Journal of Mathematical Analysis and Applications
Manuscript Draft

Manuscript Number: JMAA-15-2271

Title: Lorentz hypersurfaces in $\mathbb{E}^{n+1}_1$ satisfying $\Delta \overrightarrow{H}=\lambda \overrightarrow{H}$ with at most three distinct principal curvatures

Article Type: Regular Article

Section/Category: Applied Mathematics

Keywords: pseudo-Euclidean space,
Lorentz hypersurface,
proper mean curvature vector field,
shape operator,
constant mean curvature

Abstract: This paper is a continuation of our paper [J. Math. Anal. Appl. 419 (2014) 562--573], where we investigate hypersurface M^n_r of pseudo-Euclidean space $\mathbb{E}^{n+1}_s$ satisfying $\Delta \overrightarrow{H}=\lambda \overrightarrow{H}$, and show that if M^n_r has diagonalizable shape operator with at most three distinct principal curvatures, then it has constant mean curvature. In this paper, we prove that the same conclusion remains true for such Lorentz hypersurfaces with non-diagonalizable shape operators.

Lorentz hypersurfaces in $\mathbb{E}_1^{n+1}$ satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$ with at most three distinct principal curvatures

Jiancheng Liu, Chao Yang

College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, P. R. China
liujc@nwnu.edu.cn(J.Liu), yc963852@126.com(C.Yang)

Abstract

This paper is a continuation of our paper [J. Math. Anal. Appl. 419 (2014) 562–573], where we investigate hypersurface M_r^n of pseudo-Euclidean space $\mathbb{E}_s^{n+1}$ satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$, and show that if M_r^n has diagonalizable shape operator with at most three distinct principal curvatures, then it has constant mean curvature. In this paper, we prove that the same conclusion remains true for such Lorentz hypersurfaces with non-diagonalizable shape operators.

Keywords: pseudo-Euclidean space, Lorentz hypersurface, proper mean curvature vector field, shape operator, constant mean curvature
2000 MSC: 53C50

1. Introduction

Let $x : M_r^n \rightarrow \mathbb{E}_s^{n+1}$ be an isometric immersion of a pseudo-Riemannian hypersurface M_r^n into a pseudo-Euclidean space $\mathbb{E}_s^{n+1}$. Denote by $\overrightarrow{H}$ and Δ the mean curvature vector field of M_r^n and the Laplace operator of M_r^n with respect to the induced metric.

If the hypersurface M_r^n satisfies the equation

$$\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$$

for some real constant λ, then it is said to have *proper mean curvature vector field*. This equation is a natural generalization of the biharmonic submanifold equation $\Delta\overrightarrow{H} = 0$.

Preprint submitted to J. Math. Anal. Appl. *July 13, 2015*

Under the assumption that the shape operator of M_r^n is diagonalizable, we proved in [1] that the hypersurface M_r^n with proper mean curvature vector field and at most three distinct principal curvatures has constant mean curvature.

It is known from [5, 6] that, for Lorentz hypersurface, the shape operator has three possible non-diagonalizable forms except for diagonalizable ones. So, in this paper, we show that the same conclusion in [1] remains true for these three cases and prove the following result.

Main Theorem *Let M_1^n $(n \geq 4)$ be a nondegenerate Lorentz hypersurface with proper mean curvature vector field in $(n+1)$-dimensional pseudo-Euclidean space $\mathbb{E}_1^{n+1}$. Suppose that M_1^n has at most three distinct principal curvatures, then it has constant mean curvature.*

When $n = 3$, it is true automatically that M_1^3 has at most three distinct principal curvatures. In this case, the result has been proved by A. Arvanitoyeorgos et al. in [2, Theorem].

2. Preliminaries

2.1. Lorentz hypersurface in $\mathbb{E}_1^{n+1}$

A non-zero vector X in $\mathbb{E}_1^{n+1}$ is called *time-like*, *space-like* or *light-like*, according to whether $\langle X, X\rangle$ is negative, positive or zero.

Let M_1^n be a nondegenerate Lorentzian hypersurface in $\mathbb{E}_1^{n+1}$, $\overrightarrow{\xi}$ denote a unit normal vector field to M_1^n, then $\langle \overrightarrow{\xi}, \overrightarrow{\xi} \rangle = 1$, i.e. the normal vector to M_1^n is space-like.

Denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections of M_1^n and $\mathbb{E}_1^{n+1}$ respectively. For any vector fields X, Y tangent to M_1^n, the Gauss formula is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y)\overrightarrow{\xi},$$

where h is the scalar-valued second fundamental form. If we denote by A the shape operator of M_1^n associated to $\overrightarrow{\xi}$, the Weingarten formula is given by

$$\widetilde{\nabla}_X \overrightarrow{\xi} = -A(X),$$

where $\langle A(X), Y\rangle = h(X, Y)$. The mean curvature vector $\overrightarrow{H} = H\overrightarrow{\xi}$ with $H = \frac{1}{n}\mathrm{tr}A$, determines a well defined normal vector field to M_1^n in $\mathbb{E}_1^{n+1}$. The

Codazzi and Gauss equations are given by (cf. [3])

$$(\nabla_X A)Y = (\nabla_Y A)X, \tag{1}$$

$$R(X,Y)Z = \langle A(Y), Z\rangle A(X) - \langle A(X), Z\rangle A(Y), \tag{2}$$

where

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

According to [4], a hypersurface M_1^n of $\mathbb{E}_1^{n+1}$ is said to have proper mean curvature vector field, if and only if the following two equations hold:

$$A(\nabla H) = -\frac{n}{2}H(\nabla H), \tag{3}$$

$$\Delta H + H\mathrm{tr}A^2 = \lambda H, \tag{4}$$

where the Laplace operator Δ acting on scalar-valued function f is given by

$$\Delta f = -\sum_{i=1}^{n}\varepsilon_i(e_ie_if - \nabla_{e_i}e_if),$$

with $\{e_i\}_{i=1}^n$ be a local orthonormal frame such that $\langle e_i, e_i\rangle = \varepsilon_i = \pm 1$.

2.2. The shape operator of M_1^n

Except for diagonalizable ones, the shape operator of the Lorentz hypersurface M_1^n in $\mathbb{E}_1^{n+1}$ has three possible non-diagonalizable forms with respect to a frame at $T_xM_1^n$ (cf. [5], [6]):

$$(\mathrm{I})\quad A = \begin{pmatrix} \mu & 0 & \\ 1 & \mu & \\ & & D_{n-2} \end{pmatrix}, \qquad G = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & I_{n-2} \end{pmatrix},$$

$$(\mathrm{II})\quad A = \begin{pmatrix} \mu & 0 & 0 & \\ 0 & \mu & 1 & \\ 1 & 0 & \mu & \\ & & & D_{n-3} \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & I_{n-2} \end{pmatrix},$$

$$(\mathrm{III})\quad A = \begin{pmatrix} \xi & -\eta & \\ \eta & \xi & \\ & & D_{n-2} \end{pmatrix}, \qquad G = \begin{pmatrix} -1 & \\ & I_{n-1} \end{pmatrix}, \eta \neq 0,$$

where $D_{n-2} = \mathrm{diag}\{\lambda_3, \cdots, \lambda_n\}$, $D_{n-3} = \mathrm{diag}\{\lambda_4, \cdots, \lambda_n\}$ and I the identity matrix.

The matric G for case (Ⅲ) is referred to an orthonormal basis $\{e_1, \cdots, e_n\}$ of $T_xM_1^n$, i.e.,

$$\langle e_1, e_1\rangle = -1, \quad \langle e_i, e_j\rangle = \delta_{ij}, \quad \langle e_1, e_i\rangle = 0, \quad 2 \le i, j \le n.$$

However the matrces G for cases (I) and (Ⅱ) are referred to a pseudo-orthonormal basis $\{u_1, u_2, \cdots, u_n\}$, i.e. a basis of $T_xM_1^n$ such that

$$\langle u_1, u_2\rangle = \langle u_i, u_i\rangle = 1 \ \text{ for } \ 3 \le i \le n,$$
$$\langle u_1, u_1\rangle = \langle u_2, u_2\rangle = \langle u_i, u_j\rangle = 0 \ \text{ for } \ 3 \le i \ne j \le n.$$

For the forms (I) and (Ⅱ), we can express

$$\nabla H = u_2(H)u_1 + u_1(H)u_2 + u_3(H)u_3 + \cdots + u_n(H)u_n. \tag{5}$$

Let $\nabla_{u_B} u_C = \sum_{D=1}^n \omega_{BC}^D u_D$. Applying compatability conditions to calculate

$$\nabla_{u_D}\langle u_1, u_1\rangle, \nabla_{u_D}\langle u_2, u_2\rangle, \nabla_{u_D}\langle u_B, u_B\rangle,$$

and

$$\nabla_{u_D}\langle u_1, u_2\rangle, \nabla_{u_D}\langle u_1, u_B\rangle, \nabla_{u_D}\langle u_2, u_B\rangle, \nabla_{u_D}\langle u_B, u_C\rangle,$$

respectively, we conclude

$$\omega_{D1}^2 = \omega_{D2}^1 = \omega_{DB}^B = 0, \tag{6}$$

and

$$\omega_{D1}^1 = -\omega_{D2}^2, \ \omega_{D1}^B = -\omega_{DB}^2, \ \omega_{D2}^B = -\omega_{DB}^1, \ \omega_{DC}^B = -\omega_{DB}^C, \tag{7}$$

for $B, C \ne 1, 2$ and $1 \le D \le n$. These two equations (6) and (7) will be used many times in Sections 3 and 4.

