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*-Ricci solitons of real hypersurfaces in non-flat complex space forms

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ABSTRACT

In this paper the notion of **-Ricci soliton* is introduced and real hypersurfaces in non-flat complex space forms admitting a *-Ricci soliton with potential vector field being the structure vector field ξ are studied. At the end of the paper discussion on this new notion and ideas for further work are presented.

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1. Introduction

A *complex space form* is an n-dimensional Kaehler manifold of constant holomorphic sectional curvature c. A complete and simply connected complex space form is complex analytically isometric to

- a complex projective space $\mathbb{C}P^n$, if $c > 0$,
- a complex Euclidean space $\mathbb{C}^n$, if $c = 0$,
- or a complex hyperbolic space $\mathbb{C}H^n$, if $c < 0$.

The complex projective and complex hyperbolic spaces are called non-flat complex space forms, since $c \neq 0$ and the symbol $M_n(c)$ is used to denote them, when it is not necessary to distinguish them.

Let M be a real hypersurface in a non-flat complex space form. An almost contact metric structure (φ, ξ, η, g) is defined on M induced from the Kaehler metric G and the complex structure J on $M_n(c)$. The *structure vector field* ξ is called *principal* if $A\xi = \alpha\xi$, where A is the shape operator of M and $\alpha = \eta(A\xi)$ is a smooth function. A real hypersurface is called *Hopf hypersurface*, if ξ is principal and α is called *Hopf principal curvature*.

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The *Ricci tensor* S of a Riemannian manifold is a tensor field of type $(1, 1)$ and is given by

$$g(SX, Y) = \text{Trace}\{Z \mapsto R(Z, X)Y\}.$$

A Riemannian manifold is called *Einstein*, if the Ricci tensor satisfies the relation

$$S = \lambda g,$$

where λ is a constant. It is known that there are no Einstein real hypersurfaces in $\mathbb{C}P^n$, proved by Cecil and Ryan in [1], and in $\mathbb{C}H^n$, proved by Montiel in [2].

Motivated by Tachibana, who in [3] introduced the notion of *-*Ricci tensor* on almost Hermitian manifolds, in [4] Hamada defined the *-*Ricci tensor* of real hypersurfaces in non-flat complex space forms by

$$g(S^*X, Y) = \frac{1}{2}(\text{trace}\{\varphi \circ R(X, \varphi Y)\}), \quad \text{for } X, Y \in TM.$$

The *-*scalar curvature* is denoted by ρ^* and is defined to be the trace of S^*. If ρ^* is constant and a real hypersurface M in a non-flat complex space form satisfies the relation

$$g(S^*X, Y) = \frac{\rho^*}{2(n-1)} g(X, Y), \quad \text{for } X, Y \text{ orthogonal to } \xi,$$

then M is called *-*Einstein*. The study of *-Einstein real hypersurfaces in non-flat complex spaces forms was initiated by Hamada, who in [4] classified *-Einstein Hopf hypersurfaces in $M_n(c)$ and also proved that ruled real hypersurfaces in non-flat complex space forms are *-Einstein. Recently, in [5] Ivey and Ryan completed the previous work and also proved equivalent relations for three dimensional *-Einstein real hypersurfaces in non-flat complex space forms.

The study of Ricci solitons on Riemannian manifolds is an issue, which is of great importance in the area of differential geometry and in physics as well. Hamilton in [6] was the first to introduce the latter notion, which generalizes the notion of Einstein metric on a Riemannian manifold. More precisely, a Riemannian metric g on a manifold is called *Ricci soliton*, when the following relation is satisfied

$$\frac{1}{2}\mathcal{L}_V g + \text{Ric} - \lambda g = 0.$$

The vector field V is the *potential vector field* of the Ricci soliton, Ric is given by $\text{Ric}(X, Y) = g(SX, Y)$, where S is the Ricci tensor of the manifold and λ is a constant. A Ricci soliton depending on the value of λ is called *shrinking, steady or expanding* if $\lambda > 0,\ \lambda = 0$ or $\lambda < 0$ respectively. It is obvious that if the potential vector field is a Killing or a zero, then the Ricci soliton is an Einstein metric. More information for Ricci solitons and the work that so far has been done on them can be found in [7].

