hat{\nabla}^{(k)}\phi = 0.$$

In particular, if the shape operator of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, the generalized Tanaka-Webster connection coincides with the Tanaka-Webster connection.

We can consider the tensor field of type (1,2) given by the difference of both connections $F^{(k)}(X, Y) = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$, for any X, Y tangent to M (see [10] Proposition 7.10, pages 234-235). We will call this tensor the *k-th Cho tensor* on M. Associated to it, for any X tangent to M and any nonnull real number k we can consider the tensor field of type (1,1) $F^{(k)}_X$, given by $F^{(k)}_X Y = F^{(k)}(X, Y)$ for any $Y \in TM$. This operator will be called the *k-th Cho operator corresponding to X* and is given by

$$F^{(k)}_X Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y. \tag{1.3}$$

The torsion of the connection $\hat{\nabla}^{(k)}$ is given by $\hat{T}^{(k)}(X,Y) = F_X^{(k)}Y - F_Y^{(k)}X$ for any X, Y tangent to M. Notice that if $X \in \mathbb{D}$, the corresponding k-th Cho operator does not depend on k and is called *Cho operator* and simply denoted by F_X.

Let T be a tensor field of type $(1,1)$ on M and X a vector field tangent to M. Then it is easy to see that $\nabla_X T = \hat{\nabla}_X^{(k)} T$ if and only if $TF_X^{(k)} = F_X^{(k)}T$. That means that the eigenspaces of T are preserved by $F_X^{(k)}$. In [20] we studied the problem of commutativity of Cho operators and shape operator, obtaining that the unique real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, such that $F_X A = AF_X$ for any $X \in \mathbb{D}$ are locally congruent to ruled real hypersurfaces. Similar results were obtained in the case of structure Jacobi operator of real hypersurfaces in $M_n(c), n \geq 2$, (see [19], [18]).

The aim of this paper is to study real hypersurfaces M in $M_n(c)$ whose Cho operators commute with the Ricci tensor, i.e.

$$F_X S = SF_X, X \in \mathbb{D}. \tag{1.4}$$

First we prove the following Theorem

Theorem 1.1 *There do not exist Hopf hypersurfaces in $M_n(c), n \geq 2$, whose Ricci tensor satisfies relation (1.4).*

Next we study real hypersurfaces in $M_n(c), n \geq 2$, which in addition satisfy the relation $h = g(A\xi, \xi)$, where $h = Trace(A)$. The following Theorem is proved

Theorem 1.2 *Let M be a real hypersurface in $M_n(c), n \geq 2$, such that $h = g(A\xi, \xi)$. Then $F_X S = SF_X$ for any $X \in \mathbb{D}$ if and only if M is locally congruent to a ruled real hypersurface.*

As a direct consequence of the above Theorem we have

Corollary *There do not exist real hypersurfaces M in $M_n(c)$, $n \geq 2$, such that $F_X^{(k)} S = SF_X^{(k)}$ for any X tangent to M and some nonnull k, if $h = g(A\xi, \xi)$.*

This paper is organized as follows: In Section 2 basic results concerning real hypersurfaces in $M_n(c), n \geq 2$, are stated. In Section 3 the proof of Theorem 1.1 is provided. In Section 4 the proof of Theorem 1.2 and Corollary are given. At the end of the Section an open problem is stated.

2 Preliminaries.

Throughout this paper, all manifolds, vector fields, etc., will be considered of class C^∞ unless otherwise stated. Let M be a connected real hypersurface in $M_n(c)$, $n \geq 2$, without boundary. Let N be a locally defined unit normal vector field on M. Let ∇ be the Levi-Civita connection on M and (J, g) the Kählerian structure of $M_n(c)$.

For any vector field X tangent to M we write $JX = \phi X + \eta(X)N$ and $-JN = \xi$. Then (ϕ, ξ, η, g) is an almost contact metric structure on M (see [2]). That is, we have

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.1}$$

for any tangent vectors X, Y to M. From (2.1) we obtain

$$\phi\xi = 0, \quad \eta(X) = g(X, \xi).$$

From the parallelism of J we get

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \quad \text{and} \quad \nabla_X \xi = \phi AX \tag{2.2}$$

for any X, Y tangent to M, where A denotes the shape operator of the immersion. As the ambient space has holomorphic sectional curvature c, the equations of Gauss and Codazzi are given, respectively, by

$$\begin{aligned} R(X,Y)Z = \frac{c}{4}[&g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &-2g(\phi X, Y)\phi Z] + g(AY,Z)AX - g(AX,Z)AY, \end{aligned}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi],$$

for any tangent vectors X, Y, Z to M, where R is the curvature tensor of M. We will call the (maximal) holomorphic distribution $\mathbb{D}$ on M (if $n \geq 3$) to the following one: at any $P \in M$, $\mathbb{D}(P) = \{X \in T_P M | g(X, \xi) = 0\}$.

