PROPERTIES OF IRRESOLUTE-BITOPOLOGICAL GROUPS

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Abstract. In this paper, we introduce and study a class of bitopologized groups called (i, j)-irresolute-bitopological groups.

1. Introduction

If $(G, \star)$ is a group, and τ_1 and τ_2 are topologies on G, then we say that $(G, \star, \tau_1, \tau_2)$ is a bitopologized group. Given a bitopologized group G, a question arises about interactions and relations between algebraic and bitopological structures: which topological properties are satisfied by the multiplication mapping $m : G \times G \to G$, $(x, y) \to x \star y$, and the inverse mapping $i : G \to G$, $x \to x^{-1}$. In this paper, we introduce and study a class of bitopologized groups called (i, j)-irresolute-bitopological groups.

2. Preliminaries

Throughout this paper $(G, \star, \tau_1, \tau_2)$, or simply G, will denote a group $(G, \star)$ endowed with the topologies τ_1 and τ_2 on G. The identity element of G is denoted by e, or e_G when it is necessary, the operation $\star : G \times G \to G$, $(x, y) \to x \star y$, is called the multiplication mapping and sometimes denoted by m, and the inverse mapping $i : G \to G$, $x \to x^{-1}$ is denoted by i. For a subset A of a topological space (X, τ_i), $i\,\mathrm{Cl}(A)$ and $i\,\mathrm{Int}(A)$ denote the closure of A and the interior of A in (X, τ_i), respectively.

Definition 2.1. [2] *A subset S of a bitopological space (X, τ_1, τ_2) is said to be (i, j)-semiopen if $S \subset j\,\mathrm{Cl}(i\,\mathrm{Int}(S))$. The complement of an (i, j)-semiopen set is called an (i, j)-semiclosed set.*

Definition 2.2. [2] *The intersection of all (i, j)-semiclosed sets containing $S \subset X$ is called the (i, j)-semiclosure of S and is denoted by (i, j)-$s\,\mathrm{Cl}(S)$. The family of all (i, j)-semiopen (resp. (i, j)-semiclosed) sets of (X, τ_1, τ_2) is denoted by (i, j)-$SO(X)$ (resp. (i, j)-$SC(X)$). The family of all (i, j)-semiopen (resp. (i, j)-semiclosed) sets of (X, τ_1, τ_2) containing a point $x \in X$ is denoted by (i, j)-$SO(X, x)$ (resp. (i, j)-$SC(X, x)$).*

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Definition 2.3. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be:*

(1) *(i, j)-semicontinuous* [2] *if $f^{-1}(V) \in (i, j)$-$SO(X)$ for every $V \in \sigma_i$.*
(2) *(i, j)-irresolute* [1] *if $f^{-1}(V) \in (i, j)$-$SO(X)$ for every $V \in (i, j)$-$SO(Y)$.*
(3) *pre-(i, j)-semiopen if $f(V) \in (i, j)$-$SO(Y)$ for every $V \in (i, j)$-$SO(X)$.*
(4) *(i, j)-semihomeomorphism if f is bijective, (i, j)-irresolute and pre-(i, j)-semiopen.*

Lemma 2.4. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an (i, j)-semihomeomorphism, then:*

(1) *(i, j)-$s\,\mathrm{Cl}(f(A)) = f((i, j)$-$s\,\mathrm{Cl}(A))$ for all $A \subset X$;*
(2) *(i, j)-$s\,\mathrm{Int}(f(A)) = f((i, j)$-$s\,\mathrm{Int}(A))$ for all $A \subset X$.*

3. Irresolute-bitopological groups

Definition 3.1. *A bitopologized group $(G, \star, \tau_1, \tau_2)$ is called an (i, j)-irresolute-bitopological group if for each $x, y \in G$ and each (i, j)-semiopen neighborhood W of $x \star y^{-1}$ in G there exist (i, j)-semiopen neighborhoods U of x and V of y such that $U \star V^{-1} \subset W$.*

Lemma 3.2. *If $(G, \star, \tau_1, \tau_2)$ is an (i, j)-irresolute-bitopological group, then we have the following*