Since there are no Einstein real hypersurfaces in non-flat complex space forms, the problem of studying Ricci solitons on them raised naturally. In [8] the notion of η-*Ricci soliton* is introduced

Definition 1.1. An η-*Ricci soliton* on a real hypersurface is a pair (η, g), which satisfies the following relation

$$\frac{1}{2}\mathcal{L}_\xi g + \text{Ric} - \lambda g - \mu \eta \otimes \eta = 0,$$

where $\lambda,\ \mu$ are constants, $\text{Ric}(X, Y) = g(SX, Y)$ and η is the 1-form on M.

In this case we say that the real hypersurface admits an η-Ricci soliton. It is obvious that if $\mu = 0$, then the above relation is a Ricci soliton, whose potential vector field is the structure vector field ξ. In [8] Cho and Kimura classified real hypersurfaces in non-flat complex space forms admitting η-Ricci soliton. Moreover, they proved that there are no real hypersurfaces in $M_n(c)$ admitting Ricci soliton, whose potential vector field is the structure vector field ξ. Continuing the study of Ricci solitons for real hypersurfaces, in [9] Cho and Kimura proved that no compact Hopf hypersurface in a non-flat complex space form admits a Ricci soliton.

Motivated by the work in [8] and the existence of *-Einstein Hopf hypersurfaces, the following question raises

Question. *Are there real hypersurfaces in non-flat complex space forms, which admit other "types" of Ricci solitons?*

In this paper the following notion is introduced

Definition 1.2. A Riemannian metric g on M is called *-*Ricci soliton*, if

$$\frac{1}{2}\mathcal{L}_V g + \text{Ric}^* - \lambda g = 0, \tag{1.1}$$

where $\text{Ric}^*(X, Y) = g(S^*X, Y)$ with S^* being the *-Ricci tensor on M and λ is a constant.

If a real hypersurface M in $M_n(c)$ satisfies relation (1.1), then it is said that M admits a *-Ricci soliton.

This paper focuses on the study of real hypersurfaces, which admit a *-Ricci soliton with the potential vector field being the structure vector field ξ. More precisely the following theorems are proved.

Theorem 1.3. *There do not exist real hypersurfaces in $\mathbb{C}P^n,\ n \geq 2$, admitting a *-Ricci soliton, with potential vector being the structure vector field ξ.*

Theorem 1.4. *Let M be a real hypersurface in $\mathbb{C}H^n$, $n \geq 2$ and $c = -4$, admitting a *-Ricci soliton, with potential vector being the structure vector field ξ. Then M is locally congruent to a geodesic hypersphere with $2n = \coth^2(r)$,*

The paper is organized as follows: In Section 2 preliminaries relations and basic results for real hypersurfaces in non-flat complex space forms are presented. In Section 3 the proof of Theorems 1.3 and 1.4 are provided. Finally, in Section 4 discussion on the new notion and ideas for further research, that could be developed in this area, are included.

2. Preliminaries

Throughout this paper all manifolds, vector fields etc. are assumed to be of class C^∞ and all manifolds are assumed to be connected. Furthermore, the real hypersurfaces M are supposed to be without boundary.

Let M be a real hypersurface immersed in a non-flat complex space form $(M_n(c), G)$ with complex structure J of constant holomorphic sectional curvature c. Let N be a locally defined unit normal vector field on M and $\xi = -JN$ the structure vector field of M.

For a vector field X tangent to M relation

$$JX = \varphi X + \eta(X)N$$

holds, where φX and $\eta(X)N$ are the tangential and the normal component of JX respectively. The Riemannian connections $\overline{\nabla}$ in $M_n(c)$ and ∇ in M are related for any vector fields X, Y on M by

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N,$$

where g is the Riemannian metric induced from the metric G.

