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# Hypersurfaces in $\mathbb{E}_s^{n+1}$ satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$ with at most three distinct principal curvatures

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**Abstract**

In this paper, we investigate hypersurface $M_r^n$ of pseudo-Euclidean space $\mathbb{E}_s^{n+1}$ satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$ ($\lambda$ a constant), and show that if $M_r^n$ has diagonalizable shape operator with at most three distinct principal curvatures, then it has constant mean curvature.

*Keywords:* pseudo-Euclidean space, hypersurface, proper mean curvature vector field, shape operator, diagonalizable
*2000 MSC:* 53C50

## 1. Introduction

Let $x : M_r^m \to \mathbb{E}_s^n$ be an isometric immersion of an $m$-dimensional pseudo-Riemannian submanifold $M_r^m$ into a pseudo-Euclidean space $\mathbb{E}_s^n$. Denote by $\overrightarrow{H}$ and $\Delta$ the mean curvature vector field, the Laplace operator of $M_r^m$ with respect to the induced metric.

The submanifold $M_r^m$ is called biharmonic if it satisfies the equation

$$\Delta\overrightarrow{H} = 0. \tag{1}$$

Equation (1) is a special case of the equation

$$\Delta\overrightarrow{H} = \lambda\overrightarrow{H} \tag{2}$$

for some real constant $\lambda$. The submanifold $M_r^m$ of $\mathbb{E}_s^n$ which satisfies condition (2) is said to have *proper mean curvature vector field*. In fact, equation (2)

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can be related to the theory of harmonic and biharmonic maps, we refer to [1, p. 807] and [14, p. 58]) for explanations.

Submanifolds with proper mean curvature vector field in Riemannian manifolds were originally studied by B.Y. Chen in [4] and by several other authors thereafter in space forms, contact, and Sasakian manifolds, we refer to [5–6], [9–10], [13–15].

The study of equation (2) for submanifolds in pseudo-Euclidean spaces was originated by Ferrández and Lucas in [11], where they classified surfaces $M_r^2$ $(r = 0, 1)$ in the Lorentz-Minkowski space $\mathbb{E}_1^3$. One of the possibilities for $M_1^2$ is that of a surface with zero mean curvature. In [3], A. Arvanitoyeorgos et al. proved that every hypersurface $M_1^3$ in $\mathbb{E}_1^4$ with proper mean curvature vector field has constant mean curvature, and in [2], the same conclusion holds for every hypersurface $M_2^3$ in $\mathbb{E}_2^4$. More general, it is shown in [1] that every hypersurface $M_r^3$ $(r = 0, 1, 2, 3)$ of $\mathbb{E}_s^4$ satisfying equation (2) whose shape operator is diagonalizable, has constant mean curvature.

For higher dimensional case, the hypersurface $M_r^n$ $(r = 0, 1)$ in $\mathbb{E}_1^{n+1}$ with proper mean curvature vector field and such that the minimal polynomial of the shape operator is at most of degree two, were studied in [12], showing that $M_r^n$ has constant mean curvature.

The results of the previous two paragraphs suggest a further study of hypersurfaces of $\mathbb{E}_s^{n+1}$ $(0 \leq s \leq n+1)$ satisfying equation (2), as it is conjectured by Arvanitoyeorgos and Kaimakamis in [2] that such hypersurfaces must have constant mean curvature. If $\lambda = 0$, i.e. the hypersurface is biharmonic, then this implies that such hypersurface is minimal. In the pseudo-Euclidean setting, this is a version of a well known conjecture due to Chen [8]: *Any biharmonic submanifold in pseudo-Euclidean space* $\mathbb{E}_s^{n+1}$ *is minimal.* Towards this goal, in this paper we prove the following theorem.

**Main Theorem** *Let* $M_r^n$ $(n \geq 4)$ *be a nondegenerate hypersurface of* $(n+1)$ *dimensional pseudo-Euclidean space* $\mathbb{E}_s^{n+1}$ *satisfying* $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$. *Assume that* $M_r^n$ *has diagonalizable shape operator with at most three distinct principal curvatures, then it has constant mean curvature.*

*Remark 1* The shape operator of a Riemannian submanifold is always diagonalizable, but for pseudo-Riemannian submanifolds, there may be other forms (cf. [17]). However, the cases where $M_r^n$ is a Riemannian or a pseudo-Riemannian hypersurface, and the shape operator is diagonalizable, can be treated in a uniform way in this paper. The same problems for the other forms of the shape operator will be studied in another paper.

*Remark 2* When $n = 3$, it is true automatically that $M_r^3$ has at most three distinct principal curvatures. In this case, the result has been proved by A. Arvanitoyeorgos in [1, Theorem]. So, in this paper, we study with $n \geq 4$.