In the following, we will prove our Main Theorem according to each of the non-diagonalizable forms (I), (Ⅱ) and (Ⅲ), and refer readers to [1] when M_1^n has diagonalizable shape operator.

3. When the shape operator has the canonical form (I)

Theorem 3.1 *Let M_1^n ($n \ge 4$) be a nondegenerate Lorentzian hypersurface of the pseudo-Euclidean space $\mathbb{E}_1^{n+1}$ satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$ (λ a real constant). Suppose that the shape operator of M_1^n has the form (I) and M_1^n has at most three distinct principal curvatures, then M_1^n has constant mean curvature.*

In view of the canonical form (I) with respect to the pseudo-orthonormal basis $\mathfrak{B} = \{u_1, u_2, \cdots, u_n\}$, we have

$$A(u_1) = \mu u_1 + u_2, \quad A(u_2) = \mu u_2, \quad A(u_3) = \lambda_3 u_3, \; \cdots, \quad A(u_n) = \lambda_n u_n. \quad (8)$$

In order to prove Theorem 3.1, we start with the Codazzi equation

$$\langle(\nabla_X A)Y, Z\rangle = \langle(\nabla_Y A)X, Z\rangle.$$

After straightforward calculations, combining (6) and (7), we get the following equations:
For $X = u_1, Y = u_2, Z = u_1$,

$$u_1(\mu) = 2\omega_{22}^2. \quad (9)$$

For $X = u_1, Y = u_2, Z = u_2$,

$$u_2(\mu) = 0. \quad (10)$$

For $X = u_1, Y = u_2, Z = u_B, B \neq 1, 2$,

$$(\mu - \lambda_B)(\omega_{12}^B - \omega_{21}^B) = \omega_{22}^B. \quad (11)$$

For $X = u_1, Y = u_B, Z = u_2, B \neq 1, 2$,

$$u_B(\mu) = (\lambda_B - \mu)\omega_{1B}^1. \quad (12)$$

For $X = u_1, Y = u_B, Z = u_C, B, C \neq 1, 2$,

$$u_1(\lambda_B) = (\mu - \lambda_B)\omega_{B1}^B + \omega_{B2}^B, \quad (13)$$

$$(\lambda_B - \lambda_C)\omega_{1B}^C = (\mu - \lambda_C)\omega_{B1}^C + \omega_{B2}^C. \quad (14)$$

For $X = u_2, Y = u_B, Z = u_1, B \neq 1, 2$,

$$u_B(\mu) = (\lambda_B - \mu)\omega_{2B}^2 - \omega_{2B}^1. \quad (15)$$

For $X = u_2, Y = u_B, Z = u_2, B \neq 1, 2$,

$$(\lambda_B - \mu)\omega_{2B}^1 = 0. \quad (16)$$

For $X = u_2, Y = u_B, Z = u_C, B, C \neq 1, 2$,

$$u_2(\lambda_B) = (\mu - \lambda_B)\omega_{B2}^B, \quad (17)$$

$$(\lambda_B - \lambda_C)\omega_{2B}^C = (\mu - \lambda_C)\omega_{B2}^C. \tag{18}$$

For $X = u_B, Y = u_C, Z = u_2, B, C \neq 1, 2$,

$$(\lambda_C - \mu)\omega_{BC}^1 = (\lambda_B - \mu)\omega_{CB}^1. \tag{19}$$

For $X = u_B, Y = u_C, Z = u_D, B, C, D \neq 1, 2$,

$$u_B(\lambda_C) = (\lambda_B - \lambda_C)\omega_{CB}^C, \ B \neq C \tag{20}$$

$$(\lambda_C - \lambda_D)\omega_{BC}^D = (\lambda_B - \lambda_D)\omega_{CB}^D, \ D \neq B, C. \tag{21}$$

Proof *(of Theorem 3.1)* Assume that H is not a constant, then (3) tells us that ∇H is an eigenvector of A with corresponding eigenvalue $-\frac{n}{2}H$. In view of (8), ∇H can be chosen in the direction of u_2 (light-like), or in one of the directions $u_3, \cdots, u_n$ (space-like). We will prove that either of cases will lead to a contradiction, then complete the proof of Theorem 3.1

Case 1: ∇H is space-like.

Without loss of generality, we can choose u_n in the direction of ∇H, so $\lambda_n = -\frac{n}{2}H$, and (5) implies that

$$u_n(H) \neq 0, \quad u_1(H) = u_2(H) = \cdots = u_{n-1}(H) = 0. \tag{22}$$

Observe $(\nabla_{u_B}u_C - \nabla_{u_C}u_B)(H) = [u_B, u_C](H)$, using (22), we have

$$\omega_{BC}^n = \omega_{CB}^n, \quad B, C \neq n. \tag{23}$$

If $\mu = \lambda_n$ or $\lambda_C = \lambda_n$, by (12) and (20) with $B = n$, we have $u_n(\lambda_n) = 0$. It follows that $u_n(H) = 0$, a contradiction. So λ_n is simple, i.e.

$$\mu \neq \lambda_n, \ \lambda_C \neq \lambda_n, \ \ 3 \leq C \leq n-1. \tag{24}$$

Under the assumption that M_1^n has at most three distinct principal curvatures, combining (24), the form (I) can be written as

$$\text{(I)} \quad A = \begin{pmatrix} \mu & 0 & & & \\ 1 & \mu & & & \\ & & D_{t-2}(\mu) & & \\ & & & D_{n-t-1}(\nu) & \\ & & & & \lambda_n \end{pmatrix},$$

where $2 \leq t \leq n-1$, $D_{t-2}(\mu) = \mathrm{diag}\{\mu, \cdots, \mu\}$, $D_{n-t-1}(\nu) = \mathrm{diag}\{\nu, \cdots, \nu\}$ and μ, ν, λ_n are mutually distinct principal curvatures of M_1^n with multiplicities t, $n-t-1$ and 1 respectively.

When $t = n-1$, i.e. M_1^n has two distinct principal curvatures, we will explain this situation in Remark 1.

In the following, we suppose $2 \leq t \leq n-2$, i.e. M_1^n has three distinct principal curvatures. Since $H = \frac{1}{n}\mathrm{tr}A$ and $\lambda_n = -\frac{n}{2}H$, it follows that $\nu = \frac{\frac{3}{2}nH - t\mu}{n-t-1}$.

In this case, we shall make use of the following convention for the range of indices: $1 \leq B, C, D \leq n$, $3 \leq i, j, k \leq t$, $t+1 \leq a, b, c \leq n-1$, and $B, C, D, i, j, k, a, b, c$ are distinct. When $t = 2$, the terms about i, j, k are disappear.

Lemma 3.2 *We have the following results about* $\{\omega_{BC}^D\}$*:*

$$\omega_{2a}^1 = \omega_{2n}^1 = \omega_{ia}^1 = \omega_{in}^1 = \omega_{2i}^a = \omega_{a2}^b = \omega_{ai}^b = \omega_{an}^b = \omega_{ia}^j = \omega_{in}^j = 0,$$
$$\omega_{nB}^n = 0, \quad \omega_{aD}^n = 0, D \neq a, \quad \omega_{1i}^a = \omega_{i1}^a, \quad u_1(\mu) = \omega_{i2}^i,$$
$$u_1(\nu) = (\mu - \nu)\omega_{a1}^a + \omega_{a2}^a, \quad u_2(\nu) = (\mu - \nu)\omega_{a2}^a, \quad u_i(\nu) = (\mu - \nu)\omega_{ai}^a, \tag{25}$$
$$\omega_{1a}^1 = \omega_{2a}^2 = \cdots = \omega_{ta}^t = \frac{u_a(\mu)}{\nu - \mu}, \tag{26}$$
$$\begin{cases} \omega_{1n}^1 = \omega_{2n}^2 = \cdots = \omega_{tn}^t = \frac{u_n(\mu)}{-\frac{n}{2}H - \mu}, \\ \omega_{(t+1)n}^{t+1} = \cdots = \omega_{(n-1)n}^{n-1} = \frac{u_n(\nu)}{-\frac{n}{2}H - \nu}. \end{cases} \tag{27}$$

Proof It follows from (16) that $\omega_{2a}^1 = \omega_{2n}^1 = 0$, together with the equations (12), (15) and (20) for $B = a, n$, we get (26) and (27).

Using equations (13), (17) for $B = a, n$, (20) for $C = a, n$ and (22), we have (25) and $\omega_{nB}^n = 0$. Combining (7), (18) and (19) give $\omega_{in}^1 = \omega_{ia}^1 = \omega_{2i}^a = \omega_{a2}^b = 0$. Observe (14), combining $\omega_{ia}^1 = 0$ and (7), we have $\omega_{1i}^a = \omega_{i1}^a$.