The shape operator A of the real hypersurface M in $M_n(c)$ with respect to N is given by

$$\overline{\nabla}_X N = -AX.$$

The real hypersurface M has an almost contact metric structure (φ, ξ, η, g) induced from J on $M_n(c)$, where φ is the *structure tensor* and is a tensor field of type $(1, 1)$ and η is a 1-form on M such that

$$g(\varphi X, Y) = G(JX, Y), \qquad \eta(X) = g(X, \xi) = G(JX, N).$$

Moreover, the following relations hold

$$\varphi^2 X = -X + \eta(X)\xi, \qquad \eta \circ \varphi = 0, \qquad \varphi\xi = 0, \qquad \eta(\xi) = 1,$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad g(X, \varphi Y) = -g(\varphi X, Y).$$

The fact that J is parallel implies $\nabla J = 0$. The last relation leads to

$$\nabla_X \xi = \varphi AX, \qquad (\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{2.1}$$

The ambient space $M_n(c)$ is of constant holomorphic sectional curvature c and this results in the Gauss and Codazzi equations to be given respectively by

$$\begin{aligned} R(X, Y)Z &= \frac{c}{4}[g(Y, Z)X - g(X, Z)Y + g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z] \\ &\quad + g(AY, Z)AX - g(AX, Z)AY, \\ (\nabla_X A)Y - (\nabla_Y A)X &= \frac{c}{4}[\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi], \end{aligned} \tag{2.2}$$

where R denotes the Riemannian curvature tensor on M and X, Y, Z are any vector fields on M.

The tangent space $T_P M$, for every point $P \in M$, can be decomposed as

$$T_P M = \operatorname{span}\{\xi\} \oplus \mathbb{D},$$

where $\mathbb{D} = \ker \eta = \{X \in T_P M : \eta(X) = 0\}$ and is called *holomorphic distribution*. Due to the above decomposition, the vector field $A\xi$ can be written

$$A\xi = \alpha\xi + \beta U, \tag{2.3}$$

where $\beta = |\varphi \nabla_\xi \xi|$ and $U = -\frac{1}{\beta}\varphi\nabla_\xi\xi \in \ker(\eta)$, provided that $\beta \neq 0$.

Since the ambient space $M_n(c)$ is of constant holomorphic sectional curvature c, following similar calculations to those in Theorem 2 in [5] and taking into account relation (2.2), it is proved that the **-Ricci tensor* S^* of M is given by

$$S^* = -\left[\frac{cn}{2}\varphi^2 + (\varphi A)^2\right]. \tag{2.4}$$

In [10,11] Takagi was the first to study and classify homogeneous real hypersurfaces in $\mathbb{C}P^n$, $n \geq 2$. He proved that they could be divided into six types, namely (A_1), (A_2), (B), (C), (D) and (E). All these real hypersurfaces are Hopf hypersurfaces with constant principal curvatures, i.e. the eigenvalues of the shape operator are constant. In case of $\mathbb{C}H^n$, $n \geq 2$, the study of real hypersurfaces with constant principal curvatures was started by Montiel [2] and completed by Berndt in [12] for the Hopf case. They are divided into two types, namely (A) and (B), depending on the number of constant principal curvatures. The real hypersurfaces found by them are homogeneous. In contrast to $\mathbb{C}P^n$, in $\mathbb{C}H^n$ there are homogeneous real hypersurfaces which are not Hopf (see [13]). More information on the problem of classification of real hypersurfaces with constant principal curvatures in complex space forms can be found in [14] by Díaz-Ramos and Domínguez-Vázquez.

In the study of real hypersurfaces in $M_n(c)$ the following theorem plays an important role. In case of $\mathbb{C}P^n$ it was proved by Okumura (see [15]) and in case of $\mathbb{C}H^n$ by Montiel and Romero (see [16]). It provides the classification of real hypersurfaces satisfying relation $A\varphi = \varphi A$, i.e. the shape operator A commutes with the structure tensor φ.