## 2. Preliminaries and Lemma

Let $x : M_r^n \to \mathbb{E}_s^{n+1}$ be an isometric immersion of a hypersurface $M_r^n$ $(r = 0, 1, \cdots, n)$ into $\mathbb{E}_s^{n+1}(s = 0, 1, \cdots, n+1)$, $r \leq s$. Let $\xi$ denote a unit normal vector field with $\langle \xi, \xi \rangle = \varepsilon = \pm 1$. The hypersurface $M_r^n$ can itself be endowed with a Riemannian or a pseudo-Riemannian metric structure, depending on whether the metric induced on $M_r^n$ from the pseudo-Euclidean space $\mathbb{E}_s^{n+1}$ is positive-definite or indefinite.

Denote by $\nabla$ and $\widetilde{\nabla}$ the Levi-Civita connections of $M_r^n$ and $\mathbb{E}_s^{n+1}$ respectively. For any vector fields $X, Y$ tangent to $M_r^n$, the Gauss formula is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y)\xi,$$

where $h$ is the scalar-valued second fundamental form. If we denote by $A$ the shape operator of $M_r^n$ associated to $\xi$, then the Weingarten formula is given by

$$\widetilde{\nabla}_X \xi = -A(X),$$

where $\langle A(X), Y \rangle = \varepsilon h(X, Y)$. The mean curvature vector $\overrightarrow{H} = H\xi$ with mean curvature $H = \frac{1}{n}\varepsilon \mathrm{tr} A$, determines a well defined normal vector field to $M_r^n$ in $\mathbb{E}_s^{n+1}$. The Codazzi and Gauss equations are given by (cf. [17])

$$(\nabla_X A)Y = (\nabla_Y A)X, \tag{3}$$

$$R(X, Y)Z = \langle A(Y), Z \rangle A(X) - \langle A(X), Z \rangle A(Y), \tag{4}$$

where

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{5}$$

A hypersurface $M_r^n$ of $\mathbb{E}_s^{n+1}$ is said to have proper mean curvature vector field, if $\Delta \overrightarrow{H} = \lambda \overrightarrow{H}$ for some constant $\lambda$. Equivalently(cf. [7]),

$$\Delta \overrightarrow{H} = \big(2A(\nabla H) + n\varepsilon H(\nabla H)\big) + (\Delta H + \varepsilon H \mathrm{tr} A^2)\xi = \lambda \overrightarrow{H}.$$

By comparing the vertical and horizontal parts, the above equation is equivalent to the following two equations

$$A(\nabla H) = -\frac{n}{2}\varepsilon H(\nabla H), \tag{6}$$

$$\Delta H + \varepsilon H \mathrm{tr} A^2 = \lambda H, \tag{7}$$

where the Laplace operator $\Delta$ acting on scalar-valued function $f$ is given by (cf. [7])

$$\Delta f = -\sum_{i=1}^{n} \varepsilon_i (e_i e_i f - \nabla_{e_i} e_i f), \tag{8}$$

where $\{e_1, e_2, \cdots, e_n\}$ is a local orthonormal frame of $T_p M_r^n$ with $\langle e_i, e_i \rangle = \varepsilon_i = \pm 1$.

Assume that the mean curvature $H$ of $M_r^n$ in our main theorem is not a constant, then $\nabla H \neq 0$, and (6) shows that $\nabla H$ is an eigenvector of the shape operator $A$ with correspongding eigenvalue $\lambda_1 = -\frac{n}{2}\varepsilon H$. We can choose a suitable orthonormal frame $\{e_i\}_{i=1}^n$ such that, without loss of generality, $e_1$ is in the direction of $\nabla H$. Therefore, the diagonalizable shape operator $A$ takes the form $A = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. At that time, we have the following lemma.

**Lemma 2.1** *Suppose that $H$ is not a constant, then $\lambda_j \neq \lambda_1$ for $j \neq 1$.*

**Proof** As $\nabla H \neq 0$, let us express $\nabla H$ as $\nabla H = \sum_{i=1}^n \varepsilon_i e_i(H) e_i$. Since $\nabla H$ is in the direction of $e_1$, we get

$$e_1(H) \neq 0, \quad e_2(H) = e_3(H) = \cdots = e_n(H) = 0. \tag{9}$$

For any $i, j = 1, 2, \cdots, n$, we write $\nabla_{e_i} e_j = \sum_{k=1}^n \omega_{ij}^k e_k$. By applying compatibility conditions to $\nabla_{e_k}\langle e_i, e_i \rangle = 0$ and $\nabla_{e_k}\langle e_i, e_j \rangle = 0$, we obtain

$$\omega_{ki}^i = 0, \quad \omega_{ki}^j = -\varepsilon_i \varepsilon_j \omega_{kj}^i, \tag{10}$$

for $1 \leq i, j, k \leq n$ and $i \neq j$. Since $\lambda_1 = -\frac{n}{2}\varepsilon H$, it follows from (9) that

$$e_1(\lambda_1) \neq 0, \quad e_2(\lambda_1) = e_3(\lambda_1) = \cdots = e_n(\lambda_1) = 0. \tag{11}$$

We consider distinct $i, j, k = 1, 2, \cdots, n$, then the Codazzi equation (3) implies the equations

$$\langle (\nabla_{e_i} A) e_j, e_j \rangle = \langle (\nabla_{e_j} A) e_i, e_j \rangle,$$
$$\langle (\nabla_{e_k} A) e_i, e_j \rangle = \langle (\nabla_{e_i} A) e_k, e_j \rangle.$$

A straightforward calculation on the above two equations shows that

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j, \tag{12}$$

$$(\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j, \tag{13}$$

for distinct $i, j, k = 1, 2, \cdots, n$.

