Extended Block Backward Differentiation Formula for the Valuation of Options

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Abstract

In this paper, an extended Block Backward Differentiation Formula (BBDF) is proposed for the valuation of options on a non-dividend-paying stock. The development of the method is facilitated by the availability of a Continuous Backward Differentiation Formula (CBDF) that is defined for all values of the independent variable on the range of interest. Hence, discrete schemes which are recovered from the CBDF as by-products are combined to form a BBDF, which is then applied on the entire region as a single block matrix equation by solving a system resulting from the semi-discretization of the Black-Scholes model. The stability and convergence of the BBDF are discussed. It is demonstrated that the American put values are obtained by incorporating an additional equation that generates values for the early exercise boundary which are used to ensure that the put option will be optimal. The performance of the method is tested on some numerical examples.

Key Words: Block Backward Differentiation Formula, Extended block, Options, Black-Scholes partial differential equation.

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1 Introduction

The Black-Scholes Option Pricing model is one of the most celebrated achievements in financial economics in the last four decades. The model gives the theoretical value of European style options on a non-dividend-paying stock given the stock price, the strike price, the volatility of the stock, the time to maturity, and the risk-free rate of interest. However, since it is optimal to exercise early an American put option on a non-dividend paying stock, the Black-Scholes formula cannot be used[1]. In fact, no analytic formula for valuing American put options on non-dividend paying stocks exists. As a result, numerous analytical approximation techniques as well as numerical procedures are utilized. A discussion of some of these analytical approximation and numerical techniques is found in Hull (2012).

In addition, several other analytical approximation procedures and numerical techniques for solving the Black-Scholes model abound in the literature (Prékopa and Szántai (2010), Chawla, Al-Zanaidia, and Evans (2003), and Khaliq, Voss, and Kazmi (2006)). Since there is the possibility of an early exercise, Khaliq et al. (2006) consider the pricing of an American put option as a free boundary problem. In effect, the early exercise feature of the American put option transforms the Black-Scholes linear differential equation into a non-linear type. In order to do away with the free and moving boundary, Khaliq et al. (2006) add a small continuous penalty term to the Black-Scholes equation and treat the nonlinear penalty term explicitly. They conclude that their method maintains superior accuracy and stability properties when compared to standard methods that are based on the Newton-type iteration procedure in valuing American options.

Furthermore, Chawla et al. (2003) employ a technique based on the Generalized Trapezoidal Formulas (GTF) and compare the computational performance of the scheme obtained with the Crank-Nicolson scheme for the case of European option pricing. They note that their GTF (1/3) scheme is superior to the Crank-Nicholson scheme. While all these techniques try to accomplish the same goal by solving the Black-Scholes differential equation for a particular derivative security, they are applied only after transforming the model to be forward in time or the differential equation is transform into an integral equation (Huang, Subrahmanyam, and Yu (1996)). In this paper, we propose a BBDF that is L_0-stable and apply it to solve the model in its original form without transforming it into a forward parabolic equation or integral equation.

Thus, consider the Black-Scholes model

[1] Hull (2012) argues that it is never optimal for an American call option on a non-dividend-paying stock to be exercised early. Therefore the Black-Scholes formula can be used to value American Style call options on non-dividend-paying stocks

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \quad (1)$$

subject to the initial/boundary conditions

$$V(0,t) = X, \quad V(S,t) = 0 \text{ if } S > 0, \quad V(S,T) = max(X - S, 0)$$

where $V(S,t)$ denotes the value of the option, σ the volatility of the underlying asset, X the exercise price, T the option expiration date, and r the interest rate.

The method considered in this paper is facilitated by the method of lines approach (Lambert (1991), Ramos and Vigo-Aguiar (2007), and Cash (1984)) which involves seeking a solution in the strip $[a,b] \times [c,d]$, where a, b, c, d are real constants, by first discretizing the variable S with mesh spacings $\Delta S = 1/M$,

$$S_m = m\Delta S,\ m = 0, 1, \ldots, M.$$

We then define $v_m(t) \approx V(S_m,\ t)$, $\mathbf{v}(t) = [V_0(t), V_1(t), \ldots, V_{M-1}(t)]^T$, and replace the partial derivatives $\frac{\partial^2 V(S,\ t)}{\partial S^2}$ and $\frac{\partial V(S,t)}{\partial S}$ occurring in (1) by central difference approximations to obtain

$\frac{\partial^2 V(S_m,\ t)}{\partial S^2} = [v(S_{m+1},t) - 2v(S_m,t) + v(S_{m-1},t)]/(\Delta S)^2$; $\frac{\partial V(S_m,\ t)}{\partial S} = [v(S_{m+1},t) - v(S_{m-1},t)]/2\Delta S$, $m = 1, \ldots, M-1$.