Note that $u_1(\mu) = \omega_{i2}^i$ by (13). Then we have from (14), (18) and (21) that

$$\begin{cases} (\nu + \frac{n}{2}H)\omega_{1a}^n = (\mu + \frac{n}{2}H)\omega_{a1}^n + \omega_{a2}^n \\ (\nu + \frac{n}{2}H)\omega_{2a}^n = (\mu + \frac{n}{2}H)\omega_{a2}^n, \\ (\nu + \frac{n}{2}H)\omega_{ia}^n = (\mu + \frac{n}{2}H)\omega_{ai}^n, \end{cases}$$

which together with (23) implies that

$$\omega_{2a}^n = \omega_{a2}^n = \omega_{ia}^n = \omega_{ai}^n = \omega_{1a}^n = \omega_{a1}^n = 0.$$

Finally, it follows from (21) that $\omega_{ai}^b = \omega_{an}^b = \omega_{ia}^j = \omega_{in}^j = 0$. □

Lemma 3.3 *We have*

$$-(t\omega_{1n}^1 + (n-t-1)\omega_{(n-1)n}^{n-1})u_n(H) - u_n u_n(H) + H\mathrm{tr}A^2 = \lambda H, \tag{28}$$

where

$$\mathrm{tr}A^2 = \frac{t(n-1)}{n-t-1}\mu^2 - \frac{3nt}{n-t-1}H\mu + \frac{n-t+8}{4(n-t-1)}n^2H^2. \tag{29}$$

Proof Based upon the pseudo-orthonormal basis $\mathfrak{B} = \{u_1, u_2, \cdots, u_n\}$, we can construct an orthonormal basis $\mathfrak{E} = \{e_1, e_2, \cdots, e_n\}$, given by

$$e_1 = (u_1 - u_2)/\sqrt{2}, \quad e_2 = (u_1 + u_2)/\sqrt{2}, \quad e_3 = u_3, \quad \cdots, \quad e_n = u_n. \tag{30}$$

The shape operator A with respect to this new basis $\mathfrak{E}$ takes the form

$$A = \begin{pmatrix} \mu - \frac{1}{2} & -\frac{1}{2} & & & \\ \frac{1}{2} & \mu + \frac{1}{2} & & & \\ & & D_{t-2}(\mu) & & \\ & & & D_{n-t-1}(\nu) & \\ & & & & \lambda_n \end{pmatrix},$$

and a straightforward calculation gives (29).

With the orthonormal basis $\mathfrak{E}$, we express $\nabla H = -e_1(H)e_1 + e_2(H)e_2 + \cdots + e_n(H)e_n$. Therefore,

$$e_n(H) \neq 0, \quad e_1(H) = e_2(H) = \cdots = e_{n-1}(H) = 0.$$

It follows from (4) that

$$-\nabla_{e_1}e_1(H) + \nabla_{e_2}e_2(H) + \cdots + \nabla_{e_n}e_n(H) - e_ne_n(H) + H\mathrm{tr}A^2 = \lambda H. \tag{31}$$

According to (30), we easily obtain

$$\begin{aligned}
&\nabla_{e_1}e_1(H) = \frac{1}{2}(\omega_{11}^n - \omega_{21}^n - \omega_{12}^n + \omega_{22}^n)u_n(H),\\
&\nabla_{e_2}e_2(H) = \frac{1}{2}(\omega_{11}^n + \omega_{21}^n + \omega_{12}^n + \omega_{22}^n)u_n(H),\\
&\nabla_{e_B}e_B(H) = \nabla_{u_B}u_B(H) = \omega_{BB}^n u_n(H), \quad 3 \leq B \leq n-1,\\
&\nabla_{e_n}e_n(H) = \nabla_{u_n}u_n(H) = 0.
\end{aligned}$$

Using these equations, combining (7) and (27), then (31) can be simplified to (28) and the lemma follows. □

Lemma 3.4 *Suppose that H is not a constant. Then, for $1 \leq B \leq n-1$,*

$$u_B(\mu) = 0.$$

Proof From (10) and (12), we easily get $u_2(\mu) = u_3(\mu) = \cdots = u_t(\mu) = 0$. In the following, we need to prove: (i) $u_1(\mu) = 0$. (ii) $u_{t+1}(\mu) = \cdots = u_{n-1}(\mu) = 0$.

(i) To prove $u_1(\mu) = 0$.

Since $[u_B, u_n] = \nabla_{u_B} u_n - \nabla_{u_n} u_B$, using (6), (22) and $\omega_{nB}^n = 0$ (see Lemma 3.2), we have

$$u_B u_n(H) = 0, \quad u_B u_n u_n(H) = 0, B \neq n. \tag{32}$$

As $u_2(\mu) = u_3(\mu) = \cdots = u_t(\mu) = 0$, (25) implies that

$$\omega_{(n-1)2}^{n-1} = \omega_{(n-1)3}^{n-1} = \cdots = \omega_{(n-1)t}^{n-1} = 0 \tag{33}$$

and

$$\omega_{(n-1)1}^{n-1} = \frac{u_1(\nu)}{\mu - \nu}. \tag{34}$$

Using Gauss equation for $\langle R(u_{n-1}, u_1)u_n, u_{n-1}\rangle$, combining (6), (7), (23), (33) and Lemma 3.2, we obtain

$$u_1(\omega_{(n-1)n}^{n-1}) = (\omega_{1n}^1 - \omega_{(n-1)n}^{n-1})\omega_{(n-1)1}^{n-1}. \tag{35}$$

Putting $\omega_{(n-1)n}^{n-1} = -\frac{u_n(\nu)}{\frac{n}{2}H+\nu}$ (see Eq. (27)) and (34) into (35) gives

$$u_1 u_n(\nu) = -(\frac{n}{2}H + \nu)\omega_{1n}^1 \omega_{(n-1)1}^{n-1} + \frac{(t-3)nH + 2(n+t-1)\nu}{2t}\omega_{(n-1)n}^{n-1}\omega_{(n-1)1}^{n-1}.$$

Differentiating both sides of $\omega_{1n}^1 = \frac{u_n(3nH - 2(n-t-1)\nu)}{-(t+3)nH + 2(n-t-1)\nu}$ (cf. Eq. (27)) along the direction u_1, using (33), (35) and the above equation, we have

$$u_1(\omega_{1n}^1) = \frac{(n-t-1)((t-3)nH + 2(n+t-1)\nu)}{-t(t+3)nH + 2(n-t-1)t\nu}(\omega_{1n}^1 - \omega_{(n-1)n}^{n-1})\omega_{(n-1)1}^{n-1}. \tag{36}$$

In view of $\nu = \frac{\frac{3}{2}nH - t\mu}{n-t-1}$ and (22), we know that $u_1(\mu) = 0$ is equivalent to $u_1(\nu) = 0$. Suppose on the contrary that $u_1(\nu) \neq 0$, differentiating both sides of (28) along the direction u_1, using (22), (32), (34), (35) and (36), we deduce that

$$\frac{4t(\omega_{1n}^1 - \omega_{(n-1)n}^{n-1})u_n(H)}{-(t+3)nH + 2(n-t-1)\nu} + \frac{-3nH^2 + 2(n-1)H\nu}{t} = 0.$$

By differentiating the above equation along u_1, with $u_1(\nu) \neq 0$, we get

$$\frac{2t(((n-1)(t-6)+9t)nH + 4(n-t-1)(n-1)\nu)}{(-(t+3)nH + 2(n-t-1)\nu)^2(\frac{3}{2}nH - (n-1)\nu)}(\omega_{1n}^1 - \omega_{(n-1)n}^{n-1})u_n(H) + \frac{(n-1)H}{t} = 0.$$

Eliminating $(\omega_{1n}^1 - \omega_{(n-1)n}^{n-1})u_n(H)$ from the above two equations yields

$$H(3nH - 2(n-1)\nu)^2 = 0.$$

Since $H \neq 0$, so $\nu = \frac{3nH}{2(n-1)}$, and then $\mu = \frac{3nH}{2(n-1)}$, which contradicts to $\mu \neq \nu$. So $u_1(\nu) = 0$, then $u_1(\mu) = 0$.

(ii) To prove $u_{t+1}(\mu) = \cdots = u_{n-1}(\mu) = 0$.

When $t \neq n-2$, the equation (20) implies $u_{t+1}(\nu) = \cdots = u_{n-1}(\nu) = 0$, thus, $u_{t+1}(\mu) = \cdots = u_{n-1}(\mu) = 0$. When $t = n-2$, we need only to prove $u_{n-1}(\mu) = 0$.