If $\lambda_j = \lambda_1$ for $j \neq 1$, we have from (12) that

$$0 = (\lambda_1 - \lambda_j)\omega_{j1}^j = e_1(\lambda_j) = e_1(\lambda_1),$$

which contradicts to (11) and completes the proof of Lemma 2.1. $\square$

*Remark 3* In the proof of Lemma 2.1, we don't use the equation (13). However, it will play an important role in the following sections.

In the special case in which all the principal curvatures $\lambda_i$ of $M_r^n$ in $\mathbb{E}_s^{n+1}$ are equal, then Lemma 2.1 implies that $H$ is clearly a constant. According to our assumption that $M_r^n$ has at most three distinct principal curvatures, the remaining cases are: two or three distinct principal curvatures. We will treat each of the cases in Sections 3 and 4, separately.

## 3. $M_r^n$ has two distinct principal curvatures

**Proposition 3.1** *Let $M_r^n$ be a nondegenerate hypersurface of $\mathbb{E}_s^{n+1}$ satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$. Assume that $M_r^n$ has diagonalizable shape operator with two distinct principal curvatures, then it has constant mean curvature.*

**Proof** Assume on the contrary that $H$ is not a constant, we will end up the proof with a contradiction.

According to Lemma 2.1, we know $\lambda_j \neq \lambda_1$ for $j \neq 1$. Since the hypersurface $M_r^n$ has two distinct principal curvatures, without loss of generality, we denote $\lambda_2 = \lambda_3 = \cdots = \lambda_n = \mu$ and $\mu \neq \lambda_1$. Since $H = \frac{1}{n}\varepsilon\mathrm{tr}A$, it follows that

$$\mu = \frac{3n\varepsilon H}{2(n-1)}.$$

Let $j = 1$ in (12), combining (10) and (11), we get

$$\omega_{1i}^1 = \omega_{11}^i = 0, \quad 2 \leq i \leq n.$$

For $i=1, j\neq 1$ in (12), by using (10), we obtain

$$\omega_{j1}^{j} = -\frac{3e_1(H)}{(n+2)H}, \quad \omega_{jj}^{1} = \varepsilon_1\varepsilon_j\frac{3e_1(H)}{(n+2)H}. \tag{14}$$

Observe (13) for $i=1$ and $2\leq k\neq j\leq n$, combining (10), we have

$$\omega_{k1}^{j} = \omega_{kj}^{1} = 0.$$

Using the above equations, it is easy to check that, for $2\leq i\neq j\leq n$,

$$\nabla_{e_1}e_1 = 0, \quad \nabla_{e_1}e_i = \sum_{k\neq 1,i}\omega_{1i}^{k}e_k, \quad \nabla_{e_i}e_1 = \omega_{i1}^{i}e_i,$$
$$\nabla_{e_i}e_i = -\varepsilon_1\varepsilon_i\omega_{i1}^{i}e_1 + \sum_{k\neq 1,i}\omega_{ii}^{k}e_k, \quad \nabla_{e_i}e_j = \sum_{k\neq 1,j}\omega_{ij}^{k}e_k.$$

Applying Gauss equation (4) and the definition (5) of the curvature tensor to $\langle R(e_1,e_2)e_1,e_2\rangle$, we find that

$$e_1(\omega_{21}^{2}) = \frac{3\varepsilon_1 n^2}{4(n-1)}H^2 - (\omega_{21}^{2})^2. \tag{15}$$

Using (8), (9) and the formulas of $\nabla_{e_i}e_j$, it follows from (7) that

$$-\varepsilon_1 e_1e_1(H) - (n-1)\varepsilon_1\omega_{21}^{2}e_1(H) + \varepsilon\frac{(n+8)n^2H^3}{4(n-1)} = \lambda H. \tag{16}$$

By differentiating (14) with $j=2$ along $e_1$, and using (15), we get

$$e_1e_1(H) = \frac{(n+2)(n+5)}{9}H(\omega_{21}^{2})^2 - \varepsilon_1\frac{n^2(n+2)}{4(n-1)}H^3. \tag{17}$$

Substituting (17) into (16), combining (14) and noticing that $H\neq 0$, we have

$$\varepsilon_1\frac{(n+2)(-2n+8)}{9}(\omega_{21}^{2})^2 + \lambda - \frac{n^2(n+2)+\varepsilon n^2(n+8)}{4(n-1)}H^2 = 0. \tag{18}$$

Acting on both sides of (18) by $e_1$ and using (14) and (15), then

$$\varepsilon_1\frac{-2n+8}{9}(\omega_{21}^{2})^2 - \frac{n^2(-n+10+(n+8)\varepsilon)}{12(n-1)}H^2 = 0. \tag{19}$$