The problem (1) then leads to the resulting semi-discrete problem

$$\frac{dv_i(t)}{dt} = -\frac{1}{2}\sigma^2 S_i^2[v_{i+1}(t) - 2v_i(t) + v_{i-1}(t)]/(\Delta S)^2 - rS_i[v_{i+1}(t) - v_{i-1}(t)]/(\Delta S) + rv_i(t) = 0,$$

which can be written in the form

$$\frac{d\mathbf{v}(t)}{dt} = \mathbf{f}(t, \mathbf{v}), \quad (2)$$

where **f(y, u)=Au+g**, **A** is an $M-1 \times M-1$ matrix arising from the central difference approximations to the derivatives of S, and **g** is a vector of constants. The problem (2) is now a system of ordinary differential equations which is solved by the BBDF. The implementation of the BBDF is different from the approach recently given in Akinfenwa, Jator, and Yao, (2013).

The rest of the paper proceeds as follows. In section 2, we construct the CBDF and use it to produce the BBDF whose analysis is given in the same section. In Section 3, the implementation of the method to solve the Black-Scholes model is given. Section 4 is devoted to numerical examples and the conclusion is given in Section 5.

2 Construction of CBDF and BBDF

In this section, we state the BBDF, which is recovered from a CBDF as by-products. The BBDF is formed by combining the standard Backward Differentiation Formula with additional methods. The main method is given by

$$\sum_{j=0}^{k} \alpha_j v_{n+j} = h\beta_k f_{n+k} \tag{3}$$

and the additional methods are given by

$$\sum_{j=0}^{k-1} \alpha_{i,j} v_{n+j} = h\beta_{i,k} f_{n+k} + h\beta_{i,i} f_{n+i}, i = 1, \ldots, k-1, \quad k = 4, \tag{4}$$

where α_j, $\alpha_{i,j}$, β_j, $\beta_{i,k}$, $\beta_{i,i}$ are coefficients. We note that v_{n+j} is the numerical approximation to the analytical solution $v(t_{n+j})$, $f_{n+j} = f(t_{n+j}, v_{n+j}), j = 0, \ldots, 4$. In order to obtain equations (3) and (4), we seek an approximation $U(t)$ to the exact solution $v(t)$ on the interval $[t_n, t_n + 4h]$ of the form

$$U(t) = a_0 + a_1 t + \ldots + a_4 t^4 \tag{5}$$

where $a_j, j = 0, 1, \ldots, 4$ are coefficients that must be uniquely determined. We then impose that the interpolating function equation (5) coincides with the analytical solution at the points $t_{n+j}, j = 0, 1$ to obtain the equations

$$U(t_{n+j}) = v_{n+j}, j = 0, \ldots, 3. \tag{6}$$

We also demand that the function (4) satisfies the differential equation (2) at the end point t_{n+4} to obtain the equation

$$U'(t_{n+4}) = f_{n+4}. \tag{7}$$

Equations (6) and (7) lead to a system of five equations which is solved by Cramer's rule to obtain $a_j,\ j = 0, \ldots, 4$. Our continuous CBDF is constructed by substituting the values of a_j into equation (5). After some algebraic manipulation, the CBDF is expressed in the form

$$U(t) = \alpha_0(t) v_n + \ldots + \alpha_3(t) v_{n+3} + h\beta_4(t) f_{n+4} \tag{8}$$

whose first derivative is given by

$$U'(t) = \frac{d}{dt}(\alpha_0(t)v_n + \ldots + \alpha_3(t)v_{n+3} + h\beta_4(t)f_{n+4}) \tag{9}$$

where $\alpha_j(t), j = 0, \ldots, 3$ and $\beta_4(t)$ are continuous coefficients. The continuous method (8) is used to generate the main method of the form (3) and the additional methods of the form (4) are provided by the method (9). Thus, evaluating (8) at $t = t_{n+4}$ and (9) at $t = \{t_{n+1}, t_{n+2}, t_{n+2}\}$, we obtain the coefficients of (3) and (4) respectively.