From the above formulas we have that the Ricci tensor on M is given by

$$SX = \frac{c}{4}[(2n+1)X - 3\eta(X)\xi] + hAX - A^2 X \tag{2.3}$$

for any X tangent to M, where $h = Trace(A)$.

In the sequel we need the following result which is owed to Maeda in case of $\mathbb{C}P^n, n \geq 2$, and is owed to Montiel [14] in case of $\mathbb{C}H^n, n \geq 2$ (also Corollary 2.3 in [16]).

Theorem 2.1 *Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 2$. Then*
i) α is constant.
ii) If W is a vector field which belongs to $\mathbb{D}$ such that $AW = \lambda W$, then

$$(\lambda - \frac{\alpha}{2})A\phi W = (\frac{\lambda\alpha}{2} + \frac{c}{4})\phi W.$$

iii) If the vector field W satisfies $AW = \lambda W$ and $A\phi W = \nu\phi W$ then

$$\lambda\nu = \frac{\alpha}{2}(\lambda+\nu) + \frac{c}{4}. \tag{2.4}$$

Remark 2.1 *In case of real hypersurfaces of dimension greater than three the third case of Theorem 2.1 occurs when $\alpha^2 + c \neq 0$, since in this case relation $\lambda \neq \frac{\alpha}{2}$ holds.*

Finally we provide the following Theorem which is proved by Okumura in case of $\mathbb{C}P^n$ ([17]) and by Montiel and Romero in case of $\mathbb{C}H^n$ ([15]).

Theorem 2.2 *Let M be a real hypersurface of $M_n(c)$, $n \geq 2$. Then $A\phi = \phi A$, if and only if M is locally congruent to a homogeneous real hypersurface of type (A). More precisely:*
In case of $\mathbb{C}P^n$
(A_1) a geodesic hypersphere of radius r , where $0 < r < \frac{\pi}{2}$,
(A_2) a tube of radius r over a totally geodesic $\mathbb{C}P^k$,$(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$.
In case of $\mathbb{C}H^n$
(A_0) a horosphere in $\mathbb{C}H^n$, i.e a Montiel tube,
(A_1) a geodesic hypersphere or a tube over a totally geodesic complex hyperbolic hyperplane $\mathbb{C}H^{n-1}$,
(A_2) a tube over a totally geodesic $\mathbb{C}H^k$ $(1 \leq k \leq n-2)$.

3 Proof of Theorem 1.1

Let M be a Hopf hypersurface in $M_n(c), n \geq 2$, with $A\xi = \alpha\xi$ and whose Ricci tensor satisfies relation (1.4). Relation (1.4) taking into account relation (1.3) is written as

$$g(\phi AX, SY)\xi - \eta(SY)\phi AX = g(\phi AX, Y)S\xi - \eta(Y)S\phi AX. \tag{3.1}$$

We consider the following two cases
Case I: $\alpha^2 + c \neq 0$.

In this case relations of Theorem 2.1 and Remark 2.1 hold. Taking $W \in \mathbb{D}$ such that $AW = \lambda W$ then $A\phi W = \nu\phi W$. Relation (2.3) due to the previous relations implies

$$\begin{aligned} &S\xi = [\frac{c}{2}(n-1) + h\alpha - \alpha^2]\xi, \quad SW = [\frac{c}{4}(2n+1) + h\lambda - \lambda^2]W \text{ and} \\ &S\phi W = [\frac{c}{4}(2n+1) + h\nu - \nu^2]\phi W. \end{aligned} \tag{3.2}$$

Relation (3.1) for $Y = \xi$ implies

$$S\phi AX = [\frac{c}{2}(n-1) + h\alpha - \alpha^2]\phi AX, \text{for any } X \in \mathbb{D}.$$

The above relation for $X = W$ and $X = \phi W$ taking into account relation (3.2) yields respectively

$$\lambda[\frac{3c}{4} + h(\nu - \alpha) - (\nu^2 - \alpha^2)] = 0 \quad \nu[\frac{3c}{4} + h(\lambda - \alpha) - (\lambda^2 - \alpha^2)] = 0. \tag{3.3}$$

If $\frac{3c}{4} + h(\nu - \alpha) - (\nu^2 - \alpha^2) \neq 0$ then the first of (3.3) implies $\lambda = 0$ and relation (2.4) results in $2\alpha\nu + c = 0$. So we conclude that M has at most three different constant eigenvalues. So M is locally congruent to a real hypersurface of type (B). Substitution of the eigenvalues of these real hypersurfaces in $\lambda = 0$ leads to a contradiction.