Eliminating $\omega_{21}^{2}$ from (18) and (19), we obtain that

$$\lambda + \frac{n^2((n+2)(-n+7)+(n+8)(n-1)\varepsilon)}{12(n-1)}H^2 = 0.$$

Then, $H$ must be a constant, which contradicts to our assumption, and completes the proof of Proposition 3.1. $\square$

## 4. $M_r^n$ has three distinct principal curvatures

**Proposition 4.1** *Let $M_r^n$ be a nondegenerate hypersurface of $\mathbb{E}_s^{n+1}$ satisfying $\Delta\overrightarrow{H} = \lambda\overrightarrow{H}$. Assume that $M_r^n$ has diagonalizable shape operator with three distinct principal curvatures, then it has constant mean curvature.*

In order to prove Proposition 4.1, assume on the contrary that $H$ is not a constant, we will show that this assumption runs into a contradiction. So in the following, we start with $H \neq$ const..

From Lemma 2.1, we have $\lambda_j \neq \lambda_1$ for $j \neq 1$. Without loss of generality, we suppose that $\lambda_2 = \cdots = \lambda_t = \mu$, $\lambda_{t+1} = \cdots = \lambda_n = \nu$, where $\lambda_1,\ \mu,\ \nu$ are mutually distinct principal curvature of $M_r^n$ with multiplicities $1$, $t-1$ and $n-t$ respectively. Obviously, $2 \leq t \leq n-1$. Since $H = \frac{1}{n}\varepsilon \mathrm{tr} A$, it follows that $\nu = \frac{\frac{3}{2}n\varepsilon H-(t-1)\mu}{n-t}$.

In this section, we shall make use of the following convention for the range of indices: $1 \leq i,j,k \leq n$, $2 \leq a,b,c \leq t$, $t+1 \leq \alpha,\beta,\gamma \leq n$.

We need the following four lemmas.

**Lemma 4.2** *Let $n \geq 4$. When $t = n-1$, the covariant derivatives have the following forms:*

$$\nabla_{e_1}e_1 = \nabla_{e_1}e_n = 0, \quad \nabla_{e_1}e_a = \sum_{c\neq a}\omega_{1a}^c e_c, \quad \nabla_{e_a}e_1 = -\frac{e_1(\mu)}{\frac{n}{2}\varepsilon H + \mu}e_a,$$

$$\nabla_{e_a}e_a = \frac{\varepsilon_1\varepsilon_a e_1(\mu)}{\frac{n}{2}\varepsilon H + \mu}e_1 + \sum_{c\neq a}\omega_{aa}^c e_c - \frac{\varepsilon_a\varepsilon_n e_n(\mu)}{\frac{3}{2}n\varepsilon H - (n-1)\mu}e_n,$$

$$\nabla_{e_a}e_b = \omega_{ab}^a e_a + \sum_{c\neq a,b}\omega_{ab}^c e_c, \qquad \nabla_{e_a}e_n = \frac{e_n(\mu)}{\frac{3}{2}n\varepsilon H - (n-1)\mu}e_a,$$

$$\nabla_{e_n}e_1 = \frac{e_1(3n\varepsilon H - 2(n-2)\mu)}{-4n\varepsilon H + 2(n-2)\mu}e_n, \qquad \nabla_{e_n}e_a = \sum_{c\neq a}\omega_{na}^c e_c,$$

$$\nabla_{e_n}e_n = -\varepsilon_1\varepsilon_n\frac{e_1(3n\varepsilon H - 2(n-2)\mu)}{-4n\varepsilon H + 2(n-2)\mu}e_1.$$

**Proof** According to (11), for $i,j \neq 1$ and $i \neq j$, we obtain $[e_i,e_j](\lambda_1) = 0$. Then $(\nabla_{e_i}e_j - \nabla_{e_j}e_i)(\lambda_1) = 0$, which yields directly

$$\omega_{ij}^1 = \omega_{ji}^1. \tag{20}$$

From (12), we have $e_a(\lambda_b) = (\lambda_a - \lambda_b)\omega_{ba}^b = 0$, i.e.

$$e_a(\mu) = 0. \tag{21}$$

Consider equation (12) for $j = 1, i \neq 1$ and $j = n, i = 2, 3, \cdots, n-1$, combining (10), (11) and (21), we get

$$\omega_{na}^n = \omega_{nn}^a = \omega_{1k}^1 = \omega_{11}^k = 0, \quad k = 1, 2, \cdots, n.$$

For $i = 1, j = 2, 3, \cdots, n$ and $i = n, j = 2, 3, \cdots, n-1$ in (12), we obtain

$$\begin{aligned} \omega_{a1}^a &= -\frac{e_1(\mu)}{\frac{n}{2}\varepsilon H + \mu}, \quad \omega_{an}^a = \frac{e_n(\mu)}{\frac{3}{2}n\varepsilon H - (n-1)\mu}, \\ \omega_{n1}^n &= \frac{e_1(3n\varepsilon H - 2(n-2)\mu)}{-4n\varepsilon H + 2(n-2)\mu}. \end{aligned} \tag{22}$$