2.1 Formulation of BBDF.

The methods (3) and (4) are used to form the BBDF as follows:

$$A_1 Y_{\mu+1} = A_0 Y_\mu + h[B_1 F_{\mu+1} + B_0 F_\mu] \tag{10}$$

where $Y_{\mu+1} = (v_{n+1}, v_{n+2}, v_{n+3}, v_{n+4})^T$, $Y_\mu = (v_{n-3}, v_{n-2}, v_{n-1}, v_n)^T$

$F_\mu = (f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4})^T$, $F_{\mu-1} = (f_{n-3}, f_{n-2}, f_{n-1}, f_n)^T$

for $\mu = 0, \ldots\ \Gamma$, where $\Gamma = N/4$ is the number of blocks and $n = 0, 4, \ldots, N-4$.

and A_i, B_i, $i = 0, 1$ are 4 by 4 matrices whose entries are given by the coefficients of (3) and (4).

2.2 Consistency of BBDF.

The consistency of (10) is accomplished by rewriting it as

$$Y_{\mu+1} = A_1^{-1} A_0 Y_\mu + h A_1^{-1}[B_1 F_{\mu+1} + B_0 F_\mu] \tag{11}$$

and defining the local truncation error (LTE) of (11) as

$$Ł[z(t); h] = Z_{\mu+1} - (A_1^{-1} A_0 Z_\mu + h A_1^{-1}[B_1 \overline{F}_{\mu+1} + B_0 \overline{F}_\mu]) \tag{12}$$

where

$$Z_{\mu+1} = ((v(t_{n+1}), v(t_{n+2}), v(t_{n+3}), v(t_{n+4}))^T,$$

$$\overline{F}_{\mu+1} = ((f(t_{n+1}, v(t_{n+2})), f(t_{n+3}, v(t_{n+4})), f(t_{n+1}, v(t_{n+2})), f(t_{n+3}, v(t_{n+4})))^T,$$

$$Z_{\mu} = (v(t_{n-3}), v(t_{n-2}), v(t_{n-1}), v(t_n))^T,$$

$$\overline{F}_{\mu} = ((f(t_{n-3}, v(t_{n-3})), f(t_{n-2}, v(t_{n-2})), f(t_{n-1}, v(t_{n-1})), f(t_n, v(t_n)))^T,$$

and $Ł[z(x); h] = (Ł_1[z(x); h], \ldots, Ł_4[z(x); h])^T$ is a linear difference operator. Thus, assuming that the arbitrary function $z(t)$ is the exact solution and is sufficiently differentiable; we can expand the terms in (12) as a Taylor series about the point t_n, to obtain the expression for the local truncation error (LTE) as $Ł[z(t); h] = O(h^5)$. LTE vector is given by (14).

2.3 Stability of BBDF.

The Linear stability regions is obtained by applying (10) to the test equation $y' = \lambda y$ to give

$$Y_{\mu+1} = M(q) Y_{\mu}\ ,\ q = \lambda h, \tag{13}$$

where the stability matrix $M(q)$ is given by

$$M(q) = (A_1 - qB_1)^{-1}(A_0 + qB_0)$$

Definition 2.1. *A method is said to be (i) A-stable if for all $q \,\epsilon\, \mathbb{C}^-$, $M(q)$ has a dominant eigenvalue q_{max} such that $|q_{max}| \leq 1$; moreover, since q_{max} is a rational function, the real part of the zeros of q_{max} must be negative, while the real part of the poles of q_{max} must be positive; (ii) A_0-stable if for all $q \,\epsilon\, \Re \subset \mathbb{C}^-$, $M(q)$ has a dominant eigenvalue q_{max} such that $|q_{max}| \leq 1$; (iii) L_0-stable if it is A_0-stable and $\lim_{q \to -\infty} q_{max} = 0$; and (iv) L-stable if it is A-stable and $\lim_{q \to -\infty} q_{max} = 0$.*

Remark 2.2. *The method (10) is A_0-stable since for $q_{max} = -\frac{12+18q+11q^2+3q^3}{12-30q+35q^2-25q^3+12q^4}$, $|q_{max}| \leq 1$ with the zeros of q_{max} having negative real parts and the poles of q_{max} having positive real parts. The method (10) is also L_0-stable since it is A_0-stable and $\lim_{q \to -\infty} q_{max} = 0$.*