Calculating $\langle R(u_i, u_{n-1})u_n, u_i\rangle$, $\langle R(u_1, u_{n-1})u_n, u_2\rangle$ by Gauss equation, using (6), (7), (23) and Lemma 3.2, we obtain

$$u_{n-1}(\omega_{in}^i) = (\omega_{(n-1)n}^{n-1} - \omega_{in}^i)\omega_{i(n-1)}^i + 2\omega_{(n-1)i}^1\omega_{1n}^i, \tag{37}$$

$$u_{n-1}(\omega_{1n}^1) = (\omega_{(n-1)n}^{n-1} - \omega_{1n}^1)\omega_{1(n-1)}^1 - \sum_{k=3}^{t}\omega_{(n-1)k}^1\omega_{1n}^k. \tag{38}$$

Taking sum in (37) for i, combining $\omega_{in}^i = \omega_{1n}^1$ (cf. Eq. (27)) and $\omega_{i(n-1)}^i = \omega_{1(n-1)}^1$ (cf. Eq. (26)), we get $\sum_{k=3}^{t}\omega_{(n-1)k}^1\omega_{1n}^k = 0$, and (38) then becomes

$$u_{n-1}(\omega_{1n}^1) = (\omega_{(n-1)n}^{n-1} - \omega_{1n}^1)\omega_{1(n-1)}^1. \tag{39}$$

Suppose on the contrary that $u_{n-1}(\mu) \neq 0$, taking $t = n-2$ in (28) and differentiating along the direction u_{n-1}, making use of (22), (32), (39) and the formulas of ω_{1n}^1, $\omega_{(n-1)n}^{n-1}$ and $\omega_{1(n-1)}^1$ in Lemma 3.2, we deduce that

$$\frac{-2u_n(H)(\omega_{(n-1)n}^{n-1} - \omega_{1n}^1)}{2nH - (n-2)\mu} - 3nH^2 + 2(n-1)H\mu = 0.$$

Differentiating the above equation along u_{n-1}, we get

$$\frac{-(n(5n-14)H-4(n-1)(n-2)\mu)}{(2nH-(n-2)\mu)^2(\frac{3}{2}nH-(n-1)\mu)}(\omega^{n-1}_{(n-1)n}-\omega^1_{1n})u_n(H)+2(n-1)H=0.$$

Eliminating $(\omega^{n-1}_{(n-1)n}-\omega^1_{1n})u_n(H)$ yields $H(3nH-2(n-1)\mu)=0$. Then $\mu=\nu=\frac{3nH}{2(n-1)}$, which contradicts to $\mu\neq\nu$. So $u_{n-1}(\mu)=0$. □

Lemma 3.5 *Suppose that H is not a constant. Then we have*

$$u_n(\omega^{n-1}_{(n-1)n})+(\omega^{n-1}_{(n-1)n})^2=\frac{n}{2}H\frac{\frac{3}{2}nH-t\mu}{n-t-1}, \tag{40}$$

$$u_n(\omega^1_{1n})+(\omega^1_{1n})^2=\frac{n}{2}H\mu, \tag{41}$$

$$\omega^{n-1}_{(n-1)n}\omega^1_{1n}=\frac{t\mu^2-\frac{3}{2}nH\mu}{n-t-1}. \tag{42}$$

Proof Using Gauss equation (2) for $\langle R(u_n,u_{n-1})u_{n-1},u_n\rangle$, combining (6), (7), (23), Lemmas 3.2 and 3.4, it is not difficult to check (40).

Similarly, using Gauss equation for $\langle R(u_n,u_1)u_2,u_n\rangle$, $\langle R(u_n,u_i)u_i,u_n\rangle$, $\langle R(u_{n-1},u_1)u_{n-1},u_2\rangle$ and $\langle R(u_{n-1},u_i)u_{n-1},u_i\rangle$, by Lemmas 3.2 and 3.4, we obtain

$$u_n(\omega^1_{1n})+(\omega^1_{1n})^2=-\sum_{k=3}^{t}\omega^k_{n2}\omega^n_{1k}+\frac{n}{2}H\mu, \tag{43}$$

$$u_n(\omega^i_{in})+(\omega^i_{in})^2=-2\omega^1_{ni}\omega^n_{1i}+\frac{n}{2}H\mu,\quad i=3,\cdots,t, \tag{44}$$

$$\omega^{n-1}_{(n-1)n}\omega^1_{1n}+\sum_{k=3}^{t}\omega^k_{1(n-1)}\omega^1_{(n-1)k}=\frac{t\mu^2-\frac{3}{2}nH\mu}{n-t-1}, \tag{45}$$

$$\omega^{n-1}_{(n-1)n}\omega^i_{in}-2\omega^i_{1(n-1)}\omega^1_{(n-1)i}=\frac{t\mu^2-\frac{3}{2}nH\mu}{n-t-1}. \tag{46}$$

Taking sum in (44) and (46) for i, $3\leq i\leq t$, combining $\omega^i_{in}=\omega^1_{1n}$ (cf. Eq. (27)), then (43) and (45) can be simplified to (41) and (42). □

Now, we continue the proof of Theorem 3.1 for case 1.

Substituting $\omega^{n-1}_{(n-1)n}=\frac{u_n(3nH-2t\mu)}{-n(n-t+2)H+2t\mu}$ and $\omega^1_{1n}=\frac{u_n(\mu)}{-\frac{n}{2}-\mu}$ (cf. Eq. (27)) into (40) and (41), together with (28) and (42), we eliminate $u_nu_n(H)$ and $u_nu_n(\mu)$ and obtain

$$
\begin{aligned}
&(n-t-1)\big((-3nt+8n-4)\omega_{1n}^1+(-3n+3t+10)n\omega_{(n-1)n}^{n-1}\big)u_n(H)\\
&=\frac{3}{4}(-n+t-12)n^3H^3-\frac{1}{2}(-n^2+nt+9n+20t+10)n^2H^2\mu\\
&\quad+(-n^2-4nt-3t+4)nH\mu^2+2(n-2)t\mu^3+3n(n-t-1)\lambda H.
\end{aligned}
\tag{47}
$$

Acting on both sides of (47) by u_n, applying Lemmas 3.4 and 3.5, we obtain

$$
f_1(H,\mu)\omega_{1n}^1+g_1(H,\mu)\omega_{(n-1)n}^{n-1}=h_1(H,\mu)u_n(H),
\tag{48}
$$

where

$$
\begin{aligned}
f_1(H,\mu)=&\frac{1}{4}(-n^3+n^2t+20n^2+21nt+106n-32)n^2H^3\\
&+(\frac{1}{2}n^3t-n^3-\frac{1}{2}n^2t^2-\frac{17}{2}n^2t-nt^2-32nt+4n+12t)nH^2\mu\\
&+(n^3t+n^2t^2-n^3+7n^2t+6nt^2-25nt+4t+4n)H\mu^2\\
&-2(n-2)(t-2)t\mu^3-(11n-4)(n-t-1)\lambda H,
\end{aligned}
$$

$$
\begin{aligned}
g_1(H,\mu)=&\frac{1}{2}(11n-11t+40)n^3H^3-2(n-t)(n-2)t\mu^3\\
&+[(n-t)(\frac{1}{2}n^2-\frac{1}{2}nt-\frac{9}{2}n-t-5)-30t]n^2H^2\mu\\
&+(n^3-nt^2+16nt-6t^2-4n-6t)nH\mu^2-10n(n-t-1)\lambda H,
\end{aligned}
$$

$$
\begin{aligned}
h_1(H,\mu)=&-\frac{69}{2}n^3H^2+3n(n-t-1)\lambda\\
&+[-n^3+n^2t+5n^2+\frac{55}{2}nt+16n-2t-2+\frac{3(-3n^2+18n-4)}{2(n-t-1)}]nH\mu\\
&+[-n^3-4n^2t-3nt+4n-\frac{t(-3n^2+18n-4)}{n-t-1}]\mu^2.
\end{aligned}
$$

On the other hand, it follows from (27) that

$$
3nu_n(H)=-2(n-2)(\frac{n}{2}H+\mu)\omega_{1n}^1+[-4nH+2(n-2)\mu]\omega_{(n-1)n}^{n-1}.
\tag{49}
$$

Putting (49) into (48) and eliminating $u_n(H)$, we have

$$
f_2(H,\mu)\omega_{1n}^1+g_2(H,\mu)\omega_{(n-1)n}^{n-1}=0,
\tag{50}
$$

where

$$
\begin{aligned}
f_2(H,\mu) =& \frac{1}{4}(-n^3+n^2t-26n^2+21nt+198n-32)n^2H^3 \\
&+[-\frac{1}{3}n^4+\frac{5}{6}n^3t-\frac{1}{2}n^2t^2+\frac{4}{3}n^3-nt^2-21n^2-51nt \\
&\quad+\frac{116}{3}n+\frac{40}{3}t+\frac{4}{3}+\frac{(n-2)(-3n^2+18n-4)}{2(n-t-1)}]nH^2\mu \\
&+(-n^4+\frac{1}{3}n^3t+n^2t^2+n^3+23n^2t+6nt^2 \\
&\quad+12n^2-61nt-\frac{64}{3}n+\frac{20}{3}t+\frac{8}{3})H\mu^2 \\
&-\frac{2}{3}(n-2)[n^2+4nt+3t^2-3t-4+\frac{t(-3n^2+18n-4)}{n(n-t-1)}]\mu^3 \\
&+(n-t-1)(n^2-13n+4)\lambda H+2(n-2)(n-t-1)\mu\lambda,
\end{aligned}
$$

$$
\begin{aligned}
g_2(H,\mu) =& \frac{1}{2}(11n-11t-52)n^3H^3 \\
&+[\frac{1}{2}n^4-n^3t+\frac{1}{2}n^2t^2-\frac{13}{2}n^3+\frac{11}{2}n^2t+nt^2+\frac{68}{3}n^2+\frac{35}{3}nt \\
&\quad-\frac{74}{3}n-\frac{8}{3}t-\frac{8}{3}+\frac{2(-3n^2+18n-4)}{n-t-1}]nH^2\mu \\
&+[\frac{5}{3}n^4-\frac{2}{3}n^3t-n^2t^2-\frac{14}{3}n^3-\frac{23}{3}n^2t-6nt^2-\frac{44}{3}n^2-\frac{26}{3}nt \\
&\quad+\frac{20}{3}n+\frac{(n-2)(-3n^2+18n-4)(-n-\frac{4}{3}t+2)}{n-t-1}]H\mu^2 \\
&+2(n-2)[\frac{1}{3}n^2+\frac{1}{3}nt+t^2+t-\frac{4}{3}+\frac{t(-3n^2+18n-4)}{3n(n-t-1)}]\mu^3 \\
&-6n(n-t-1)\lambda H-2(n-2)(n-t-1)\lambda\mu.
\end{aligned}
$$