Using equation (13) for $i = 1, j, k = 2, 3, \cdots, n-1$; $i = n, j = 1, 2, \cdots, n-1, k = 2, 3, \cdots, n-1$, we have

$$\omega_{b1}^a = \omega_{bn}^a = 0, \quad a \neq b,$$

$$(2n\varepsilon H - (n-2)\mu)\omega_{an}^1 = \Big(\mu + \frac{n}{2}\varepsilon H\Big)\omega_{na}^1.$$

Combining (10) and (20), we obtain

$$\omega_{ab}^1 = \omega_{an}^1 = \omega_{na}^1 = \omega_{n1}^a = \omega_{ab}^n = \omega_{a1}^n = 0, \quad a \neq b.$$

Let $j = n, k = 1$ in (13), using relation $\omega_{a1}^n = 0$ and (10), we get $\omega_{1a}^n = \omega_{1n}^a = 0$. Applying the above results about $\{\omega_{ij}^k\}$ and (10), a straightforward calculation completes the proof of Lemma 4.2. □

**Lemma 4.3** *Let $n \geq 4$. Suppose that $H$ is not a constant. Then*

$$e_i(\mu) = 0, \quad \forall\, i = 2, \cdots, n.$$

**Proof** When $t \neq n-1$, it follows from (12) that $e_a(\lambda_b) = (\lambda_a - \lambda_b)\omega_{ba}^b$ and $e_\alpha(\lambda_\beta) = (\lambda_\alpha - \lambda_\beta)\omega_{\beta\alpha}^\beta$, which implies $e_i(\mu) = 0$ for $2 \leq i \leq n$.

When $t = n-1$, according to (21), we need only to prove $e_n(\mu) = 0$.

Computing $\langle R(e_2, e_n)e_1, e_2\rangle$ by using Gauss equation (4) and the definition (5) of the curvature tensor, we obtain

$$e_n(\omega_{21}^2) = (\omega_{n1}^n - \omega_{21}^2)\omega_{2n}^2. \tag{23}$$

Putting $\omega_{21}^2 = -\frac{e_1(\mu)}{\frac{n}{2}\varepsilon H+\mu}$ and $\omega_{2n}^2 = \frac{e_n(\mu)}{\frac{3}{2}n\varepsilon H-(n-1)\mu}$ (see Eq. (22)) into (23) gives

$$e_n e_1(\mu) = -(\frac{n}{2}\varepsilon H + \mu)\omega_{n1}^n \omega_{2n}^2 - (n\varepsilon H - n\mu)\omega_{21}^2\omega_{2n}^2.$$

Since $[e_1, e_k](H) = (\nabla_{e_1} e_k - \nabla_{e_k} e_1)(H)$, for $k \neq 1$, using (9) and the covariant derivatives $\nabla_{e_i} e_j$ in Lemma 4.2, we have

$$e_k e_1(H) = 0. \tag{24}$$

Differentiating both sides of $\omega_{n1}^n = \frac{e_1(3n\varepsilon H-2(n-2)\mu)}{-4n\varepsilon H+2(n-2)\mu}$ (see Eq. (22)) along the direction $e_n$, using the above two formulas, we have

$$e_n(\omega_{n1}^n) = \frac{(n-2)(\varepsilon nH - n\mu)}{2n\varepsilon H - (n-2)\mu}(\omega_{n1}^n - \omega_{21}^2)\omega_{2n}^2. \tag{25}$$

Using (8), (9) and formulas of $\nabla_{e_i} e_j$ in Lemma 4.2, we calculate $\Delta H$ directly and put the result into (7),

$$-\varepsilon_1 e_1 e_1(H) - \varepsilon_1((n-2)\omega_{21}^2 + \omega_{n1}^n)e_1(H) + \varepsilon H \mathrm{tr}A^2 = \lambda H. \tag{26}$$

Note that $\lambda_1 = -\frac{n}{2}\varepsilon H$ and $\nu = \frac{3}{2}n\varepsilon H - (n-2)\mu$ for $t = n-1$, it is easy to check that

$$\mathrm{tr}A^2 = \frac{5}{2}n^2H^2 + (n-1)(n-2)\mu^2 - 3n(n-2)\varepsilon H\mu. \tag{27}$$

According to $[e_1, e_n](e_1(H)) = (\nabla_{e_1} e_n - \nabla_{e_n} e_1)(e_1(H))$, combining (24) and the covariant derivatives $\nabla_{e_i} e_j$ in Lemma 4.2, it follows that

$$e_n e_1 e_1(H) = 0. \tag{28}$$

Differentiating both sides of (26) along the direction $e_n$, using (9), (23), (25), (27) and(28), we deduce that

$$\left(\frac{2\varepsilon_1 e_1(H)(\omega_{21}^2 - \omega_{n1}^n)}{2n\varepsilon H - (n-2)\mu} - 3nH^2 + 2(n-1)\varepsilon H\mu\right) e_n(\mu) = 0. \tag{29}$$