Therefore, on M we have $\frac{3c}{4} + h(\nu - \alpha) - (\nu^2 - \alpha^2) = 0$. Following similar steps as in the above case we conclude that the second relation of (3.3) implies $\frac{3c}{4} + h(\lambda - \alpha) - (\lambda^2 - \alpha^2) = 0$. Combination of the last two relation yields

$$(\nu - \lambda)(h - \nu - \lambda) = 0.$$

Suppose that $\nu \neq \lambda$ then $h = \lambda + \nu$. So relation $\frac{3c}{4} = (\nu^2 - \alpha^2) - h(\nu - \alpha)$ because of (2.4) results in $\lambda\nu = \frac{5c}{4} + \alpha^2$. Substitution of the latter in (2.4) implies $\alpha(\lambda + \nu) = 2(\alpha^2 + c)$. So $\lambda + \nu$ and $\lambda\nu$ are constant. Thus, λ, ν are constant and the real hypersurface has at most three different eigenvalues. So it is locally congruent to a real hypersurface of type (B). Substitution of the eigenvalues of these real hypersurfaces in $\lambda\nu = \frac{5c}{4} + \alpha^2$ leads to a contradiction.

Therefore, on M relation $\lambda = \nu$ holds and this implies that $A\phi = \phi A$ and because of Theorem 2.2 we conclude that M is locally congruent to a real hypersurface of type (A). So relation (2.4) becomes

$$\lambda^2 = \alpha\lambda + \frac{c}{4}.$$

Furthermore, we have $h = \alpha + (2n - 2)\lambda$. Relation $\frac{3c}{4} = (\lambda^2 - \alpha^2) - h(\lambda - \alpha)$ because of the latter results in $c = 0$, which is impossible.

Case II: $\alpha^2 + c = 0$.

This case occurs when the ambient space is the complex hyperbolic space $\mathbb{C}H^n, n \geq 2$. So we have that $c = -4$ and $\alpha^2 = 4$.

Take a unit $W \in \mathbb{D}$ such that $AW = \lambda W$ and suppose that $\lambda \neq \frac{\alpha}{2}$. Then $A\phi W = \nu\phi W$ and relation (2.4) owing to $\alpha^2 - 4 = 0$ yields $\nu = \frac{\alpha}{2}$ and the real hypersurface has three distinct eigenvalues α, λ and $\nu = \frac{\alpha}{2}$. If p is the multiplicity of λ and q is the multiplicity of ν we have that $h = \alpha + p\lambda + q\nu$.

Relation (3.2) holds. The inner product of the first of relation (3.2) with ξ implies $\eta(S\xi) = \frac{c}{2}(n-1) + h\alpha - \alpha^2$. Moreover, relation (3.1) for $X = W$ and $Y = \xi$ due to relation (3.2), $\eta(S\xi) = \frac{c}{2}(n-1) + h\alpha - \alpha^2$, $\nu = \frac{\alpha}{2}$ and $\alpha^2 = 4$ results in $h\lambda = 0$.

Suppose that $\lambda \neq 0$ then $h = 0$. Moreover, relation (3.1) for $X = \phi W$ and $Y = \xi$ because of the relation (3.2) and all the above relations yields $\lambda = -\frac{\alpha}{2}$. Thus, M has three constant principal curvatures. So M is locally congruent to a real hypersurface of type (B). Substitution of the eigenvalues of such real hypersurface in $\lambda = -\frac{\alpha}{2}$ leads to a contradiction.

So $\lambda = 0$. Furthermore, relation (3.1) for $X = \phi W$ and $Y = \xi$ because of relation (3.2) and all the above relations yields $h = \frac{\alpha}{4}$. The latter due to $h = \alpha + p\lambda + q\nu$, $\nu = \frac{\alpha}{2}$ and $\lambda = 0$ implies $\alpha = 0$ which is a contradiction.

Therefore, we conclude that $\lambda = \frac{\alpha}{2}$ will be the only eigenvalue for all vectors in $\mathbb{D}$. In this case the real hypersurface is a horosphere. In the same way as in the previous case we obtain $\eta(S\xi) = \frac{c}{2}(n-1) + h\alpha - \alpha^2$. Moreover, relation (3.1) for $X = W$ and $Y = \xi$ due to (3.2), $\eta(S\xi) = \frac{c}{2}(n-1) + h\alpha - \alpha^2$, $\lambda = \frac{\alpha}{2}$ and $\alpha^2 = 4$ yields $h = 0$. In this case we have $h = (2n-1)\alpha$, so $\alpha = 0$ which is impossible and this completes the proof of Theorem.

4 Proof of Theorem 1.2

In order to prove Theorem 1.2 the steps below are followed:

- As a consequence of Theorem 1.1 we conclude

  Proposition 4.1 *There do not exist Hopf hypersurfaces in $M_n(c)$, $n \geq 2$, with $h = g(A\xi, \xi)$ and whose Ricci tensor satisfies relation (1.4).*

- Next we study non-Hopf hypersurfaces satisfying the above conditions and the shape operator on U and ϕU orthogonal to ξ is characterized (see Lemma 4.1). In case of three dimensional real hypersurfaces Lemma 4.1 leads to the conclusion that the real hypersurface is a ruled one (see Proposition 4.2).