Combining (42), (47) with (49), we get that

$$
f_3(H,\mu)(\omega_{1n}^1)^2+g_3(H,\mu)(\omega_{(n-1)n}^{n-1})^2=h_3(H,\mu), \tag{51}
$$

where

$$
\begin{aligned}
f_3(H,\mu) &= -\frac{2(n-2)}{3n}(n-t-1)(3nt-8n+4)(\frac{n}{2}H+\mu), \\
g_3(H,\mu) &= \frac{2}{3}(n-t-1)(-3n+3t+10)(-2nH+(n-2)\mu),
\end{aligned}
$$

$$h_3(H,\mu) = \frac{3}{4}(-n+t-12)n^3H^3 + \frac{1}{2}(2n^2-2nt-7n+38t-2)n^2H^2\mu$$
$$+ (-n^3t+n^2t^2+2n^3-\frac{14}{3}n^2t-6nt^2-8n^2+13nt+12n-8)H\mu^2$$
$$-\frac{2}{3n}t(n-2)(3n^2-6nt-5n+4)\mu^3 + 3n(n-t-1)\lambda H.$$

Using relations (42), (50) and (51), by a long computation, we obtain an algebraic equation of H and μ, and denoted by $f(H,\mu)=0$. Acting on $f(H,\mu)=0$ with u_n twice, and using (40), (41), (42), (49) and (50), we get another algebraic equation of H and μ with the form $g(H,\mu)=0$. Applying Lemma 4.6 in [1], μ can be eliminated from $f(H,\mu)=0$ and $g(H,\mu)=0$, and we obtain an algebraic equation for H with constant coefficients. So, H must be a constant, a contradiction. We prove Theorem 3.1 for case 1.

Remark 1 When $t=n-1$, $\mu=\frac{3nH}{2(n-1)}$ and many terms in the proof of Theorem 3.1 are disappear. Meanwhile, Lemma 3.3 becomes more simple and Lemma 3.5 is simplified to (41). Substitute $\omega_{1n}^1=\frac{u_n(\mu)}{-\frac{n}{2}H-\mu}$ into (41) and combine Lemma 3.3, we get the equation

$$\frac{2(n+2)(n-4)}{9}(\omega_{1n}^1)^2 + \frac{n^2(n+5)H^2}{2(n-1)} = \lambda.$$

Applying u_n to both sides of the above equation, we obtain

$$\frac{4(n-4)}{9}(\omega_{1n}^1)^2 + \frac{3n^2H^2}{n-1} = 0,$$

which together with the previous equation leads to H be a constant. It is a contradiction.

Case 2: ∇H is light-like.

In this case, ∇H is in the direction of u_2. By our assumption that M_1^n has at most three distinct principal curvatures, the form (I) can be written as

$$\text{(I)}\quad A = \begin{pmatrix} \mu & 0 & & & \\ 1 & \mu & & & \\ & & D_{t-2}(\mu) & & \\ & & & D_p(\nu) & \\ & & & & D_{n-t-p}(\tau) \end{pmatrix},$$

where $2 \le t \le n$, μ,ν,τ are mutually distinct principal curvatures of M_1^n with multiplicities t, p and $n-t-p$ respectively.

When $t = n$, there are only one principal curvature; when $t+p = n, t \neq n$, there are two distinct principal curvatures. In the following, we consider that M_1^n has three distinct principal curvatures, i.e. $t+p \neq n$. Indeed, for $t+p = n$, many terms in the following process are disappear, we can easily get $H = 0$, a contradiction.

As ∇H is in the direction of u_2, we know $\mu = -\frac{n}{2}H$. Since $H = \frac{1}{n}\mathrm{tr}A$, it follows that $\nu = \frac{(\frac{t}{2}+1)nH-(n-t-p)\tau}{p}$. We have from (5) that

$$u_1(H) \neq 0, \quad u_2(H) = u_3(H) = \cdots = u_n(H) = 0, \tag{52}$$

which implies that $(\nabla_{u_B} u_C - \nabla_{u_C} u_B)(H) = [u_B, u_C](H) = 0$, $B, C \neq 1$. Thus

$$\omega_{BC}^1 = \omega_{CB}^1, \quad B, C \neq 1. \tag{53}$$

Lemma 3.6 *We have*

$$8(t-2)(\omega_{22}^2)^2 + \frac{4(n-t-p)(\tau-\nu)}{\mu-\nu}\omega_{22}^2\omega_{n2}^n + nH\mathrm{tr}A^2 + n\lambda H = 0, \tag{54}$$

where

$$\mathrm{tr}A^2 = t\mu^2 + p\nu^2 + (n-t-p)\tau^2. \tag{55}$$

Proof We construct an orthonormal basis $\mathfrak{E}$ as (30). According to the form of A with respect to this new basis, by a straightforward calculation, we get (55).

From (30) and (52), we have

$$\begin{cases} e_1e_1(H) = (u_1u_1(H) - u_2u_1(H))/2, \\ e_2e_2(H) = (u_1u_1(H) + u_2u_1(H))/2. \end{cases} \tag{56}$$

Using (30) and (56), (4) can be rewritten as

$$-u_2u_1(H) + \Big(\omega_{21}^1 + \sum_{B=3}^{n} \omega_{BB}^1\Big)u_1(H) + H\mathrm{tr}A^2 = \lambda H. \tag{57}$$

Since $[u_1, u_2](H) = \nabla_{u_1}u_2(H) - \nabla_{u_2}u_1(H)$, combining (6) and (53), we have

$$u_2u_1(H) = \omega_{21}^1 u_1(H) = -\omega_{22}^2 u_1(H). \tag{58}$$

From (9), (13) and (17), it follows that

$$u_1(\mu) = 2\omega_{22}^2 = \omega_{32}^3 = \cdots = \omega_{t2}^t. \tag{59}$$

and

$$\begin{cases} \omega_{b2}^b = \frac{u_2(\nu)}{\mu-\nu}, \ t+1 \leq b \leq t+p, \\ \omega_{\alpha 2}^\alpha = \frac{u_2(\tau)}{\mu-\tau}, \ t+p+1 \leq \alpha \leq n. \end{cases} \tag{60}$$

Noting that $\nu = \frac{(\frac{t}{2}+1)nH-(n-t-p)\tau}{p}$, the above equations imply that

$$\omega_{b2}^b = -\frac{(n-t-p)(\mu-\tau)}{p(\mu-\nu)}\omega_{\alpha 2}^\alpha. \tag{61}$$

Also, we have from (59) and $\mu = -\frac{n}{2}H$ that

$$u_1(H) = -\frac{4}{n}\omega_{22}^2. \tag{62}$$

Substituting (58) and (62) into (57), using (59)∼(61), then (54) follows. □

Now, we continue the proof of Theorem 3.1 for case 2.

Differentiating both sides of $u_1(\mu) = 2\omega_{22}^2$ (cf. Eq. (59)) along the direction u_2, combining $\mu = -\frac{n}{2}H$ and (58), we get

$$u_2(\omega_{22}^2) = -(\omega_{22}^2)^2. \tag{63}$$

From (11), (12), (16), (18) and (19), combining (7), (52) and (53), we obtain

$$\omega_{22}^i = \omega_{1b}^1 = \omega_{1\alpha}^1 = \omega_{2b}^1 = \omega_{2\alpha}^1 = \omega_{\alpha 2}^\beta = \omega_{i\alpha}^1 = \omega_{b\alpha}^1 = 0, \tag{64}$$

with $3 \leq i \leq t, \ t+1 \leq b \leq t+p$ and $\alpha \neq \beta$, $t+p+1 \leq \alpha, \beta \leq n$.