Suppose on the contrary that $e_n(\mu) \neq 0$, then (29) implies that

$$\frac{2\varepsilon_1(\omega_{21}^2 - \omega_{n1}^n)e_1(H)}{2n\varepsilon H - (n-2)\mu} - 3nH^2 + 2(n-1)\varepsilon H\mu = 0. \tag{30}$$

By differentiating (30) along $e_n$, $e_n(\mu) \neq 0$, combining (9), (23), (24) and (25), we get

$$\begin{aligned}\frac{\varepsilon_1(n(5n-14)\varepsilon H - 4(n-1)(n-2)\mu)}{(2n\varepsilon H-(n-2)\mu)^2(\frac{3}{2}n\varepsilon H-(n-1)\mu)}&(\omega_{21}^2 - \omega_{n1}^n)e_1(H) \\ &+ 2(n-1)\varepsilon H = 0.\end{aligned} \tag{31}$$

Eliminating $(\omega_{21}^2 - \omega_{n1}^n)e_1(H)$ from (30) and (31) yields

$$\varepsilon H(3n\varepsilon H - 2(n-1)\mu) = 0,$$

Since $H \neq 0$, so $\mu = \frac{3n\varepsilon H}{2(n-1)}$. It follows that $\nu = \frac{3n\varepsilon H}{2(n-1)}$, which contradicts to $\mu \neq \nu$. So $e_n(\mu) = 0$. □

According to Lemma 4.3, $e_n(\mu) = 0$. Using this fact and taking similar method to the proof of Lemma 4.2, we can easily obtain the following Lemma.

**Lemma 4.4** *Let $n \geq 4$. The covariant derivatives have the following forms:*

$$\begin{aligned}
&\nabla_{e_1}e_1 = 0, \quad \nabla_{e_1}e_a = \sum_{c\neq a}\omega_{1a}^c e_c, \quad \nabla_{e_1}e_\alpha = \sum_{\gamma\neq\alpha}\omega_{1\alpha}^\gamma e_\gamma,\\
&\nabla_{e_a}e_1 = -\frac{e_1(\mu)}{\frac{n}{2}\varepsilon H + \mu}e_a, \nabla_{e_a}e_a = \varepsilon_1\varepsilon_a\frac{e_1(\mu)}{\frac{n}{2}\varepsilon H+\mu}e_1 + \sum_{c\neq a}\omega_{ac}^c e_c,\\
&\nabla_{e_a}e_b = \omega_{ab}^a e_a + \sum_{c\neq a,b}\omega_{ab}^c e_c, \qquad \nabla_{e_a}e_\alpha = \sum_{\gamma\neq\alpha}\omega_{a\alpha}^\gamma e_\gamma,\\
&\nabla_{e_\alpha}e_1 = \frac{e_1(3n\varepsilon H - 2(t-1)\mu)}{-(n-t+3)n\varepsilon H + 2(t-1)\mu}e_\alpha, \qquad \nabla_{e_\alpha}e_a = \sum_{k\neq a}\omega_{\alpha a}^c e_c,\\
&\nabla_{e_\alpha}e_\alpha = -\varepsilon_1\varepsilon_\alpha\frac{e_1(3n\varepsilon H - 2(t-1)\mu)}{-(n-t+3)n\varepsilon H + 2(t-1)\mu}e_1 + \sum_{\gamma\neq\alpha}\omega_{\alpha\alpha}^\gamma e_\gamma,\\
&\nabla_{e_\alpha}e_\beta = \omega_{\alpha\beta}^\alpha e_\alpha + \sum_{\gamma\neq\alpha,\beta}\omega_{\alpha\beta}^\gamma e_\gamma.
\end{aligned}$$

*When $t = n-1$, all of them can be simplified to that of Lemma 4.2.*

**Lemma 4.5** *Let $n \geq 4$. We have from Lemma 4.4 that*

$$\omega_{21}^2 = -\frac{e_1(\mu)}{\frac{n}{2}\varepsilon H + \mu}, \quad \omega_{n1}^n = \frac{e_1(3n\varepsilon H - 2(t-1)\mu)}{-(n-t+3)n\varepsilon H + 2(t-1)\mu}.$$

*Furthermore, $\omega_{21}^2$ and $\omega_{n1}^n$ satisfy the following equations:*

$$e_1(\omega_{21}^2)+(\omega_{21}^2)^2=\frac{n}{2}\varepsilon_1\varepsilon H\mu, \tag{32}$$

$$e_1(\omega_{n1}^n)+(\omega_{n1}^n)^2=\varepsilon_1\varepsilon H\frac{\frac{3}{2}n^2\varepsilon H-(t-1)n\mu}{2(n-t)}, \tag{33}$$

$$-\varepsilon_1\omega_{n1}^n\omega_{21}^2=\frac{\frac{3}{2}n\varepsilon H\mu-(t-1)\mu^2}{n-t}. \tag{34}$$

**Proof** Using Gauss equation (4), Lemma 4.4 and the definition (5) of the curvature tensor, a straightforward calculation for $\langle R(e_1,e_2)e_1,e_2\rangle$, $\langle R(e_1,e_n)e_1,e_n\rangle$ and $\langle R(e_n,e_2)e_n,e_2\rangle$ proves (32), (33) and (34), respectively. $\square$

Now, we can prove Proposition 4.1.