Remark 2.3. *The method (10) can be used to solve initial value systems in a block by block fashion as in Akinfenwa, Jator, and Yao, (2013). In this paper, the approach in*

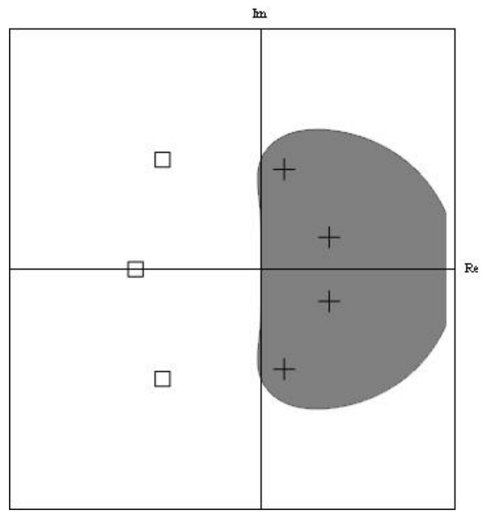


Figure 1: Stability Region for the BBDF confirming A_0-stability

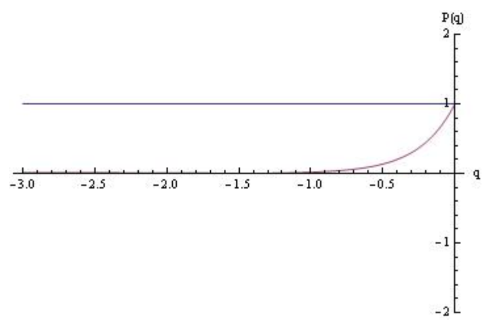


Figure 2: Plot confirming L_0-stability

Akinfenwa et. al (2013) is modified by first generating the blocks on the entire interval and then using these blocks to form a single block matrix equation which is solved to provide the global solution of (2).

2.4 Block Extension

Convergence of BBDF. Since the block extension of (10) gives a global method, we discuss its convergence in what follows.

Theorem 2.4. *Let Y and Z be solution vectors formed by extending (10) from the interval $[t_0, t_4]$, to the intervals $[t_4, t_8], \ldots, [t_{N-4}, t_N]$, and $E = Z - Y$, where Y is interpreted as an approximation of the solution vector for the system formed from the block extension of (10) whose exact solution is Z. If $e_i = |v(t_i) - v_i|$, where the exact solution $v(t)$ is several times differentiable on $[a, b]$ and if $\|E\| = \|Z - Y\|$, then, the*

BBDF is a fourth-order convergent method. That is $\|E\| = O(h^4)$.

Proof.

Let the matrices obtained from the block extension of (10) be defined as follows:

$$A = \begin{bmatrix} A_{11} & & & \\ & A_{22} & & \\ & & \ddots & \\ & & & A_{NN} \end{bmatrix},$$

where the elements of A are 4×4 matrices given as

$$A_{ij} = \begin{bmatrix} -\frac{39}{50} & \frac{69}{50} & -\frac{17}{50} & 0 \\ -\frac{18}{25} & \frac{3}{25} & \frac{38}{75} & 0 \\ \frac{33}{50} & -\frac{93}{50} & \frac{197}{150} & 0 \\ \frac{16}{25} & -\frac{36}{25} & \frac{48}{25} & -1 \end{bmatrix}, i = j = 1, 2, \ldots, N,$$

$$B = h \begin{bmatrix} B_{11} & & & \\ & B_{22} & & \\ & & \ddots & \\ & & & B_{NN} \end{bmatrix},$$ where the elements of B are 4×4 matrices given as

$$B_{ij} = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{25} \\ 0 & 1 & 0 & \frac{1}{25} \\ 0 & 0 & 1 & -\frac{3}{25} \\ 0 & 0 & 0 & -\frac{12}{25} \end{bmatrix}, i = j = 1, 2, \ldots, N,$$

$C = (-\frac{13}{50}v_0, \frac{7}{75}v_0, -\frac{17}{150}v_0, -\frac{3}{25}v_0, \ldots, 0)^T$

The local truncation errors are given by

$$\begin{cases} \tau_{i+1} = \frac{29}{500}h^5 y^{(5)}(t_i + \theta_i) + O(h^6), \\ \tau_{i+2} = -\frac{31}{750}h^5 y^{(5)}(t_i + \theta_i) + O(h^6), \\ \tau_{i+3} = \frac{37}{500}h^5 y^{(5)}(t_i + \theta_i) + O(h^6), \\ \tau_{i+4} = -\frac{12}{125}h^5 y^{(5)}(t_i + \theta_i) + O(h^6), \quad i = 0, 4, \ldots, N-4, \quad |\theta_i| \leq 1. \end{cases} \tag{14}$$