(1) *$A \in (i, j)$-$SO(G)$ if, and only if $A^{-1} \in (i, j)$-$SO(G)$.*
(2) *If $A \in (i, j)$-$SO(G)$ and $B \subset G$, then $A \star B, B \star A \in (i, j)$-$SO(G)$.*

Proof. The proof is clear. □

Theorem 3.3. *If $(G, \star, \tau_1, \tau_2)$ is an (i, j)-irresolute-bitopological group, then the multiplication mapping $m : G \times G \to G$ and the inverse mapping $i : G \to G$ are (i, j)-irresolute.*

Proof. Let $(x, y) \in G \times G$ and let $W \subset G$ be an (i, j)-semiopen neighbourhood of $m(x, y) = x \star y$. Since G is an (i, j)-irresolute-bitopological group, and W is an (i, j)-semiopen neighbourhood of $x \star (y^{-1})^{-1}$, there exist (i, j)-semiopen neighborhoods U of x and V of y^{-1} such that $U \star V^{-1} \subset W$. From Lemma 3.2, V^{-1} is an (i, j)-semiopen neighbourhood of y, and $U \star V^{-1}$ is an (i, j)-semiopen neighbourhood of $(x, y) \in G \times G$. Then $m(U \star V^{-1}) \subset W$ just means that m is irresoute at (x, y), and, as (x, y) was an arbitrary point in $G \times G$, that m is (i, j)-irresolute on $G \times G$. For i is (i, j)-irresolute, let $x \in G$ be a point and let W be an (i, j)-semiopen neighbourhood of $i(x) = x^{-1}$. Then by Lemma 3.2, W^{-1} is an (i, j)-semiopen neighbourhood of x satisfying $i(W^{-1}) = W$, which means that i is (i, j)-irresolute at x, hence on G. □

Remark 3.4. *If $(G, \star, \tau_1, \tau_2)$ is an (i,j)-irresolute-bitopological group such that the family (i,j)-$SO(G)$ is a topology on G with (i,j)-$SO(G) \neq \tau_i$, then $(G, \star, (i,j)$-$SO(G))$ is a topological group.*

Theorem 3.5. *Let $(G, \star, \tau_1, \tau_2)$ be an (i,j)-irresolute-bitopological group and $(H, \circ, \tau_1|H, \tau_2|H)$ a bitopological group. If $f : G \to H$ is a pairwise homeomorphism and (i,j)-semihomeomorphism, then H is also an (i,j)-irresolute-bitopological group.*

Proof. Let x and y be any two points in H and let $W \subset H$ be an (i,j)-semiopen neighbourhood of $x \circ y^{-1}$. Let $a = f^{-1}(x)$, $b = f^{-1}(y)$. Since f is an (i,j)-irresolute mapping, the set $f^{-1}(W)$ is an (i,j)-semiopen neighbourhood of $a \star b^{-1}$, and as G is an (i,j)-irresolute-bitopological group there are (i,j)-semiopen neighbourhoods A and B of a and b, respectively with $A \star B^{-1} \subset f^{-1}(W)$. Pre-$(i,j)$-semiopenness of f implies that the sets $U = f(A)$ and $V = f(B)$ are (i,j)-semiopen neighbourhoods of x and y that satisfy $U \circ V^{-1} = f(A) \circ (f(B))^{-1} = f(A \star B^{-1}) \subset W$. This means that $(H, \circ, \tau_1|H, \tau_2|H)$ is an (i,j)-irresolute-bitopological group. □

Theorem 3.6. *Let $f : G \to H$ be a pairwise homeomorphism, where $(G, \star, \tau_1, \tau_2)$ and $(H, \circ, \tau_1|H, \tau_2|H)$ be (i,j)-irresolute bitopological groups. If f is (i,j)-irresolute at the neutral element e_G of G, then f is (i,j)-irresolute on G.*

Proof. Let $x \in G$ be an arbitrary element, and let W be an (i,j)-semiopen neighbourhood of $y = f(x)$ in H. Since the left translations in H are (i,j)-semihomeomorphisms, there is an (i,j)-semiopen neighbourhood V of the neutral element e_H of H such that $l_y(V) = y \circ V \subset W$. From (i,j)-irresoluteness of f at e_G it follows the existence of an (i,j)-semiopen neighbourhood U of e_G such that $f(U) \subset V$. But $l_x : G \to G$ is a pre-(i,j)-semiopen mapping, so that the set $x \star U$ is an (i,j)-semiopen neighbourhood of x, for which we have $f(x \star U) = f(x) \circ f(U) = y \circ f(U) \subset y \circ V \subset W$: Hence f is (i,j)-irresolute at the point x of G, and since x was an arbitrary element in G, f is (i,j)-irresolute on G. □