Computing $\langle R(u_2, u_n)u_2, u_n\rangle$ by using Gauss equation, combining (6), (7), (53) and (64), we obtain

$$u_2(\omega_{n2}^n) = (\omega_{22}^2 - \omega_{n2}^n)\omega_{n2}^n.$$

Acting on (54) with u_2, using (60), (61), (63) and the above equation, it follows that

$$\begin{aligned} 8(t-2)(\omega_{22}^2)^3 - 2(n-t-p)\frac{p(\nu-\tau)^2 + (n-t)(\mu-\tau)^2}{p(\mu-\nu)^2}(\omega_{n2}^n)^2\omega_{22}^2 \\ + n(n-t-p)H(\nu-\tau)(\mu-\tau)\omega_{n2}^n = 0. \end{aligned} \tag{65}$$

Eliminating ω_{n2}^n from (65) and (54), we get

$$F(H,\tau,\omega_{22}^2) := f_4(H,\tau)(\omega_{22}^2)^4 + f_2(H,\tau)(\omega_{22}^2)^2 + f_0(H,\tau) = 0,$$

where $f_i(H,\tau), i = 0,2,4$, are algebraic expressions of H and τ. Differentiating $F(H,\tau,\omega_{22}^2) = 0$ along u_2, combining (54) and (60), we get

$$G(H,\tau,\omega_{22}^2) := g_6(H,\tau)(\omega_{22}^2)^6 + g_4(H,\tau)(\omega_{22}^2)^4 + g_2(H,\tau)(\omega_{22}^2)^2 + g_0(H,\tau) = 0,$$

where $g_i(H,\tau), i = 0,2,4,6$, are algebraic expressions of H and τ.

Eliminating ω_{22}^2 from $F(H,\tau,\omega_{22}^2) = 0$ and $G(H,\tau,\omega_{22}^2) = 0$ by using Lemma 4.6 in [1], we obtain an algebraic expression of H and τ, denote by $f(H,\tau) = 0$. Acting on $f(H,\tau) = 0$ with u_2, using $u_2(H) = 0$ and $u_2(\tau) = (\mu - \tau)\omega_{n2}^n$, we have

$$g(H,\tau)\omega_{n2}^n = 0, \tag{66}$$

where $g(H,\tau)$ is an algebraic equation of H and τ.

We claim that $\omega_{n2}^n \neq 0$.

Indeed, If $\omega_{n2}^n = 0$, from (65), we get $(t-2)\omega_{22}^2 = 0$. Then (54) implies that $\mathrm{tr}A^2 = \lambda$. Differentiating along u_B for $3 \leq B \leq n$, we get $2(n-t-p)(\tau - \nu)u_B(\tau) = 0$. As $\nu \neq \tau$, it follows that $u_B(\tau) = 0$, $3 \leq B \leq n$. Combining (20), we finally get

$$\omega_{(t+1)B}^{t+1} = \omega_{nB}^n = 0,\ 3 \leq B \leq t+p+1.$$

Also, if $\omega_{n2}^n = 0$, then (61) implies that $\omega_{(t+1)2}^{t+1} = 0$. Applying Gauss equation to calculate $\langle R(u_1,u_n)u_2,u_n\rangle$ and $\langle R(u_1,u_{t+1})u_2,u_{t+1}\rangle$, combining (53), (64) and the above equation, we find $\mu\tau = 0$ and $\mu\nu = 0$. It follows from $\mu = -\frac{n}{2}H$ that $\nu = \tau = 0$, and then $H = 0$, a contradiction.

Since $\omega_{n2}^n \neq 0$, we conclude from (66) that $g(H,\tau) = 0$. Combining $f(H,\tau) = 0$ and $g(H,\tau) = 0$, we obtain an algebraic equation for H with constant coefficients. So, H must be a constant, a contradiction.

We finish the proof of Theorem 3.1. □

4. When the shape operator has the canonical form (II)

Theorem 4.1 *Let M_1^n ($n \geq 4$) be a nondegenerate Lorentzian hypersurface of the pseudo-Euclidean space $\mathbb{E}_1^{n+1}$ satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$ (λ a real constant).*

Suppose that the shape operator of M_1^n has the form (II) and M_1^n has at most three distinct principal curvatures, then M_1^n has constant mean curvature.

The proof ideas of Theorem 4.1 are the same as that of Theorem 3.1. So in the following, we give the outline of the proof of Theorem 4.1 and leave the details to the readers. First of all, we need some preliminaries.

When the shape operator A has the canonical form (II), we have

$$A(u_1) = \mu u_1 + u_3, \quad A(u_2) = \mu u_2, \quad A(u_3) = u_2 + \mu u_3,$$
$$A(u_B) = \lambda_B u_B, \quad B = 4, \cdots, n.$$

From $\langle (\nabla_X A)Y, Z\rangle = \langle (\nabla_Y A)X, Z\rangle$, using (6) and (7), we get the following equations: for $4 \le B, C, D \le n$ and B, C, D are distinct,

$$\begin{aligned}
&X = u_1, Y = u_2, Z = u_2, \text{and } X = u_2, Y = u_3, Z = u_3 \Longrightarrow u_2(\mu) = \omega_{23}^1 = 0;\\
&X = u_1, Y = u_2, Z = u_B \Longrightarrow (\mu - \lambda_B)(\omega_{12}^B - \omega_{21}^B) = \omega_{23}^B;\\
&X = u_1, Y = u_3, Z = u_B \Longrightarrow (\mu - \lambda_B)(\omega_{13}^B - \omega_{31}^B) = \omega_{33}^B - \omega_{12}^B;\\
&X = u_1, Y = u_B, Z = u_2 \Longrightarrow u_B(\mu) = (\lambda_B - \mu)\omega_{1B}^1 - \omega_{B3}^1;\\
&X = u_1, Y = u_B, Z = u_B \Longrightarrow u_1(\lambda_B) = (\mu - \lambda_B)\omega_{B1}^B + \omega_{B3}^B;\\
&X = u_1, Y = u_B, Z = u_C \Longrightarrow (\lambda_B - \lambda_C)\omega_{1B}^C = (\mu - \lambda_C)\omega_{B1}^C + \omega_{B3}^C;\\
&X = u_2, Y = u_B, Z = u_1 \Longrightarrow u_B(\mu) = (\lambda_B - \mu)\omega_{2B}^2 - \omega_{2B}^3 + \omega_{B2}^3;\\
&X = u_2, Y = u_B, Z = u_2 \Longrightarrow (\lambda_B - \mu)\omega_{2B}^1 = 0;\\
&X = u_2, Y = u_B, Z = u_3 \Longrightarrow (\lambda_B - \mu)\omega_{2B}^3 = \omega_{2B}^1;\\
&X = u_2, Y = u_B, Z = u_B \Longrightarrow u_2(\lambda_B) = (\mu - \lambda_B)\omega_{B2}^B;\\
&X = u_2, Y = u_B, Z = u_C \Longrightarrow (\lambda_B - \lambda_C)\omega_{2B}^C = (\mu - \lambda_C)\omega_{B2}^C;\\
&X = u_3, Y = u_B, Z = u_2 \Longrightarrow (\lambda_B - \mu)\omega_{3B}^1 = 0;\\
&X = u_3, Y = u_B, Z = u_3 \Longrightarrow u_B(\mu) = (\lambda_B - \mu)\omega_{3B}^3 - \omega_{3B}^1 + 2\omega_{B3}^1;\\
&X = u_3, Y = u_B, Z = u_B \Longrightarrow u_3(\lambda_B) = (\mu - \lambda_B)\omega_{B3}^B + \omega_{B2}^B;\\
&X = u_3, Y = u_B, Z = u_C \Longrightarrow (\lambda_B - \lambda_C)\omega_{3B}^C = (\mu - \lambda_C)\omega_{B3}^C + \omega_{B2}^C;\\
&X = u_B, Y = u_C, Z = u_1 \Longrightarrow (\lambda_C - \mu)\omega_{BC}^2 = (\lambda_B - \mu)\omega_{CB}^2;\\
&X = u_B, Y = u_C, Z = u_2 \Longrightarrow (\lambda_C - \mu)\omega_{BC}^1 = (\lambda_B - \mu)\omega_{CB}^1;\\
&X = u_B, Y = u_C, Z = u_3 \Longrightarrow (\lambda_C - \mu)\omega_{BC}^3 - \omega_{BC}^1 = (\lambda_B - \mu)\omega_{CB}^3 - \omega_{CB}^1;\\
&X = u_B, Y = u_C, Z = u_B \Longrightarrow u_C(\lambda_B) = (\lambda_C - \lambda_B)\omega_{BC}^B;\\
&X = u_B, Y = u_C, Z = u_D \Longrightarrow (\lambda_C - \lambda_D)\omega_{BC}^D = (\lambda_B - \lambda_D)\omega_{CB}^D,
\end{aligned} \tag{67}$$

Proof *(of Theorem 4.1)* Assume that H is not a constant, then ∇H can be in the direction of u_2 (light-like), or in one of the directions $u_4, \cdots, u_n$ (space-like). Next, we will study the two cases separately, and show that each of the cases will lead to a contradiction.

Case 1: ∇H is space-like.

We choose u_n in the direction of ∇H, so $\lambda_n = -\frac{n}{2}H$. In this case, (22) and (23) also hold.