We further define the following vectors:

$Z = (v(t_1), \ldots, v(t_N))^T$, $Y = (v_1, \ldots, v_N)^T$, $F = (f_1, \ldots, f_N)^T$, $L(h) = (\tau_1, \ldots, \tau_N)^T$, $E = Y - Z = (e_1, \ldots, e_N)^T$,

where $L(h)$ is the local truncation error.

The exact form of the system given by the block extension of (10) is

$$AZ - BF(Z) + C + L(h) = 0, \tag{15}$$

and the approximate form of the system is given by

$$AY - BF(Y) + C = 0, \tag{16}$$

where Y is the approximation of the solution vector Z. Subtracting (16) from (17) we obtain

$$AE - BF(Y) + BF(Z) = L(h), \tag{17}$$

Using the mean-value theorem, we write (18) as

$$(A - BJ)E = L(h),$$

where the Jacobian matrix J is a diagonal matrix defined as follows:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial v_1} & & \\ & \ddots & \\ & & \frac{\partial f_N}{\partial v_N} \end{bmatrix},$$

Let $M = -BJ$ be a matrix of dimension N. We have

$$(A + M)E = L(h), \tag{18}$$

and for sufficiently small h, $A + M$ is a monotone matrix and thus invertible (Jain and Aziz (1983); Jator and Li (2012)). Therefore,

$$(A + M)^{-1} = D = (d_{i,j}) \geq 0, \qquad and \quad \sum_{j=1}^{N} d_{i,j} = O(h^{-1}). \tag{19}$$

If $\|E\|$ is the norm of maximum global error and from (19), $E = (A+M)^{-1}L(h)$, using (20) and the truncation error vector $L(h)$, it follows that

$$\|E\| = O(h^4).$$

Therefore, the BBDF is an fourth-order convergent method.

$\square$

3 Implementation

We summarize the process as follows:

Recall that the system of ODEs is obtained on the partition

$$\pi_M : \{c = S_0 < S_1 < \ldots < S_M = d, \qquad S_m = S_{m-1} + \Delta S\}$$

$\Delta S = \frac{d-c}{M}$ is a constant step-size of the partition of π_M, $m = 1, 2, \ldots, M$, M is a positive integer and m the grid index.

The resulting system of ODEs (3) is then solved on the partition

$$\pi_N : \{a = t_0 < t_1 < \ldots < t_N = b, \qquad t_n = t_{n-1} + h\}$$

$h = \Delta t = \frac{b-a}{N}$ is a constant step-size of the partition of π_N, $n = 1, 2, \ldots, N$, N is a positive integer and n the grid index.

We emphasize the block extension of (10) lead to a single matrix of finite difference equations, which is solved to provide all the solutions of (2) on the entire grid given by the rectangle $[a, b] \times [c, d]$.

Step 1: Use the block extension of (10) to generate from the rectangles $[t_0, t_4] \times [c, d]$, to the rectangle $[t_4, t_8] \times [c, d], \ldots, [t_{N-4}, t_N] \times [c, d]$.

Step 2: Solve the system obtained in step 1 to obtain $\mathbf{v}_n = [V_{0,n}, V_{1,n}, \ldots, V_{M-1,n}]^T, n = 1, 2, \ldots, N$.

Step 3: The solution of (1) is approximated by the solutions in step 2 as $\mathbf{v}$, where $\mathbf{v}(t_n) = [V(S_0, t_n), V(S_1, t_n), \ldots, V(S_{M-1}, t_n)]^T, n = 1, 2, \ldots, N$, where $\mathbf{v}(t_n) = \mathbf{v}_n$.

4 Numerical example

In this section, we give a numerical example to illustrate the accuracy of the BBDF. Computations were carried out using Mathematica 8.0 enhanced by the feature $NSolve[\ ]$.