- We go on with the study of real hypersurfaces of dimension greater than three. In this case it is proved that the eigenvalues of the shape operator on $\mathbb{D}_U$, which consist of the vector fields orthogonal to $\{\xi, U, \phi U\}$, can be:
  either all are equal to zero,
  or zero and two non-zero λ_1 and λ_2. It is proved that this case can not occur.

Therefore, the only case that occurs is the first one and this leads to the conclusion that M is a ruled real hypersurface.

We are now focused on the study of non-Hopf real hypersurfaces satisfying relation (1.4) and $h = g(A\xi, \xi)$. In this case also relation (3.1) holds. First, the scalar product of relation (3.1) for $Y \in \mathbb{D}$ with Y yields

$$\eta(SY)g(\phi AX, Y) = 0, \text{for any } X, Y \in \mathbb{D}. \tag{4.1}$$

Suppose that $g(\phi AX, Y) = 0$ for any $X, Y \in \mathbb{D}$. Then M is a ruled hypersurface.

Next we examine the case of $\eta(SY) = 0$, for any $Y \in \mathbb{D}$. The previous relation implies $S\xi = \mu\xi$, for a certain function μ on M . Since M is a non-Hopf real hypersurface we locally have

$$A\xi = \alpha\xi + \beta U,$$

where we denote by $\alpha = g(A\xi, \xi)$, U is a unit vector field in $\mathbb{D}$, α and β are functions on M with $\beta \neq 0$. Furthermore, we denote by $\mathbb{D}_U$ the orthogonal complementary distribution in $\mathbb{D}$ to the one spanned by U and ϕU (this holds in case of real hypersurfaces with dimension greater than 3).

Lemma 4.1 *Let M be a real hypersurface in $M_n(c), n \geq 2$, whose Ricci tensor satisfies relation (1.4) and $h = \alpha$. Then the shape operator A of M satisfies the relation*

$$AU = \beta\xi \quad A\phi U = 0. \tag{4.2}$$

Proof: Relation (3.1) for $Y = \xi$ implies $\eta(S\xi)\phi AX = S\phi AX$, for any $X \in \mathbb{D}$. As $S\xi = \mu\xi = \frac{c}{4}(2n-2)\xi + \alpha A\xi - A^2\xi = [\frac{c}{4}(2n-2)]\xi - \beta AU$, its scalar product with a vector field Z, orthogonal to ξ and U, gives $\beta g(AU, Z) = 0$. Moreover, the scalar product with U yields $\beta g(AU, U) = 0$ from our hypothesis. This implies

$$AU = \beta\xi.$$

The scalar product of (3.1) with U yields $\eta(SY)g(A\phi U, X) = \eta(Y)g(A\phi SU, X)$. Taking $Y = \xi$ it becomes

$$\eta(S\xi)g(A\phi U, X) = g(A\phi SU, X) \tag{4.3}$$

for any $X \in \mathbb{D}$. Since $SU = (\frac{c}{4}(2n+1) - \beta^2)U$ and $\eta(S\xi) = \frac{c}{4}(2n-2) - \beta^2$, from (4.3) we have

$$A\phi U = 0.$$

$\square$

Due to Lemma 4.1 and Proposition 4.1 we conclude that

Proposition 4.2 *Let M be a real hypersurface in $M_2(c)$, with $h = \alpha$ and whose Ricci tensor satisfies relation (1.4). Then M is locally congruent to a ruled real hypersurface.*

From now on we suppose that the dimension of the real hypersurface is greater than 3. From Lemma 4.1 we know now that $\mathbb{D}_U$ is A-invariant. Take now a unit $Y \in \mathbb{D}_U$ such that $AY = \lambda Y$. From (3.1) we get $\lambda(g(\phi Y, SZ)\xi - \eta(SZ)\phi Y) = \lambda(g(\phi Y, Z)S\xi - \eta(Z)S\phi Y)$, for any Z tangent to M. Therefore either $\lambda = 0$, or, if $\lambda \neq 0$, taking $Z = \xi$, we have $S\phi Y = \eta(S\xi)\phi Y$.

Now we have that if $AY = 0$ for any $Y \in \mathbb{D}_U$ we obtain a ruled real hypersurface.