Theorem 2.1. *Let M be a real hypersurface of $M_n(c)$, $n \geq 2$. Then $A\varphi = \varphi A$, if and only if M is locally congruent to a homogeneous real hypersurface of type* (A). *More precisely:*

- *In case of $\mathbb{C}P^n$*
  (A_1) *a geodesic hypersphere of radius r, where $0 < r < \frac{\pi}{2}$,*
  (A_2) *a tube of radius r over a totally geodesic $\mathbb{C}P^k$, $(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$.*
- *In case of $\mathbb{C}H^n$*
  (A_0) *a horosphere in $\mathbb{C}H^n$, i.e. a Montiel tube,*
  (A_1) *a geodesic hypersphere or a tube over a totally geodesic complex hyperbolic hyperplane $\mathbb{C}H^{n-1}$,*
  (A_2) *a tube over a totally geodesic $\mathbb{C}H^k$ $(1 \leq k \leq n-2)$.*

In the bibliography and the present paper all the above real hypersurfaces are called real hypersurfaces of type (A) and are Hopf hypersurfaces with constant principal curvatures.

3. Proof of theorems

Let M be a real hypersurface in $M_n(c)$, $n \geq 2$, admitting a $*$-Ricci soliton with potential vector field ξ. Generally, the Lie derivative of the Riemannian metric g is given by

$$\mathcal{L}_V g = g(\nabla_X V, Y) + g(\nabla_Y V, X).$$

Therefore, relation (1.1) due to (2.4) and the first of (2.1) since the potential vector field is ξ becomes

$$g(\varphi AX, Y) + g(\varphi AY, X) + ncg(X, Y) - nc\eta(X)\eta(Y) + 2g(\varphi AX, A\varphi Y) - 2\lambda g(X, Y) = 0. \tag{3.1}$$

Let $\mathcal{N}$ be the open subset of M such that

$$\mathcal{N} = \{P \in M : \ \beta \neq 0 \text{ in a neighborhood of } P\}.$$

So on $\mathcal{N}$ relation (2.3) holds. The relation (3.1) for $X = Y = \xi$ and because of (2.3) implies

$$\lambda = 0. \tag{3.2}$$

Furthermore, (3.1) for $X = \xi$ and $Y = U$ and for $X = \xi$ and $Y = \varphi U$ due to (3.2) and $\beta \neq 0$ yields respectively

$$g(A\varphi U, \varphi U) = 0, \tag{3.3}$$

$$g(AU, \varphi U) = g(A\varphi U, U) = \frac{1}{2}. \tag{3.4}$$

Relation (3.1) for $X = Y = U$ and $X = Y = \varphi U$ because of relations (3.2)–(3.4) results in the following relations respectively

$$-1 + nc + 2g(\varphi AU, A\varphi U) = 0,$$

$$1 + nc + 2g(A\varphi U, \varphi AU) = 0.$$

The combination of the last two relations leads to $2 = 0$, which is a contradiction.

Therefore, the following proposition is proved

Proposition 3.1. *Every real hypersurface in $M_n(c)$, $n \geq 2$, admitting a $*$-Ricci soliton with potential vector field ξ is a Hopf hypersurface.*

Since M is a Hopf hypersurface, due to Theorem 2.1 [17] we have that α is a constant. We consider a unit vector field $Z \in \mathbb{D}$, such that $AZ = \mu Z$, then $(\mu - \frac{\alpha}{2})A\varphi Z = (\frac{\mu\alpha}{2} + \frac{c}{4})\varphi Z$ at some point $P \in M$, (Corollary 2.3 [17]).

- Case I: $\alpha^2 + c \neq 0$.

In this case we have that $\mu \neq \frac{\alpha}{2}$, so $A\varphi Z = \nu\varphi Z$. Moreover, on M the following relation holds (Corollary 2.3 [17])

$$\mu\nu = \frac{\alpha}{2}(\mu + \nu) + \frac{c}{4}. \tag{3.5}$$

Relation (3.1) for $X = Y = \xi$, due to $A\xi = \alpha\xi$ implies $\lambda = 0$. Relation (3.1) for $X = Y = Z$ due to $\lambda = 0$, the first of (2.1) and $AZ = \mu Z$, implies

$$nc + 2\mu\nu = 0. \tag{3.6}$$

Furthermore, relation (3.1) for $X = Z$ and $Y = \varphi Z$ because of the first of (2.1), $\lambda = 0$, $AZ = \mu Z$ and $A\varphi Z = \nu\varphi Z$ results in

$$\mu = \nu. \tag{3.7}$$

Substitution of (3.7) in (3.6) leads to $nc + 2\mu^2 = 0$, which results in $c < 0$. So there are no real hypersurfaces in $\mathbb{C}P^n$ admitting a *-Ricci soliton with potential vector field ξ and this completes the proof of Theorem 1.3.