Example 4.1. *Consider a five-month European call and put options on a non-dividend-paying stock when the stock price is \$50, the strike price is \$50, the risk-free interest rate is 10% per annum, and the volatility is 40% per annum. This example is taken from Hull (2012). In order to compute the call and put options, we use the standard notations to denote* $X = 50$, $S = 50$, $r = 0.10$, $\sigma = 0.40$, *and* $T = 0.4167$. *The theoretical solutions for the prices of the European call and put options are given in Hull (2012) as follows.*

$$c = SN(d_1) - Xe^{-r(T-t)}N(d_2)$$

$$p = Xe^{-r(T-t)}N(-d_2 - SN(-d_1)$$

where $d_1 = \frac{\ln(S/X)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$, $\quad d_2 = \frac{\ln(S/X)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$,

and $N(x)$ *is the cumulative probability distribution function for the standard normal variable.*

The following acronyms are used in the Figures:

- BBDFC is Block Backward Differentiation Formula for the call option
- BBDFP is Block Backward Differentiation Formula for the put option
- ECALL is the exact solution for the call option
- EPUT is the exact solution for the put option

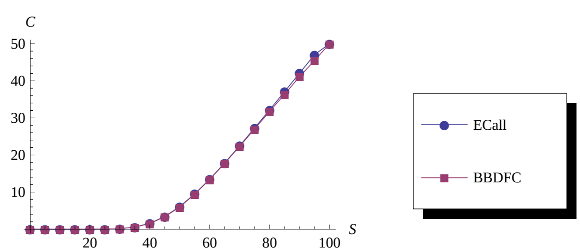


Figure 3: Call Curves for Example 1.

The call and put options obtained using BBDF and the analytical solution are presented in Figure 3 and Figure 4. It is observed from Figure 3 and Figure 4 that the BBDF fairly approximates the analytical solution.

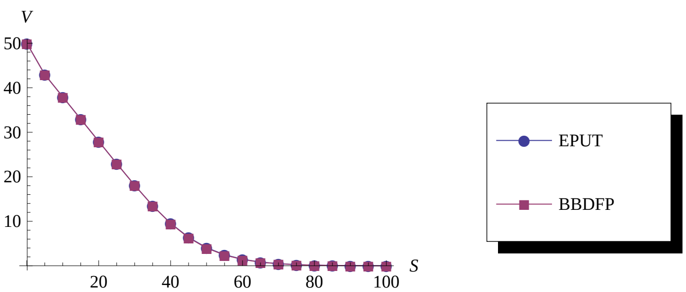


Figure 4: Put Curves for Example 1.

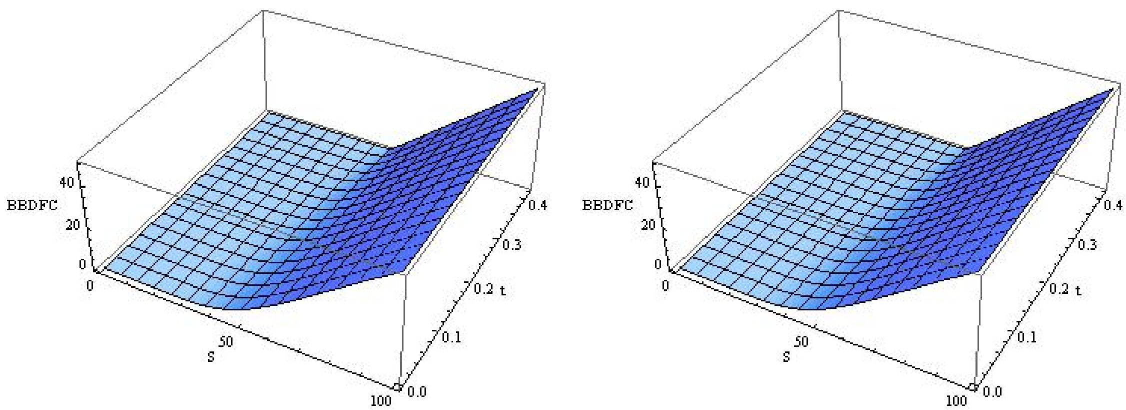


Figure 5: Approximate and exact solutions for the call option for Example 4.1, $h = 1/12, \Delta S = 20$.

Example 4.2. *As our second test example, we solve the given stiff parabolic equation given in Cash (1984).*

$$\frac{\partial V}{\partial t} = \kappa \frac{\partial^2 V}{\partial S^2}, \quad u(0,t) = u(1,t) = 0, \quad u(S,0) = \sin \pi S + \sin \omega \pi S, \ \omega \gg 1.$$

The exact solution $V(S,t) = e^{-\pi^2 \kappa t} \sin \pi S + e^{-\omega^2 \pi^2 \kappa t} \sin \omega \pi S$.