Let us suppose that $AY = 0$. Then $SY = \frac{c}{4}(2n+1)Y$. For any $X \in \mathbb{D}$ it follows $\frac{c}{4}(2n+1)g(\phi AX, Y)\xi = g(\phi AX, Y)S\xi$. Therefore, for any $X \in \mathbb{D}$, $[\frac{c}{4}(2n+1) - \eta(S\xi)]g(\phi AX, Y) = 0$. As $\frac{c}{4}(2n+1) - \eta(S\xi) = \frac{3c}{4} + \beta^2 \neq 0$, we obtain $A\phi Y = 0$. If we denote by T_0 the distribution in $\mathbb{D}_U$ corresponding to the eigenvalue 0, we have that T_0 is ϕ-invariant. Thus the complementary distribution of T_0 in $\mathbb{D}_U$ is also ϕ-invariant.

Let $\{E_1, ...E_{2p}\}$ be an orthonormal basis of eigenvectors in the complementary distribution. Then relation $S\phi Y = \eta(S\xi)\phi Y$ implies for any $i = 1, ..., 2p$ $S\phi E_i = \eta(S\xi)\phi E_i$. As $\{\phi E_1, ..., \phi E_{2p}\}$ is also an orthonormal basis of the distribution, we obtain that for any $X \in \mathbb{D}_U$ such that $AX \neq 0$, $SX = \eta(S\xi)X$. If X is an eigenvector with eigenvalue $\lambda \neq 0$, it follows $SX = [\frac{c}{4}(2n+1) + \alpha\lambda - \lambda^2]X = [\frac{c}{4}(2n-2) - \beta^2]X$. This yields

$$\frac{3c}{4} + \lambda(\alpha - \lambda) + \beta^2 = 0. \tag{4.4}$$

Relation (4.4) implies that the unique possible nonnull eigenvalues in $\mathbb{D}_U$ are $\lambda_1 = \frac{\alpha}{2} + \sqrt{(\frac{\alpha}{2})^2 + \frac{3c}{4} + \beta^2}$ and $\lambda_2 = \frac{\alpha}{2} - \sqrt{(\frac{\alpha}{2})^2 + \frac{3c}{4} + \beta^2}$. If λ_2 does not appear, as $h = \alpha$ and if p is the multiplicity of λ_1 relation $h = \alpha$ results in

$$\alpha = \alpha + p(\frac{\alpha}{2} + \sqrt{(\frac{\alpha}{2})^2 + \frac{3c}{4} + \beta^2}). \tag{4.5}$$

Similarly, if λ_1 does not appear and q is the multiplicity of λ_2

$$\alpha = \alpha + q(\frac{\alpha}{2} - \sqrt{(\frac{\alpha}{2})^2 + \frac{3c}{4} + \beta^2}). \tag{4.6}$$

Combining relations (4.5) and (4.6) yields $\frac{\alpha}{2} = \pm\sqrt{(\frac{\alpha}{2})^2 + \frac{3c}{4} + \beta^2}$, and this results in $(\frac{\alpha}{2})^2 = (\frac{\alpha}{2})^2 + \frac{3c}{4} + \beta^2$. In case the ambient space is $\mathbb{C}P^n$ the previous relation is impossible. In case the ambient space is $\mathbb{C}H^n$ the previous relation implies $\beta^2 = -\frac{3c}{4}$ and since $c = -4$ we obtain $\lambda_1 = \frac{\alpha}{2} + \sqrt{(\frac{\alpha}{2})^2} = \alpha$ and $\lambda_2 = \frac{\alpha}{2} - \sqrt{(\frac{\alpha}{2})^2} = 0$. Thus $p\alpha = 0$. Therefore, either $p = 0$ and M is ruled or $\alpha = 0$, which implies $h = 0$. So M is ruled and minimal.

From now on we suppose that both of the eigenvalues λ_1 and λ_2 do appear as eigenvalues in $\mathbb{D}_U$. Furthermore, suppose that there exists $Y \in \mathbb{D}_U$ such that $AY = A\phi Y = 0$. From the Codazzi equation $(\nabla_Y A)\xi - (\nabla_\xi A)Y = -\frac{c}{4}\phi Y$. Developing it we get $Y(\alpha)\xi + Y(\beta)U + \beta\nabla_Y U + A\nabla_\xi Y = -\frac{c}{4}\phi Y$. Its scalar product with ξ yields

$$Y(\alpha) + \beta g(\nabla_\xi Y, U) = 0 \tag{4.7}$$

and its scalar product with U gives

$$Y(\beta) = 0. \tag{4.8}$$

Let $Z \in \mathbb{D}_U$ such that $AZ = \lambda Z$ (where either $\lambda = \lambda_1$ or $\lambda = \lambda_2$). As above, $(\nabla_Z A)\xi - (\nabla_\xi A)Z = -\frac{c}{4}\phi Z$ implies $Z(\alpha)\xi + \alpha\phi AZ + Z(\beta)U + \beta\nabla_Z U - A\phi AZ - (\xi)(\lambda)Z - \lambda\nabla_\xi Z + A\nabla_\xi Z = -\frac{c}{4}\phi Z$. Its scalar product with ξ implies