Relation (3.7) implies that Theorem 2.1 holds and M is locally congruent to a real hypersurface of type (A) in $\mathbb{C}H^n$. It remains to check if these real hypersurfaces admit a *-Ricci soliton with potential vector field ξ.

Substitution of $\mu = \nu$ and $c = -\frac{2\mu^2}{n}$ in (3.5) implies

$$\mu[(2n+1)\mu - 2\alpha n] = 0.$$

If $\mu = 0$ then relation $c = -\frac{2\mu^2}{n}$ results in $c = 0$, which is a contradiction. Therefore, the following relation holds

$$(2n+1)\mu = 2n\alpha. \tag{3.8}$$

Suppose that M is a geodesic hypersphere in $\mathbb{C}H^n$. Substituting the eigenvalues of the geodesic hypersphere in relation (3.8) leads to $n = \frac{\coth^2(r)}{2}$ (for the eigenvalues see [12]). So the geodesic hypersphere admits a *-Ricci soliton with potential vector field ξ and the radius r satisfies the relation $2n = \coth^2(r)$.

Suppose that M is a tube over totally geodesic $\mathbb{C}H^{n-1}$. Substituting the eigenvalues of the tube in relation (3.8) leads to $n = \frac{\tanh^2 r}{2}$, which is a contradiction since $0 < \tanh(r) < 1$. So the tube over $\mathbb{C}H^{n-1}$ does not admit a *-Ricci soliton with potential vector field ξ.

Finally, suppose that M is a tube over totally geodesic $\mathbb{C}H^{\kappa}$, where $1 \le \kappa \le n-2$. In this case the following hold

$$A\xi = 2\coth(2r)\xi \qquad AX = \tanh(r)X \quad \text{and} \quad AY = \coth(r)Y.$$

Substitution in (3.8) the previous eigenvalues imply

$$\coth^2(r) = 2n \quad \text{and} \quad \tanh^2(r) = 2n.$$

The combination of the last two relations leads to a contradiction and the tube over totally geodesic $\mathbb{C}H^{\kappa}$, where $1 \le \kappa \le n-2$ does not admit a *-Ricci soliton with potential vector field ξ.

- Case II: $\alpha^2 + c = 0$.

In this case the ambient space is only $\mathbb{C}H^n$, since $c = -\alpha^2$. So $\alpha \neq 0$. Suppose that $\mu \neq \frac{\alpha}{2}$. Then relation (3.5), owing to $\alpha^2 + c = 0$, results in $\nu = \frac{\alpha}{2}$, where ν is the eigenvalue corresponding to φZ, i.e. $A\varphi Z = \nu\varphi Z$. Relation (3.1) for $X = Y = \xi$ implies $\lambda = 0$ and for $X = Z$ and $Y = \varphi Z$ because of the first of (2.1), $AZ = \mu Z$ and $A\varphi Z = \frac{\alpha}{2}\varphi Z$ results in

$$\mu = \frac{\alpha}{2},$$

which is a contradiction.

So the remaining case is that of $\lambda = \frac{\alpha}{2}$ being the only eigenvalue for all vectors in the holomorphic distribution $\mathbb{D}$. In this case the real hypersurface is a horosphere. Relation (3.1) for $X = Y = \xi$ implies $\lambda = 0$ and for $X = Y = Z$ taking into that $\frac{\alpha}{2}$ is the only eigenvalue in $\mathbb{D}$ results in

$$n = \frac{1}{2},$$

which is impossible and this completes the proof of Theorem 1.4.

4. Discussion-open problems

In this paper it is proved that the geodesic hypersphere in $\mathbb{C}H^n$ is the only real hypersurface which admits a *-Ricci soliton with potential vector field being ξ erase. The purpose of this section is to present open problems, which raise in a natural way. They are divided into two categories.