Theorem 3.7. *Every biopen subgroup H of an (i,j)-irresolute-bitopological group $(G, \star, \tau_1, \tau_2)$ is also an (i,j)-irresolute-bitopological group.*

Proof. We prove that for each $x, y \in H$ and each (i,j)-semiopen neighbourhood $W \subset H$ of $x \star y^{-1}$ there exist (i,j)-semiopen neighbourhoods $U \subset H$ of x and $V \subset H$ of y such that $U \star V^{-1} \subset W$. Since H is biopen, and W is (i,j)-semiopen in G, the set $W \cap H = W$ is an (i,j)-semiopen in G and contains $x \star y^{-1}$. Apply now the fact that G is an (i,j)-irresolute-bitopological group to find (i,j)-semiopen neighbourhoods A of x and B of y such that $A \star B^{-1} \subset W$. The sets $U = A \cap H$ and $V = B \cap H$ are (i,j)-semiopen subsets of H which contain x and

y and satisfy $U \star V^{-1} \subset A \star B^{-1} \subset W$, which means that H is an (i,j)-irresolute-bitopological group. □

Theorem 3.8. *A nonempty subgroup H of an (i,j)-irresolute-bitopological group G is (i,j)-semiopen if, and only if its (i,j)-semiinterior is nonempty.*

Proof. Assume $x \in (i,j)\text{-}s\,\mathrm{Int}(H)$. There is an (i,j)-semiopen set V such that $x \in V \subset H$. This implies $x \star V \subset H$. For every $y \in H$ we have $y \star V = y \star x^{-1} \star x \star V \subset y \star x^{-1} \star H = H$. Since $y \star V$ is (i,j)-semiopen we conclude that $H = \cup\{y \star V : y \in H\}$ is (i,j)-semiopen as the union of (i,j)-semiopen sets. The converse is trivial. □

Theorem 3.9. *Let $(G, \star, \tau_1, \tau_2)$ be an (i,j)-irresolute-bitopological group. Then every biopen subgroup of G is (i,j)-semiclosed in G.*

Proof. Let H be a biopen subgroup of G. Then every left coset $x \star H$ of H is (i,j)-semiopen because l_x is a pre-(i,j)-semiopen mapping. Thus, $Y = \underset{x \in G\setminus H}{\cup} x \star H$ is also (i,j)-semiopen as a union of (i,j)-semiopen sets. Then $H = G \setminus Y$ is (i,j)-semiclosed. □

Theorem 3.10. *Let U be a symmetric (i,j)-semiopen neighbourhood of the identity e in an (i,j)-irresolute-bitopological group G. Then the set $L = \overset{\infty}{\underset{n=1}{\cup}} U^n$ is an (i,j)-semiopen and (i,j)-semiclosed subgroup of G.*

Proof. It is easy to see that L is a subgroup of G: if $x \in U^p$, $y \in U^q$ are elements from L, then $x \star y \in U^{p+q} \subset L$; if $x \in L$, say $x \in U^k$, then $x^{-1} \in (U^{-1})^k = U^k \subset L$. Because all translations in G are pre-(i,j)-semiopen mappings, U^n is (i,j)-semiopen for each $n \in N$, and thus also L is (i,j)-semiopen in G. By Theorem 3.9, L is (i,j)-semiclosed. □

Corollary 3.11. *Let G be an (i,j)-irresolute bitopological group, and μ_e the collection of all (i,j)-semiopen neighbourhoods of e. Then*

(1) *for every $U \in \mu_e$, there is a $V \in \mu_e$ such that $V^{-1} \subset U$;*
(2) *for every $U \in \mu_e$ and every $x \in U$, there is $V \in \mu_e$ such that $x \star V \subset U$.*