$$Z(\alpha) + \beta g(\nabla_\xi Z, U) = 0 \tag{4.9}$$

and its scalar product with U yields

$$Z(\beta) - \lambda g(\nabla_\xi Z, U) = 0. \tag{4.10}$$

From (4.9) and (4.10) we have

$$\lambda Z(\alpha) + \beta Z(\beta) = 0. \tag{4.11}$$

Moreover, $(\nabla_Z A)U - (\nabla_U A)Z = 0$ yields $Z(\beta)\xi + \beta\phi AZ - A\nabla_Z U - U(\lambda)Z - \lambda\nabla_U Z + A\nabla_U Z = 0$. Taking its scalar product with ξ we obtain

$$Z(\beta) + \beta g(\nabla_U Z, U) = 0 \tag{4.12}$$

and its scalar product with U gives

$$\lambda g(\nabla_U Z, U) = 0. \tag{4.13}$$

As $\lambda \neq 0$, from (4.12) and (4.13) we have $Z(\beta) = 0$ and from (4.11)

$$Z(\alpha) = Z(\beta) = 0. \tag{4.14}$$

On the other hand, $(\nabla_\xi A)U - (\nabla_U A)\xi = \frac{c}{4}\phi U$ implies $\xi(\beta)\xi + \beta\phi A\xi - A\nabla_\xi U - U(\alpha)\xi - U(\beta)U - \beta\nabla_U U = \frac{c}{4}\phi U$. Its scalar product with ξ yields $\xi(\beta) - U(\alpha) = 0$ and the scalar product with U implies $U(\beta) = 0$. Therefore

$$\begin{gathered} \xi(\beta) = U(\alpha) \\ U(\beta) = 0. \end{gathered} \tag{4.15}$$

Analogously, developing $(\nabla_\xi A)\phi U - (\nabla_{\phi U} A)\xi = -\frac{c}{4}U$ and taking its scalar product with ξ, respectively with U, we obtain

$$(\phi U)(\alpha) = \alpha\beta - \beta g(\nabla_\xi \phi U, U) \tag{4.16}$$

and

$$(\phi U)(\beta) = \beta^2 + \frac{c}{4}. \tag{4.17}$$

Let p be the multiplicity of λ_1 and q the multiplicity of λ_2. As $h = \alpha$ we have $(p+q)\frac{\alpha}{2} + (p-q)\sqrt{(\frac{\alpha}{2})^2 + \beta^2 + \frac{3c}{4}} = 0$. As $U(\beta) = 0$, differentiating the latter with respect to U we get $(\frac{p+q}{2} + \frac{p-q}{4\sqrt{(\frac{\alpha}{2})^2 + \beta^2 + \frac{3c}{4}}}\alpha)U(\alpha) = 0$. If we suppose $U(\alpha) \neq 0$, then we have $2(p+q)\sqrt{(\frac{\alpha}{2})^2 + \beta^2 + \frac{3c}{4}} = (q-p)\alpha$. This yields $((p+q)^2 - (q-p)^2)\alpha^2 + (p+q)^2(4\beta^2 + 3c) = 0$. Taking the derivative of this expression in the direction of U we get $2\alpha((p+q)^2 - (q-p)^2)U(\alpha) = 0$, and as we are supposing $U(\alpha) \neq 0$ the fact that $(p+q)^2 - (q-p)^2 = 4pq \neq 0$ yields $\alpha = 0$. This contradicts $U(\alpha) \neq 0$, and we have proved that $U(\alpha) = 0$. So the first of (4.15) yields

$$U(\alpha) = \xi(\beta) = 0. \tag{4.18}$$

Following similar steps it is proved that $\xi(\alpha) = 0$.

Relations (4.8), (4.14), (4.15) and (4.18) result in

$$grad(\beta) = (\beta^2 + \frac{c}{4})\phi U. \tag{4.19}$$

As $g(\nabla_X grad(\beta), Y) = g(\nabla_Y grad(\beta), X)$ for any X, Y tangent to M, we have $X(\beta^2+\frac{c}{4})g(\phi U, Y)+(\beta^2+\frac{c}{4})g(\nabla_X \phi U, Y) = Y(\beta^2+\frac{c}{4})g(\phi U, X)+(\beta^2+\frac{c}{4})g(\nabla_Y \phi U, X)$, for any X, Y tangent to M. Taking $X = \xi$ we obtain $(\beta^2 + \frac{c}{4})[g(\nabla_\xi \phi U, Y) + g(U, AY)] = 0$ for any Y tangent to M.