Furthermore, λ_n is a simple root. In order to explain this fact, we find from (67) that $\omega_{23}^n = 0$ and

$$u_n(\mu) = (\lambda_n - \mu)\omega_{2n}^2 + \omega_{n2}^3, \quad u_n(\mu) = (\lambda_n - \mu)\omega_{3n}^3 - 2\omega_{n2}^3, \tag{68}$$

If $\lambda_C = \lambda_n$ or $\mu = \lambda_n$, it follows from the last but one equation of (67) or (68) that $u_n(H) = 0$, which contradicts to $u_n(H) \neq 0$ (cf. Eq. (22)). Then λ_n is simple, i.e. $\mu \neq \lambda_n$ and $\lambda_C \neq \lambda_n$, $4 \leq C \leq n-1$.

Note that our assumption that M_1^n has at most three distinct principal curvatures and the fact that λ_n is a simple root, we suppose that μ, ν, λ_n are mutually distinct principal curvatures of M_1^n with multiplicities t, $n-t-1$ and 1 respectively, where, $3 \leq t \leq n-1$. Since $H = \frac{1}{n}\mathrm{tr}A$ and $\lambda_n = -\frac{n}{2}H$, it follows that $t\mu + (n-t-1)\nu = \frac{3}{2}nH$.

In the following, we suppose $3 \leq t \leq n-2$, (i.e. M_1^n has three distinct principal curvatures). After we complete the proof of Theorem 4.1, the readers may check that Theorem 4.1 is also true when $t = n-1$ (i.e. M_1^n has two distinct principal curvatures).

We make use of the following convention for the range of indices: $1 \leq B, C, D \leq n$, $4 \leq i, j, k \leq t$, $t+1 \leq a, b, c \leq n-1$, and $B, C, D, i, j, k, a, b, c$ are distinct. When $t = 3$, the terms about i, j, k are disappear.

By using (7), (22), (23) and (67), we have the following.

Lemma 4.2 *When the shape operator A takes the form (II), we have*

$$\begin{aligned}
&\omega_{nB}^n = 0; \quad \omega_{Ca}^1 = 0,\ C \neq 1, a, n; \quad \omega_{aD}^n = 0,\ D \neq a; \quad \omega_{ia}^i = \frac{u_a(\mu)}{\nu - \mu};\\
&\omega_{2n}^1 = \omega_{in}^1 = \omega_{in}^2 = \omega_{23}^n = \omega_{i3}^n = \omega_{n3}^1 = \omega_{ni}^1 = \omega_{in}^j = 0,\\
&\omega_{23}^a = \omega_{2i}^a = \omega_{a1}^b = \omega_{a2}^b = \omega_{a3}^b = 0,\\
&u_1(\nu) = (\mu - \nu)\omega_{a1}^a + \omega_{a3}^a, \quad u_2(\nu) = (\mu - \nu)\omega_{a2}^a,\\
&u_3(\nu) = (\mu - \nu)\omega_{a3}^a + \omega_{a2}^a, \quad u_i(\nu) = (\mu - \nu)\omega_{ai}^a,
\end{aligned}$$

$$\begin{cases}\omega_{1n}^1 = \omega_{2n}^2 = \cdots = \omega_{tn}^t = \frac{u_n(\mu)}{-\frac{n}{2}H-\mu}, \\ \omega_{(t+1)n}^{t+1} = \cdots = \omega_{(n-1)n}^{n-1} = \frac{u_n(\nu)}{-\frac{n}{2}H-\nu}.\end{cases} \tag{69}$$

Similar to Lemma 3.3, we also can construct an orthonormal basis $\mathfrak{E}$ as (30). Then using (4) and (69), we have the following Lemma.

Lemma 4.3 *When the shape operator A takes the form (II), and ∇H is space-like, then the equations (28) and (29) also hold.*

Lemma 4.4 *Suppose that H is not a constant. Then, for $1 \leq B \leq n-1$,*

$$u_B(\mu) = 0.$$

Proof Observe (67), we have $u_2(\mu) = 0$.

(i) To prove $u_3(\mu) = 0$.

As $u_2(\mu) = 0$, we get from Lemma 4.2 that $\omega_{(n-1)2}^{n-1} = 0$ and $\omega_{(n-1)3}^{n-1} = \frac{u_3(\nu)}{\mu-\nu}$. Calculate $\langle R(u_{n-1}, u_3)u_n, u_{n-1}\rangle$ by using Gauss equation (2), combining $\omega_{(n-1)2}^{n-1} = 0$ and Lemma 4.2, we get

$$u_3(\omega_{(n-1)n}^{n-1}) = (\omega_{3n}^3 - \omega_{(n-1)n}^{n-1})\omega_{(n-1)3}^{n-1}.$$

Suppose on the contrary that $u_3(\nu) \neq 0$, similar to the proof of $u_1(\mu) = 0$ in Lemma 3.4, we finally get $\mu = \nu = \frac{3nH}{2(n-1)}$, a contradiction to $\mu \neq \nu$. So $u_3(\nu) = 0$. Furthermore, $u_3(\mu) = 0$.

(ii) To prove $u_i(\mu) = 0$, $4 \leq i \leq t$.

When $t \neq 4$, the last but one equation of (67) implies $u_i(\mu) = 0$. When $t = 4$, using Gauss equation for $\langle R(u_{n-1}, u_4)u_n, u_{n-1}\rangle$, combining $\omega_{(n-1)2}^{n-1} = 0$ and Lemma 4.2, we obtain

$$u_4(\omega_{(n-1)n}^{n-1}) = (\omega_{4n}^4 - \omega_{(n-1)n}^{n-1})\omega_{(n-1)4}^{n-1}. \tag{70}$$

Following the similar method to the proof of $u_1(\mu) = 0$ in Lemma 3.4, we can check that $u_4(\mu) = 0$ is true. Thus, we conclude $u_i(\mu) = 0$, $4 \leq i \leq t$.

(iii) To prove $u_1(\mu) = 0$.

Combining $u_3(\mu) = u_i(\mu) = 0$ and Lemma 4.2, we find $\omega_{(n-1)3}^{n-1} = \omega_{(n-1)i}^{n-1} = 0$ and $\omega_{(n-1)1}^{n-1} = \frac{u_1(\nu)}{\mu-\nu}$. Taking the same process as that in Lemma 3.4, then $u_1(\mu) = 0$ follows.

Summarize, we obtain $u_1(\mu) = u_2(\mu) = \cdots = u_t(\mu) = 0$. Following the similar process as Lemma 3.4, we also have $u_{t+1}(\mu) = \cdots = u_{n-1}(\mu) = 0$. So, Lemma 4.4 is proved. $\square$

Now, we continue the proof of Theorem 4.1 for case 1.

Using Gauss equation (2) for $\langle R(u_n, u_{n-1})u_n, u_{n-1}\rangle$ and $\langle R(u_n, u_1)u_2, u_n\rangle$, combining Lemmas 4.2 and 4.4, we conclude that (40) and (41) hold. In the following, we will prove that (42) also hold in this case.

By (67), together with Lemma 4.4, we obtain

$$\omega_{3i}^a = \omega_{i3}^a,\ (\mu - \nu)(\omega_{1i}^a - \omega_{i1}^a) = \omega_{i3}^a,\ (\nu - \mu)\omega_{3a}^i = \omega_{a2}^i,$$
$$(\mu - \nu)(\omega_{13}^a - \omega_{31}^a) = (\omega_{1a}^1 - \omega_{3a}^3),\ \omega_{1a}^1 = \omega_{2a}^2 \tag{71}$$

and

$$\begin{cases} (\nu-\mu)\omega_{1a}^1 - \omega_{a3}^1 = 0, \\ (\nu-\mu)\omega_{2a}^2 + \omega_{a2}^3 = 0, \\ (\nu-\mu)\omega_{3a}^3 - \omega_{a2}^3 + \omega_{a3}^1 = 0. \end{cases} \tag{72}$$

Clearly, (72) implies that $\omega_{1a}^1 + \omega_{2a}^2 + \omega_{3a}^3 = 0$. Notice that $\omega_{1a}^1 = \omega_{2a}^2$ (cf. Eq. (71)), we have

$$\omega_{3a}^3 - \omega_{1a}^1 = -3\omega_{1a}^1. \tag{73}$$

Taking into account the components ω_{BC}^D and Lemma 4.4, after applying Gauss equation to $\langle R(u_a, u_1)u_a, u_2\rangle$ and $\langle R(u_a, u_2)u_a, u_1\rangle$, we calculate and obtain

$$\begin{cases} u_a(\omega_{1a}^1) + \omega_{1a}^3\omega_{a3}^1 + \sum\limits_{k=4}^{t}\omega_{1a}^k\omega_{ak}^1 - \sum\limits_{b=t+1}^{n-1}\omega_{aa}^b\omega_{1b}^1 + \omega_{an}^a\omega_{1n}^1 + (\omega_{1a}^1)^2 = \frac{t\mu^2 - \frac{3}{2}nH\mu}{n-t-1}, \\ u_a(\omega_{2a}^2) - \omega_{a2}^3\omega_{3a}^2 - \sum\limits_{k=3}^{t}\omega_{a2}^k\omega_{ka}^2 - \sum\limits_{b=t+1}^{n-1}\omega_{aa}^b\omega_{2b}^2 + \omega_{an}^a\omega_{2n}^2 + (\omega_{2a}^2)^2 = \frac{t\mu^2 - \frac{3}{2}nH\mu}{n-t-1}. \end{cases} \tag{74}$$

Because of $\omega_{1a}^1 = \omega_{2a}^2$ (cf. Eq. (71)) and $\omega_{1n}^1 = \omega_{2n}^2$ (cf. Eq. (69)), it follows from (74) that $\omega_{a3}^1(\omega_{13}^a - \omega_{31}^a) + \sum_{k=3}^{t}\omega_{ak}^1(\omega_{1k}^a - \omega_{k1}^a) = 0$. Combining (71), (72) and (73), we have $3(\omega_{1a}^1)^2 + \sum_{k=3}^{t}(\omega_{k3}^a)^2 = 0$, which implies that $\omega_{1a}^1 = \omega_{k3}^a = 0$. Furthermore, $\omega_{a3}^1 = \omega_{3k}^a = \omega_{ai}^1 = 0$. Substituting into (74), we know that equation (42) also hold.