CASE I: Other types of potential vector field V.

In this case the open problems concern real hypersurfaces in $M_n(c)$ admitting a *-Ricci soliton with potential vector field being other special types of vector fields.

- The potential vector field is a Killing or a zero vector field.

In this case, since the potential vector field V is a Killing or a zero one, it implies that $\mathcal{L}_V g = 0$ and relation (1.1) becomes

$$\text{Ric}^* = \lambda g,$$

where λ is constant. Taking into account the relation $\text{Ric}^*(X, Y) = g(S^*X, Y)$, the former relation leads to

$$S^*X = \lambda X, \quad \text{where } X \in TM \text{ and } \lambda = \text{ constant}. \tag{4.1}$$

Therefore, in order to answer this question one should study real hypersurfaces in $M_n(c)$ satisfying (4.1). It should be mentioned here that the class of such real hypersurfaces is a subclass of *-Einstein real hypersurfaces mentioned in the Introduction.

- The potential vector field is not a Killing or a zero vector field, but it is other types of special vector fields.

In this case there are two questions that are interesting to be answered.

1. Are there real hypersurfaces in $M_n(c)$ admitting a *-Ricci soliton whose potential vector field V belongs to the holomorphic distribution?
2. Are there real hypersurfaces in $M_n(c)$ admitting a *-Ricci soliton whose potential vector field V is a principal vector field of the real hypersurface?

CASE II: The condition of *-Ricci soliton is satisfied only in the holomorphic distribution of the real hypersurface.

Suppose that the real hypersurface M in $M_n(c)$ is equipped with a Riemannian metric g, which is a *-Ricci soliton on the holomorphic distribution, i.e. relation (1.1) holds for vector fields $X, Y \in \mathbb{D}$. If the potential vector field V is a Killing or a zero one, it implies that $\mathcal{L}_V g = 0$ and this results in

$$g(S^*X, Y) = \lambda g(X, Y) \quad \text{where } X, Y \in \mathbb{D} \text{ and } \lambda = \text{ constant.} \tag{4.2}$$

Since λ is constant, M is a *-Einstein real hypersurface. If M is a Hopf hypersurface, then taking into account the classification of *-Einstein Hopf hypersurfaces in $M_n(c)$, $n \geq 2$ in [4,5], the following holds

Proposition 4.1. *A Hopf hypersurface in $M_n(c)$, $n \geq 2$, admitting a *-Ricci soliton on the holomorphic distribution, whose potential vector field is Killing or zero is locally congruent to one of the following*

- *in case of $\mathbb{C}P^n$,*
  either a geodesic hypersphere of radius r, where $0 < r < \frac{\pi}{2}$,
  or a tube over a complex quadric Q^{n-1} and $\mathbb{R}P^n$,
  or a Hopf hypersurface with $A\xi = 0$,
- *in case of $\mathbb{C}H^n$*
  either a horosphere,
  or a geodesic hypersphere, where $r > 0$,
  or a tube over a totally geodesic $\mathbb{C}H^{n-1}$, where $r > 0$,
  or a tube over a totally real hyperbolic space $\mathbb{R}H^n$,
  or a Hopf hypersurface with $A\xi = 0$.

Therefore, the question which raises from the above remark is the following

Are there real hypersurfaces in $M_n(c)$, whose Riemannian metric g is a *-Ricci soliton in the holomorphic distribution and the potential vector field is not Killing?

Remark 4.2. The *-Ricci soliton is a new notion not only in the area of Mathematics but in the area of Physics as well. This paper aims to introduce this notion and to be a motivation for further research in both areas. It is known that the definition of *-Ricci tensor, which is connected to the definition of *-Einstein and to that of *-Ricci soliton, is based on the property that the Ricci tensor has with respect to the complex structure J of Hermitian manifolds. Furthermore, as mentioned in the section open problems of the present paper the *-Ricci soliton can be considered as a generalization of *-Einstein, in the same way as the notion of Ricci soliton is a generalization of the notion of Einstein. Therefore, exact physical applications of this notion are still unknown, but they are still under investigation by the authors.

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