Suppose that $g(\nabla_\xi \phi U, Y)+g(U, AY) \neq 0$ then the above relation implies $\beta^2+\frac{c}{4} = 0$, This case occurs when the ambient space is the complex hyperbolic space. So we have that the nonnull eigenvalues in $\mathbb{D}_U$ are $\lambda_1 = \frac{\alpha}{2}+\sqrt{(\frac{\alpha}{2})^2 + \frac{c}{2}}$ and $\lambda_2 = \frac{\alpha}{2}-\sqrt{(\frac{\alpha}{2})^2 + \frac{c}{2}}$ with multiplicity p and q respectively. Since, $h = \alpha$ we obtain $4pq(\frac{\alpha}{2})^2 = \frac{c}{2}(q-p)^2$, which is a contradiction, since $c < 0$.

So on M, we have $g(\nabla_\xi \phi U, Y) = -g(U, AY)$ for any Y tangent to M. If $Y = U$ it follows $g(\nabla_\xi \phi U, U) = 0$ and from (4.16)

$$(\phi U)(\alpha) = \alpha\beta. \tag{4.20}$$

Moreover, from the above relation we also know that $g(\nabla_\xi \phi U, \phi Y) = -g(U, A\phi Y)$ for any Y tangent to M. If $Y \in \mathbb{D}_U$ satisfies $AY = A\phi Y = 0$, this and (2.2) yield $g(\nabla_\xi U, Y) = 0$ and from (4.7) we get

$$Y(\alpha) = 0. \tag{4.21}$$

From (4.14), (4.17), (4.18), (4.20) and (4.21) we assure

$$grad(\alpha) = \alpha\beta\phi U. \tag{4.22}$$

Recall that $(p+q)\frac{\alpha}{2} + (p-q)\sqrt{(\frac{\alpha}{2})^2 + \beta^2 + \frac{3c}{4}} = 0$. Taking its derivative in the direction of ϕU and bearing in mind (4.19) and (4.20) we obtain $\frac{p+q}{2}\alpha\beta + \frac{p-q}{2\sqrt{(\frac{\alpha}{2})^2 + \beta^2 + \frac{3c}{4}}}(\frac{1}{2}\alpha^2\beta + 2\beta(\beta^2 + \frac{c}{4})) = 0$. From this we arrive to

$$\begin{aligned}((p+q)^2-(q-p)^2)\frac{\alpha^4}{4}+(p+q)^2\alpha^2(\frac{3c}{4}+\beta^2)\\ =(q-p)^2(\beta^2+\frac{c}{4})(2\alpha^2+4\beta^2+c).\end{aligned} \tag{4.23}$$

Derivating (4.23) in the direction of ϕU and bearing in mind (4.19) and (4.20) we obtain

$$\begin{aligned}((p+q)^2-(q-p)^2)\alpha^4+2(p+q)^2\alpha^2(2\beta^2+c)\\ =4(q-p)^2(\beta^2+\frac{c}{4})(2\alpha^2+4\beta^2+c).\end{aligned} \tag{4.24}$$

From (4.23) and (4.24) it follows $c(p+q)^2\alpha^2=0$. This yields $\alpha=0$.

Relation (4.23) gives

$$4(q-p)^2(\beta^2+\frac{c}{4})^2=0.$$

Suppose that $p\neq q$ then the above relation implies $\beta^2+\dfrac{c}{4}=0$. This case occurs when the ambient space is complex hyperbolic space and the nonnull eigenvalues in D_U are $\lambda_1=\sqrt{\dfrac{c}{2}}$ and $\lambda_2=-\sqrt{\dfrac{c}{2}}$, which is a contradiction, since $c<0$.

Therefore, on M we have $p=q$ and $\mathbb{D}_U$ can be written as follows

$$\mathbb{D}_U=T_0\bigoplus T_{\sqrt{\beta^2+\frac{3c}{4}}}\bigoplus T_{-\sqrt{\beta^2+\frac{3c}{4}}},$$

and the last two eigenspaces have the same dimension.

Let $\{Z_1,...,Z_p\}$ an orthonormal basis of $T_{\sqrt{\beta^2+\frac{3c}{4}}}$. Take $i,j\in\{1,...,p\}$, $i\neq j$ (we suppose that $p\geq 2$). The Codazzi equation yields $(\nabla_{Z_i}A)Z_j-(\nabla_{Z_j}A)Z_i=-\dfrac{c}{2}g(\phi Z_i,Z_j)\xi$. As β is constant along the directions in $T_{\sqrt{\beta^2+\frac{3c}{4}}}$ we obtain

$$\sqrt{\beta^2+\frac{3c}{4}}\nabla_{Z_i}Z_j-A\nabla_{Z_i}Z_j-\sqrt{\beta^2+\frac{3c}{4}}\nabla_{Z_j}Z_i+A\nabla_{Z_j}Z_i=-\frac{c}{2}g(\phi Z_i,Z_j)\xi.$$