Proof. (1). It follows from the fact that the inverse mapping i is (i,j)-irresolute and $i(e) = e$.
(2). This is follows from: ‘x is an (i,j)-semihomeomorphism, and $l_x(e) = x$, so that there is a neighbourhood V of e such that $l_x(V) \subset U$, that is, $x \star V \subset U$. □

Theorem 3.12. *Let G be an (i,j)-irresolute bitopological group, $A \subset G$ and μ_e the system of all (i,j)-semiopen neighbourhoods of e. Then $(i,j)\text{-}s\,\mathrm{Cl}(A) = \cap\{A \star U : U \in \mu_e\}$.*

Proof. Let $x \in (i,j)\text{-}s\,\mathrm{Cl}(A)$ and let U be an (i,j)-semiopen neighborhood of e. The inverse mapping i is (i,j)-irresolute, so that $i(e) = e$

implies the existence of an (i,j)-semiopen neighbourhood V of e such that $i(V) = V^{-1} \subset U$. From $x \star V \cap A \neq \emptyset$, it follows that for some $v \in V$ and some $a \in A$ it holds $a = x \star v$, that is, $x = a \star v^{-1} \in A \star V^{-1} \subset A \star U$. As U is an arbitrary element from μ_e it follows (i,j)-$s\,\mathrm{Cl}(A) \subset \cap\{A \star U : U \in \mu_e\}$. Conversely, let $x \notin (i,j)$-$s\,\mathrm{Cl}(A)$. Then there exists $V \in \mu_e$ such that $x \star V \cap A = \emptyset$. Pick an (i,j)-semiopen neighbourhood U of e such that $U^{-1} \subset V$; the existence of such a set U follows from the (i,j)-irresoluteness of the inverse mapping i at e. Then $(x \star U^{-1}) \cap A = \emptyset$, thus $x \neq A \star U$. Therefore, $\cap\{A \star U : U \in \mu_e\} \subset (i,j)$-$s\,\mathrm{Cl}(A)$, and the theorem is proved. □

Theorem 3.13. *Let A and B be subsets of an (i,j)-irresolute-bitopological group G. Then we have the following*

(1) (i,j)-$s\,\mathrm{Cl}(A) \star (i,j)$-$s\,\mathrm{Cl}(B) \subset (i,j)$-$s\,\mathrm{Cl}(A \star B)$;
(2) $((i,j)$-$s\,\mathrm{Cl}(A))^{-1} = (i,j)$-$s\,\mathrm{Cl}(A^{-1})$;
(3) $((i,j)$-$s\,\mathrm{Int}(A))^{-1} = (i,j)$-$s\,\mathrm{Int}(A^{-1})$.

Proof. (1). Suppose that $x \in (i,j)$-$s\,\mathrm{Cl}(A)$, $y \in (i,j)$-$s\,\mathrm{Cl}(B)$. Let W be an (i,j)-semiopen neighbourhood of $x \star y$. Then there are (i,j)-semiopen neighbourhoods U and V of x and y such that $U \star V \subset W$. Since $x \in (i,j)$-$s\,\mathrm{Cl}(A)$, $y \in (i,j)$-$s\,\mathrm{Cl}(B)$, there are $a \in A \cap U$ and $b \in B \cap V$. Then $a \star b \in (A \star B) \cap (U \star V) \subset (A \star B) \subset W$. This means $x \star y \in (i,j)$-$s\,\mathrm{Cl}(A \star B)$, that is, we have (i,j)-$s\,\mathrm{Cl}(A) \star (i,j)$-$s\,\mathrm{Cl}(B) \subset (i,j)$-$s\,\mathrm{Cl}(A \star B)$.
(2). Since the inverse mapping $i : G \to G$ is an (i,j)-semihomeomorphism, then by Lemma 2.4 we have (i,j)-$s\,\mathrm{Cl}(i(A)) = (i,j)$-$s\,\mathrm{Cl}(A^{-1}) = i((i,j)$-$s\,\mathrm{Cl}(A)) = ((i,j)$-$s\,\mathrm{Cl}(A))^{-1}$.
(3). Since the inverse mapping i is an (i,j)-semihomeomorphism, which by Lemma 2.4 gives (i,j)-$s\,\mathrm{Int}(A^{-1}) = (i,j)$-$s\,\mathrm{Int}(A))^{-1}$. □

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