**Proof** (*of Proposition 4.1*) Note that $\lambda_1=-\frac{n}{2}\varepsilon H$, $\nu=\frac{\frac{3}{2}n\varepsilon H-(t-1)\mu}{n-t}$, so

$$\mathrm{tr}A^2=\frac{n-t+9}{4(n-t)}n^2H^2+\frac{(n-1)(t-1)}{n-t}\mu^2-\frac{3n(t-1)}{n-t}\varepsilon H\mu. \tag{35}$$

Applying (8), (9) and the formulas of $\nabla_{e_i}e_j$, we compute (7) directly and obtain

$$-\varepsilon_1e_1e_1(H)-\varepsilon_1\big((t-1)\omega_{21}^2+(n-t)\omega_{n1}^n\big)e_1(H)+\varepsilon H\mathrm{tr}A^2=\lambda H. \tag{36}$$

Eliminating $e_1e_1(H)$ and $e_1e_1(\mu)$ by using (32), (33), (34), (35) and (36), together with the formulas of $\omega_{21}^2$ and $\omega_{n1}^n$ (see Lemma 4.5), we obtain

$$\begin{aligned}&2\varepsilon_1\varepsilon(n-t)\big((t-4)\omega_{21}^2+(n-t-3)\omega_{n1}^n\big)e_1(H)\\&=\frac{3}{4}[n-t+9+(n-t+3)\varepsilon]n^2H^3-3(n-t)\varepsilon\lambda H\\&\quad-3[n+t+1+3(t-1)\varepsilon]nH^2\mu+3[(n-1)+(n+1)\varepsilon](t-1)H\mu^2.\end{aligned} \tag{37}$$

Acting $e_1$ on both sides of (37) and using (32)$\sim$(37), we obtain

$$f_1(H,\mu)\omega_{21}^2+g_1(H,\mu)\omega_{n1}^n=h_1(H,\mu)e_1(H), \tag{38}$$

where

$$\begin{aligned}&f_1(H,\mu)\\&=[(n-t)(-\frac{1}{4}t-2)-\frac{27}{4}(t+2)-(\frac{3}{4}t(n-t)+\frac{3}{2}(n+\frac{5}{2}t+1))\varepsilon]n^2H^3\\&\quad+3[2n+t+2+(n+t+4)\varepsilon](t-1)nH^2\mu+(t+8)(n-t)\varepsilon\lambda H\\&\quad-[(n-1)(t+2)+3(n+1)(t-2)\varepsilon](t-1)H\mu^2,\end{aligned}$$

$$
\begin{aligned}
g_1(H,\mu) =& -\frac{1}{4}[(n-t+9)^2+3(n-t+1)(n-t+3)\varepsilon]n^2H^3\\
&+3[(n+1)^2-t^2+(t-1)(n-t+9)\varepsilon]nH^2\mu\\
&-[(n-1)(n-t+9)+3(n+1)(n-t+1)\varepsilon](t-1)H\mu^2\\
&+(n-t+9)(n-t)\varepsilon\lambda H,
\end{aligned}
$$

$$
\begin{aligned}
h_1(H,\mu) =& \frac{3}{4}[3(n-t+9)+(n-t+15)\varepsilon]n^2H^2\\
&-6[2t+4+3(t-1)\varepsilon]nH\mu\\
&+[3(n-1)+(n+17)\varepsilon](t-1)\mu^2-3(n-t)\varepsilon\lambda.
\end{aligned}
$$

It follows from the formulas of $\omega_{21}^2$ and $\omega_{n1}^n$ that

$$
3n\varepsilon e_1(H) = -2(t-1)(\frac{n}{2}\varepsilon H+\mu)\omega_{21}^2-[(n-t+3)n\varepsilon H-2(t-1)\mu]\omega_{n1}^n. \quad (39)
$$

Putting (39) into (38) and eliminating $e_1(H)$, we have

$$
f_2(H,\mu)\omega_{21}^2+g_2(H,\mu)\omega_{n1}^n=0, \quad (40)
$$

where

$$
\begin{aligned}
f_2(H,\mu) =& \frac{1}{4}[(n-t)(2t-11)-81-(2t(n-t)+7n-t+21)\varepsilon]n^2H^3\\
&+\frac{1}{2}[13n-3t+11+(9n-9t+63)\varepsilon](t-1)nH^2\mu\\
&+[-3n-12t+15-\frac{1}{3}(8nt-17n+16t+47)\varepsilon](t-1)H\mu^2\\
&+2\frac{\frac{1}{3}(n+17)+(n-1)\varepsilon}{n}(t-1)^2\mu^3+9(n-t)\varepsilon\lambda H\\
&-2\frac{(n-t)(t-1)}{n}\mu\lambda,
\end{aligned}
$$