According to Cash (1984) as ω increases, equations of the type given in example 4.2 exhibit characteristics similar to model stiff equations. Hence, the methods such as the Crank-Nicolson method which are not L_0-stable are expected to perform poorly. Since the BBDF is L_0-stable it performs very well when applied to this problem. In Table 1, we display the results for $\kappa = 1$ and a range of values for ω.

In order to test for convergence, example 4.2 was solved for various values of $h = \Delta S$ and the results for the global maximum absolute errors ($Err = Max|v_m(t_n) -$

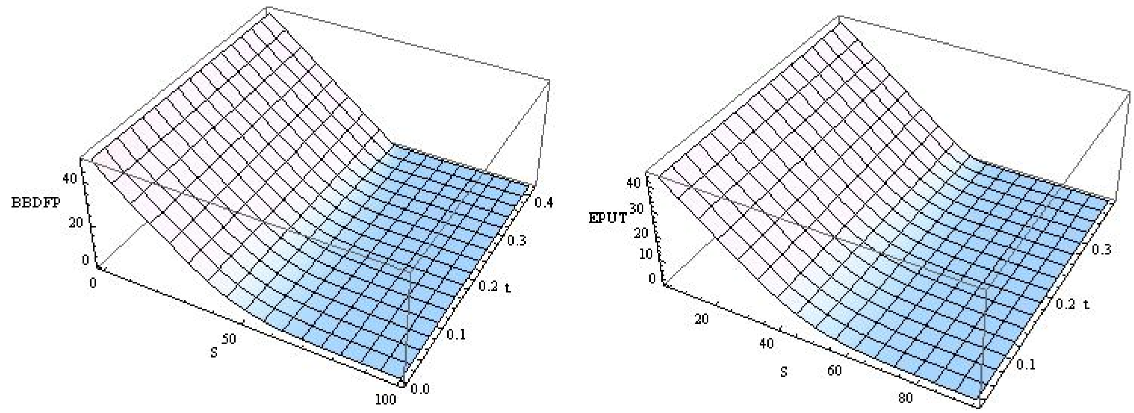


Figure 6: Approximate and exact solutions for the put option for Example 4.1, $h = 1/12, \Delta S = 20$.

$\Delta S = 1/10,\ h = 1/12$		$\Delta S = 1/10,\ h = 1/10$	
ω	(BBDF)	Crank-Nicolson	Cash(abc)
1	8.47×10^{-6}	6.20×10^{-5}	1.5×10^{-5}
2	5.17×10^{-6}	3.83×10^{-5}	7.4×10^{-6}
3	7.89×10^{-6}	9.30×10^{-3}	7.4×10^{-6}
5	4.61×10^{-6}	1.80×10^{-1}	7.4×10^{-6}
10	4.24×10^{-6}	6.10×10^{-1}	7.4×10^{-6}

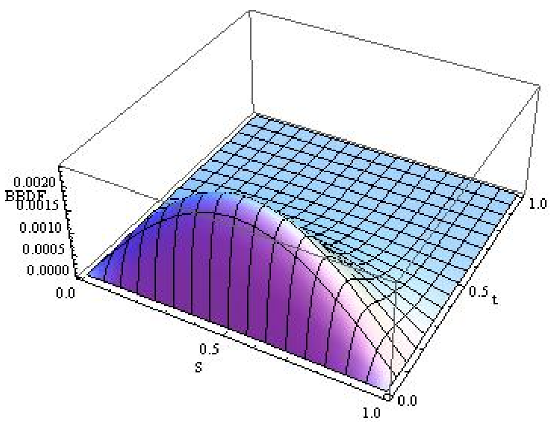
Table 1: A comparison of errors of methods for Example 4.2 at $t = 1$ and $\omega = 1$.

$V(S_m,\ t_n)|)$ are reproduced in Table 2. We also give the rate of convergence (ROC) which is calculated using the formula $ROC = \log_2(Err^{2h}/Err^h)$, Err^h is the error obtained using the step size h. In general, the ROC shows that the order of the method is slightly greater than 2. This is expected since the central difference method used for the spatial discretization is of order 2 and hence affects the convergence of the BBDF which is of order 4 with respect to the time variable. In Fig. 7, the solutions obtained using the BBDF are plotted versus S and t and compared with the plots given by the standard finite difference method (FDM). It is obvious from Figure 7 that the BBDF is more accurate as it produces smaller errors.

