Invariant Means on σ-Dedekind complete totally ordered Riesz Spaces

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Abstract

In this paper we consider the set *B* of all countable bounded subsets of *V*, where *V* is a totally ordered σ-Dedekind complete Riesz space equipped with the order topology. We show that on *B* there exists a function that in a sense behaves as an invariant "mean". To do this, we construct a set of "approximately invariant means" and we show, using the Ultrafilter Theorem, that this set has a cluster point. This cluster point is the "invariant mean" on *B* that we are looking for.

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1. Introduction

A *Riesz space* V (see [5]) is a vector space endowed with a partial order "$\preceq$" that:

(a) If $x \preceq y$, then $x+z \preceq y+z$, $\forall x, y, z \in V$.
(b) If $x \preceq y$, then $\alpha x \preceq \alpha y$, $\forall x, y \in V$ and $\forall\, 0 \leq a \in \mathbb{R}$.
(c) For all $x, y \in V$, their "supremum" $x \vee y$ and their "infimum" $x \wedge y$ with respect to "$\preceq$" exist and are elements of V.

Here we let *V* to be a totally ordered σ-Dedekind complete Riesz space. The total ordering ensures that all elements are comparable with respect to $\preceq$ and hence the order topology is well defined on *V*. The σ-Dedekind completeness means that every countable subset of *V* which is bounded above has a least upper bound (supremum). We set the order topology T_{ord} on *V*. Thus, (V, T_{ord}) is in a sense a generalization of $(\mathbb{R}, T_{ord})$. Hence, (V, T_{ord}) becomes a regular space (we refer to [7] for details regarding the order topology). It is well known (see [2]) that every element x in a Riesz space V has unique positive $x^{+} = x \vee 0$ and negative $x^{-} = -x \vee 0$ parts. Also, $x = x^{+} - x^{-}$ and an absolute value is defined as $|x| = x^{+} + x^{-}$. Regarding convergence in a Riesz space V, a net $\{x_{\alpha}\}$ in V is said to converge monotonely, if it is a monotone decreasing (respectively increasing) net and its infimum (respectively supremum) x exists in V and is denoted by $x_{\alpha} \downarrow x$ (respectively $x_{\alpha} \uparrow x$). A net $\{x_{\alpha}\}$ in a Riesz space V is said to *converge in order* to x if there exists a net $\{a_{\alpha}\}$ in V such that $a_{\alpha} \downarrow 0$ and $|x_n - x| \preceq a_{\alpha}$.

Example 1.1: Consider $\mathbb{R}^n$ with the usual vector operations, its usual norm derived by the order topology, and the partial order defined by:

$$x \preceq y \Leftrightarrow x_k \leq_{\mathbb{R}} y_k \ , k = 1, 2, ..., n .$$

Then, $\mathbb{R}^n$ is a totally ordered σ-Dedekind complete Riesz space.

It is worth noting that we can also define a norm $\|\cdot\|$ on a Riesz space V in such a way that it becomes a *normed lattice* $(V, \|\cdot\|)$. That is, if $x \preceq y$, then $\|x\| \leq \|y\|$, $\forall x, y \in V$.
A normed lattice which is also a Banach space is called a *Banach Lattice* (see [2]) .

Example 1.2: All of the classical Banach Spaces, l_p, c, c_0, $C(K)$, $L^p(\mu)$ are Banach lattices in their usual norm and the pointwise (almost everywhere in the case of $L^p(\mu)$) order.

The next result shows that there exists a relation between (V, T_{ord}) and $(V, \|\cdot\|)$. Our proof-and hence the result- also holds for order intervals in general partially ordered Riesz spaces.

Lemma 1.3
The closed order intervals of (V, T_{ord}) are also closed in $(V, \|\cdot\|)$.

Proof: At first we show that the positive cone (V^{+}, T_{ord}) is normed closed. To see this, assume that a net $\{y_{\lambda}\} \in V^{+}$ satisfies $y_{\lambda} \to y$ in $(V, \|\cdot\|)$. We have to show that $y \in V^{+}$. Indeed, since $|y_{\lambda} - y^{+}| \preceq |y_{\lambda} - y| \in V^{+}$ and since $\|\cdot\|$ is a lattice norm, $\|y_{\lambda} - y^{+}\| \leq \|y_{\lambda} - y\|$ and hence $y_{\lambda} \to y^{+}$. From the uniqueness of the limit point (since $V, \|\cdot\|$ is Hausdorff), $y = y^{+}$. Hence, $y \in V^{+}$. Thus, V^{+} is closed. Now let $x, y \in V$ and assume w.l.o.g. that $x \preceq y$. Then, it is easy to observe (see [5]) that the order closed interval $[x, y]$ is written as $[x, y] = (x + V^{+}) \bigcap (y - V^{+})$. Since (V^{+}, T_{ord}) is normed closed, so are the sets $(x+V^{+}),(y-V^{+})$, and hence their intersection. Thus, $[x,y]$ is normed closed. □

2. The Mean of a Set of Sequences

The question that we answer affirmatively here is the following: Is there any natural way of associating a shift invariant "mean"-like function to the set of countable bounded subsets of (V,T_{ord})? Intuitively, this function should be analogous and a generalization of the standard (weight-free) mean value/average on finite sets of $(\mathbb{R},T_{ord})$.