$$
\begin{aligned}
g_2(H,\mu) =& \frac{1}{2}[(n-t)(n-t+9)-(n-t+3)(n-t-6)\varepsilon]n^2H^3\\
&+[3n^2-\frac{9}{2}nt+\frac{3}{2}t^2-\frac{3}{2}n-12t-\frac{27}{2}-\frac{9}{2}(t-1)(n-t+1)\varepsilon]nH^2\mu\\
&+[-6n+12t-6+\frac{2}{3}(-4n(n-t)+n+8t+45)\varepsilon](t-1)H\mu^2\\
&-\frac{\frac{2(n+17)}{3}+2(n-1)\varepsilon}{n}(t-1)^2\mu^3+6(n-t)\varepsilon\lambda H+\frac{2(t-1)(n-t)}{n}\lambda\mu.
\end{aligned}
$$

Combining (34), (37) with (39), we get that

$$f_3(H,\mu)(\omega_{21}^2)^2 + g_3(H,\mu)(\omega_{n1}^n)^2 = h_3(H,\mu), \tag{41}$$

where

$$\begin{aligned} f_3(H,\mu) &= \frac{2}{3}(n-t)(t-1)(t-4)(\varepsilon H + \frac{2}{n}\mu)\varepsilon_1, \\ g_3(H,\mu) &= \frac{2}{3}(n-t)(n-t-3)((n-t+3)\varepsilon H - 2\frac{t-1}{n}\mu)\varepsilon_1, \\ h_3(H,\mu) &= -\frac{3}{4}[n-t+9+(n-t+3)\varepsilon]n^2H^3 \\ &\quad + [2t(n-t)-2n+8t-6+9(t-1)\varepsilon]nH^2\mu \\ &\quad - [3n-3+(\frac{4}{3}t(n-t)-\frac{7}{3}n+\frac{22}{3}t-5)\varepsilon](t-1)H\mu^2 \\ &\quad - \frac{4}{3n}(n-2t+1)(t-1)^2\mu^3 + 3(n-t)\varepsilon\lambda H. \end{aligned}$$

Multiplying Eq. (40) by $\omega_{21}^2$, or $\omega_{n1}^n$, separately, and substituting (34) into the resulting equations, we find $(\omega_{21}^2)^2$ and $(\omega_{n1}^n)^2$ can be written as algebraic expressions of $H$ and $\mu$. Putting these algebraic expressions into (41), we finally obtain a polynomial equation of $H$ and $\mu$ with constant coefficients, denoted by

$$f(H,\mu) = 0. \tag{42}$$

Acting on (42) with $e_1$ twice, and using (32)$\sim$(34), (39) and (40), we get another polynomial equation of $H$ and $\mu$ with constant coefficients

$$g(H,\mu) = 0.$$

In order to complete the proof of Proposition 4.1, we need the following algebraic lemma.

**Lemma 4.6** ([16, Theorem 4.4, pp.58–59]) *Let $D$ be a unique factorization domain, and let*

$$\begin{aligned} f(X) &= a_0X^m + a_1X^{m-1} + \cdots + a_m, \\ g(X) &= b_0X^n + b_1X^{n-1} + \cdots + b_n \end{aligned}$$

*be two polynomials in $D[X]$. Assume that the leading coefficients $a_0$ and $b_0$ of $f(X)$ and $g(X)$ are not both zero. Then $f(X)$ and $g(X)$ have a nonconstant* *common factor iff the resultant $\mathscr{R}(f,g)$ of $f$ and $g$ is zero, i.e.,*

$$\mathscr{R}(f,g) := \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_m & & & \\ & a_0 & a_1 & \cdots & \cdots & a_m & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & a_0 & a_1 & a_2 & \cdots & a_m \\ b_0 & b_1 & b_2 & \cdots & b_n & & & \\ & b_0 & b_1 & \cdots & \cdots & b_n & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & b_0 & b_1 & b_2 & \cdots & b_n \end{vmatrix} = 0.$$

Coming back to the proof of Proposition 4.1. Rewrite $f(H,\mu), g(H,\mu)$ as polynomials $f_{_H}(\mu), g_{_H}(\mu)$ of $\mu$ with coefficients in polynomial ring $\mathbb{R}[H]$ over real field $\mathbb{R}$. According to Lemma 4.6, the equations $f_{_H}(\mu) = 0$ and $g_{_H}(\mu) = 0$ has common roots iff $\mathscr{R}(f_{_H}, g_{_H}) = 0$. It is obviously that $\mathscr{R}(f_{_H}, g_{_H})$ is a polynomial of $H$ with constant coefficients. So, $\mathscr{R}(f_{_H}, g_{_H}) = 0$ implies that $H$ must be a constant, a contradiction, which completes the proof of Proposition 4.1. $\square$

Finally, combining Propositions 3.1 and 4.1, the main theorem stated in introduction is proved.

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