Example 4.3. *As our third test example, we solve example 1 for the American put by incorporating an additional equation given by (20) that generates values for the early exercise boundary and hence, ensures that the put option will be optimal. We restate (1) as given in Huang et. al (1996).*

	EBBDF		FDM	
$\Delta S = h$	Err	ROC	Err	ROC
8	6.47×10^{-2}		1.34×10^{-1}	
16	1.14×10^{-2}	2.51	4.26×10^{-2}	1.65
32	1.10×10^{-3}	3.38	1.18×10^{-2}	1.85
64	1.80×10^{-4}	2.61	3.03×10^{-3}	1.96
128	3.91×10^{-5}	2.21	7.64×10^{-4}	1.99
256	9.36×10^{-6}	2.06	1.91×10^{-4}	2.00

Table 2: A comparison of errors of methods for Example 4.2 at $t = 1$ and $\omega = 1$.



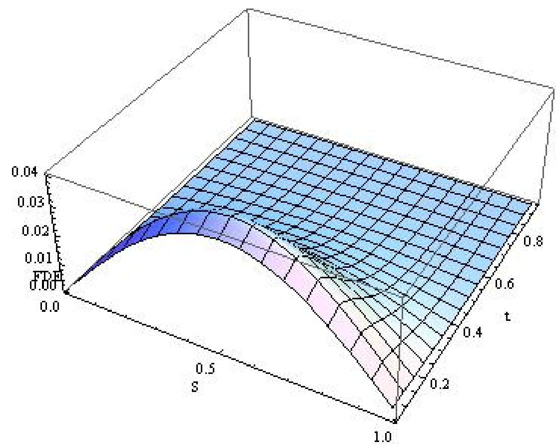


Figure 7: Approximate and exact solutions for Example 4.2, $h = 1/64, \Delta S = 1/16$.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

subject to the initial/boundary conditions

$\lim_{S \to \infty} V(S,t) = 0$

$$V(S,T) = max(X - S, 0)$$

$$\begin{cases} V(S,t) = X - S_t, \\ \frac{\partial V(S,t)}{\partial S} = -1, \quad S = S_t, \end{cases} \qquad (20)$$

where S_t is the free boundary separating the holding and the early exercise regions.

The strike prices were computed by solving the system generated by incorporating (20) into (10). The critical strike price was specified as the lowest price S_t at which the put is exercised early, Carr and Hirsa (2003). In this example, $S_t = Max(X - V(S_t, t)) = \$45$ and the corresponding put value was determined to be \$5. In Figure 8, we plot American-style put values against strike price and time.

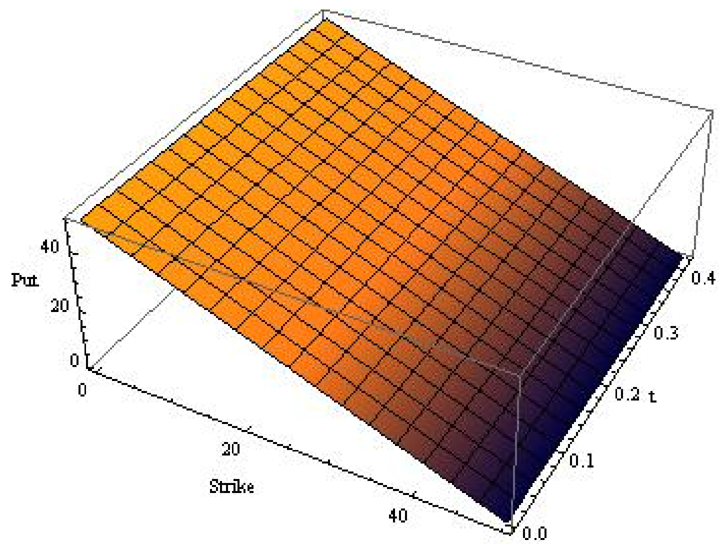


Figure 8: Put values against strike and time for Example 3.

5 Conclusion

We have proposed a BBDF for solving the the Black-Scholes partial differential equation. It is shown that the American put values given in Figure 8 are obtained by incorporating an additional equation that generates values for the early exercise boundary which are used to ensure that the put option will be optimal. It is also shown that the method is L_0-stable and convergent of order 4, hence the BBDF is viable candidate for large stiff systems.

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