Summing up, we have proved that (40), (41) and (42) hold under the assumptions of Theorem 4.1 with the case of ∇H be space-like. Starting with these three equations, following the same discussion as case 1 of Theorem 3.1, we get a contradiction.

Case 2: ∇H is light-like.

In this case, we review and find that the equations (52) and (53) hold.

We suppose $\mu = -\frac{n}{2}H, \nu, \tau$ are principal curvatures of M_1^n with multiplicities t, p and $n-t-p$ respectively, and $3 \leq t \leq n$. When $t = n$, there are only one principal curvature. When $t+p = n, t \neq n$, there are two distinct principal curvatures. In the following, we suppose that $t+p \neq n$. From $\mathrm{tr}A = nH$ and $\mu = -\frac{n}{2}H$, we have $\nu = \frac{(\frac{t}{2}+1)nH-(n-t-p)\tau}{p}$.

The equation $\langle(\nabla_X A)Y, Z\rangle = \langle(\nabla_Y A)X, Z\rangle$ for (X, Y, Z) evaluating in the set $\{(u_1, u_2, u_3), (u_1, u_3, u_2), (u_2, u_3, u_i)\}$, $4 \leq i \leq t$, implies $\omega_{21}^1 = \omega_{33}^1 = \omega_{22}^i = 0$. From (52), (53) and (67), we find that

$$\omega_{23}^1 = \omega_{i2}^i = \omega_{2b}^1 = \omega_{2\alpha}^1 = \omega_{\alpha 2}^\beta = \omega_{3b}^1 = \omega_{3\alpha}^1 = \omega_{1b}^1 = \omega_{1\alpha}^1 = \omega_{i\alpha}^1 = \omega_{b\alpha}^1 = 0 \tag{75}$$

and

$$u_2(\nu) = (\mu-\nu)\omega_{b2}^b, \quad u_2(\tau) = (\mu-\tau)\omega_{\alpha 2}^\alpha,$$

where, $t+1 \leq b \leq t+p$, and $\alpha \neq \beta$ for $t+p+1 \leq \alpha, \beta \leq n$.

Note that $\nu = \frac{(\frac{t}{2}+1)nH-(n-t-p)\tau}{p}$, the above two equations with (52) give

$$\omega_{b2}^b = \frac{(n-t-p)(\tau-\mu)}{p(\mu-\nu)}\omega_{\alpha 2}^\alpha. \tag{76}$$

Using $[u_1, u_2](H) = \nabla_{u_1}u_2(H) - \nabla_{u_2}u_1(H)$ and $\omega_{21}^1 = 0$, we have

$$u_2u_1(H) = 0.$$

Then, taking similar method to Lemma 3.6, we get the following Lemma.

Lemma 4.5 *We have*

$$\frac{(\nu-\tau)(n-t-p)}{\mu-\nu}\omega_{n2}^n u_1(H) + H\mathrm{tr}A^2 = \lambda H. \tag{77}$$

where

$$\mathrm{tr}A^2 = t\mu^2 + p\nu^2 + (n-t-p)\tau^2.$$

Now, we continue the proof of Theorem 4.1 for case 2. Using Gauss equation to calculate $\langle R(u_2, u_n)u_2, u_n\rangle$, combining (6), (7), (53) and (75), we obtain $u_2(\omega_{n2}^n) = -(\omega_{n2}^n)^2$. Acting on (77) with u_2, it follows that

$$\frac{p(\nu-\tau)^2 - (n-t)(\mu-\tau)^2}{p(\mu-\nu)^2}\omega_{n2}^n u_1(H) + 2H(\tau-\nu)(\mu-\tau) = 0,$$

from which and (77), we eliminate $\omega_{n2}^n u_1(H)$ and obtain

$$2p(n-t-p)(\mu-\nu)(\tau-\nu)^2(\mu-\tau) = (\lambda-\mathrm{tr}A^2)(p(\nu-\tau)^2-(n-t)(\mu-\tau)^2).$$

Because of $\mu = -\frac{n}{2}H$ and $\nu = \frac{(\frac{t}{2}+1)nH-(n-t-p)\tau}{p}$, the above algebraic equation is of H and τ and denoted by $f(H,\tau)=0$. Acting on $f(H,\tau)=0$ with u_2, we have $g(H,\tau)\omega_{n2}^n = 0$ with $g(H,\tau)$ is an algebraic expression of H and τ.

Similar to the proof of Theorem 3.1 for case 2, we can prove $\omega_{n2}^n \neq 0$, and conclude $g(H,\tau)=0$. Combining $f(H,\tau)=0$ and $g(H,\tau)=0$, we obtain an algebraic equation for H with constant coefficients. So, H must be a constant, a contradiction.

We finish the proof of Theorem 4.1. □

5. When the shape operator has the canonical form (Ⅲ)

Theorem 5.1 *Let M_1^n $(n \geq 4)$ be a nondegenerate Lorentzian hypersurface of the pseudo-Euclidean space $\mathbb{E}_1^{n+1}$ satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$ (λ a real constant). Suppose that the shape operator of M_1^n has the form (Ⅲ) and M_1^n has at most three distinct principal curvatures, then M_1^n has constant mean curvature.*

Proof According to the form (Ⅲ), we easily find that the shape operator has at least three distinct eigenvalues: $\xi+\eta i, \xi-\eta i, \lambda_3$. In view of our assumption, we only need to discuss the case that M_1^n has three distinct principal curvatures.

With the form (Ⅲ), we have

$$A(e_1) = \xi e_1 + \eta e_2, A(e_2) = -\eta e_1 + \xi e_2, A(e_3) = \lambda_3 e_3, \cdots, A(e_n) = \lambda_n e_n$$

with $\lambda_3 = \cdots = \lambda_n$. Assume that H is not a constant, then $\nabla H \neq 0$. From (3), we conclude that ∇H is an eigenvector of A, so it can be in one of the directions $e_3, \cdots, e_n$ (space-like). Without loss of generality, we can choose e_n in the direction of ∇H, so $\lambda_n = -\frac{n}{2}H$. Express ∇H as $\nabla H = e_1(H)e_1 + e_2(H)e_2 + \cdots + e_n(H)e_n$, we get

$$e_n(H) \neq 0, \quad e_1(H) = e_2(H) = \cdots = e_{n-1}(H) = 0.$$

On the other hand, it follows from $\langle(\nabla_{e_n}A)e_i, e_i\rangle = \langle(\nabla_{e_i}A)e_n, e_i\rangle$ that $e_n(\lambda_i) = (\lambda_n - \lambda_i)\omega_{in}^i, 3 \leq i \leq n-1$. Since $\lambda_3 = \cdots = \lambda_n = -\frac{n}{2}H$, we have $e_n(H) = 0$, which contradicts to $e_n(H) \neq 0$. Theorem 5.1 follows. □

Finally, the Main Theorem follows from Theorems 3.1, 4.1 and 5.1.

Acknowledgement This work was supported by the National Natural Science Foundation of China (Grant No. 11261051, 11171246), and 2012 Fundamental Research Funds of the Gansu Universities.

References

[1] J.-C. Liu, C. Yang, Hypersurfaces in $\mathbb{E}_s^{n+1}$ satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$ with at most three distinct principal curvatures, J. Math. Anal. Appl. **419** (2014), 562–573.

[2] A. Arvanitoyeorgos, G. Kaimakamis, M. Magid, Lorentz hypersurfaces in E_1^4 satisfying $\Delta\overrightarrow{H} = \alpha\overrightarrow{H}$, Illinois J. Math. **53**:2 (2009), 581–590.

[3] B. O'Neill, Semi-Riemannian Geometry: with Applications to Relativity, Pure and Applied Mathematics 103, Academic Press, New York, 1983.

[4] B. Y. Chen, S. Ishikawa, Biharmonic surfaces in pseudo-Euclidean spaces, Mem. Fac. Sci. Kyushu Univ. Ser. A. **45**:2 (1991), 323–347.

[5] M. A. Magid, Isometric immersions of Lorentz space with parallel second fundamental forms, Tsukuba J. Math. **8**:1 (1984), 31–54.

[6] M. A. Magid, Lorentzian isoparametric hypersurfaces, Pacific J. Math. **118** (1985), 165–197.

[7] A. Z. Petrov, Einstein spaces, Pergamon Press, Oxford, 1969.