Its scalar product with ξ yields

$$\beta g([Z_j,Z_i],U)=\frac{c}{2}(\beta^2+\frac{c}{2})g(\phi Z_i,Z_j) \tag{4.25}$$

and its scalar product with U implies

$$g([Z_j,Z_i],U)=\frac{c}{2}\beta g(\phi Z_i,Z_j). \tag{4.26}$$

From (4.25) and (4.26) we obtain $g(\phi Z_i, Z_j) = 0$. This means that for any $Z \in T_{\sqrt{\beta^2+\frac{3c}{4}}}$, $\phi Z \in T_{-\sqrt{\beta^2+\frac{3c}{4}}}$. Call $\lambda = \sqrt{\beta^2 + \frac{3c}{4}}$. Take $Z \in T_\lambda$. The Codazzi equation yields $-\lambda\nabla_Z\phi Z - A\nabla_Z\phi Z - \lambda\nabla_{\phi Z}Z + A\nabla_{\phi Z}Z = -\frac{c}{2}\xi$. Its scalar product with ξ yields

$$\beta g(\nabla_{\phi Z}Z, U) - \beta g(\nabla_Z\phi Z, U) = -\frac{c}{2}(\beta^2 + c) \tag{4.27}$$

and its scalar product with U, bearing in mind that $\lambda \neq 0$, gives

$$g(\nabla_{\phi Z}Z, U) + g(\nabla_Z\phi Z, U) = 0. \tag{4.28}$$

From (4.27) and (4.28) we obtain

$$g(\nabla_Z\phi Z, U) = -g(\nabla_{\phi Z}Z, U) = \frac{\beta^2 + c}{\beta}. \tag{4.29}$$

On the other hand $(\nabla_{\phi U}A)\phi Z - (\nabla_{\phi Z}A)\phi U = 0$. This yields $-(\phi U)(\lambda)\phi Z - \lambda\nabla_{\phi U}\phi Z - A\nabla_{\phi U}\phi Z + A\nabla_{\phi Z}\phi U = 0$. Its scalar product with ϕZ yields $-(\phi U)(\lambda) - \lambda g(\nabla_{\phi Z}\phi U, \phi Z) = 0$. From (2.2) $g(\nabla_{\phi Z}\phi U, \phi Z) = g(\nabla_{\phi Z}U, Z)$. Bearing in mind the value of λ, from (4.29) it follows $\beta^2(\beta^2 + c) + (\beta^2 + \frac{3c}{4})(\beta^2 + c) = 0$. That is, $\beta^4 + \frac{5c}{2}\beta^2 + \frac{3c^2}{4} = 0$. Thus β is constant and this results in $grad(\beta) = 0$. So relation (4.19) implies $\beta^2 + \frac{c}{4} = 0$, which occurs in case the ambient space is $\mathbb{C}H^n$. In this case, substitution of the last one in $\beta^4 + \frac{5c}{2}\beta^2 + \frac{3c^2}{4} = 0$ implies $c = 0$, which is impossible. This means that our non Hopf real hypersurfaces must be ruled and this completes the proof of Theorem 1.2. $\square$

In order to prove the Corollary, suppose that M is a ruled real hypersurface such that for some nonnull k, $F_\xi^{(k)}SY = SF_\xi^{(k)}Y$ for any Y tangent to M. The previous relation because of $F_X^{(k)}Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$ becomes

$$g(\phi A\xi, SY)\xi - \eta(SY)\phi A\xi - k\phi SY = g(\phi A\xi, Y)S\xi - \eta(Y)S\phi A\xi - kS\phi Y \tag{4.30}$$

for any Y tangent to M.

The shape operator of a ruled real hypersurface M is given by

$$A\xi = \alpha\xi + \beta U, \quad AU = \beta\xi \quad \text{and} \quad AY = 0, \quad \text{for any } Y \text{ orthogonal to } \{\xi, U\}. \tag{4.31}$$

The Ricci tensor (2.3) for $X = \xi$, $X = U$ and $X = Y$, where Y is any orthogonal vector to $\{\xi, U\}$, because of $h = g(A\xi, \xi) = \alpha$ and relation (4.31) becomes respectively

$$S\xi = (\frac{c}{2}(n-1) - \beta^2)\xi, \quad SU = \frac{c}{4}(2n+1)U - \beta^2\xi \quad \text{and} \quad SY = \frac{c}{4}(2n+1)Y. \tag{4.32}$$

Relation (4.30) for $Y = U$ bearing in mind the first of relation (4.31) and the second of relation (4.32) leads to $\beta = 0$, which is a contradiction since M is ruled and this completes the proof of the Corollary.

Remark If in our Theorem we suppose $h \neq g(A\xi, \xi)$, it is easy to see that $\beta^2 = g(A\xi, \xi)(h - g(A\xi, \xi)) - 3$ for a non Hopf real hypersurface. This might produce a new kind of real hypersurfaces.
Conjecture (open problem): Such real hypersurfaces in complex space forms do not exist.

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