Formulation of the problem: Consider doubly infinite sequences $\{x_n\}_{n\in\mathbb{Z}}$ in (V,T_{ord}), that is, elements of $V^{\mathbb{Z}}$. Now let $B\subseteq V^{\mathbb{Z}}$ be the set of all such sequences that are bounded. That is, if $x=\{x_n\}_{n\in\mathbb{Z}}\in B$, then there is a $b\in V$ with $\sup\{|x_n|\}_{n\in\mathbb{Z}}\preceq b$. This set is well defined, since by assumption V is totally ordered σ-Dedekind complete. We set the product topology on $V^{\mathbb{Z}}$, where (V,T_{ord}) has the order topology, and we consider $B\subseteq V^{\mathbb{Z}}$ with the subspace topology induced by the topology of $V^{\mathbb{Z}}$. It is worth mentioning that apart from the topology considered in our problem, one can put a norm on B, by setting $\|x\|_{\infty}=\sup_{n\in N}|x_n|$. This norm induces a metric $d(x,y)=\|x-y\|$ and hence another topology.

Definition 2.1: A *mean* on the set *B* is a linear map $\mu:B\to V$ such that for every $x=\{x_n\}_{n\in\mathbb{Z}}\in B$ we have, $\inf_{n\in\mathbb{Z}} x_n\preceq\mu(x)\preceq\sup_{n\in\mathbb{Z}} x_n$.

Notice that the *inf* and the *sup* are well defined, since $\{x_n\}_{n\in\mathbb{Z}}$ is bounded from above and below and (V,T_{ord}) is totally ordered and σ-Dedekind complete.

Example 2.2: Consider the set B of doubly bounded real valued sequences.
(i) Define $\mu:B\to\mathbb{R}$ by $\mu(a)=\sum_{n\in\mathbb{Z}} w_n a_n$, where $w_n\in\mathbb{R}$ is a weight such that $\sum_{n\in\mathbb{Z}} w_n=1$. Then, according to Definition 2.1, μ is a mean on *B*.

(ii) Define $\mu:B\to\mathbb{R}$ by $\mu(a)=\sum_{n\in K} a_n$, where $K\subset\mathbb{Z}$ is a finite subset of $\mathbb{Z}$. Then, according to Definition 2.1, μ is a mean on *B*.

The two "means" in the above example are fundamentally different, in the sense that the second, in contrast to the first one, is in addition *shift invariant* according to the following definition.

Definition 2.3: Let $B \subseteq V^{\mathbb{Z}}$ be the set of doubly infinite bounded sequences in (V, T_{ord}). The *shift operator* $S : B \to B$, is defined by $S(x_n) = x_{n+1}$, where $x = \{x_n\}_{n\in\mathbb{Z}} \in B$. A mean is said to be *shift invariant* if for all $x \in B$ it holds that $\mu(S(x)) = \mu(x)$.

Here note that the second function in Example 2.2, as being *shift invariant*, can be considered as an averaging procedure, where the mean value is not affected by which element of the sequence is considered as the 'center' (position "0") of the sequence.

The question that we discuss here it now becomes concrete: Does such a shift invariant mean function on $B \subseteq V^{\mathbb{Z}}$ exist? And if it does, can we provide a method of constructing it? We pursue this question in the following section and with the aid of a *weak version of the Axiom of Choice* (namely the *Ultrafilter theorem*) we show that such a function exists on $B \subseteq V^{\mathbb{Z}}$. Since our proof is purely an existential one, as it uses a version of the Axiom of Choice, most probably a constructive method is hard to be given.

3. Invariant Means-Main Result

In this section we present the main result of this article. At first, we construct "approximately invariant means" and then we show the existence of shift invariant means from the approximately invariant means.

Next we present some preliminary results necessary for the proof of the main theorem.

Definition 3.1: A sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of means is approximately invariant if for all $x \in B$ $\lim_{n\to\infty} \mu_n(S(x)) = \mu_n(x)$.

The existence of such approximate means is guaranteed by the following Lemma.

Lemma 3.2: There is a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of means such that for all $x \in B$, $\lim_{n\to\infty} \mu_n(S(x)) = \mu_n(x)$.

Proof: For $x = \{x_n\}_{n\in\mathbb{Z}} \in B$, let $\mu_n(x) = \frac{1}{n}\sum_{i=1}^{n} x_i$. Then, by definition 2.3,

$$|\mu_n(S(x)) - \mu_n(x)| = \frac{|x_{n+1} - x_1|}{n} \le \frac{2\|x\|}{n}, \text{ where } \|x\| = \sup_{n\in N}|x_n| < \infty .$$

Thus, $\lim_{n\to\infty} |\mu_n(S(x)) - \mu_n(x)| = 0$. □

How to conclude the existence of invariant means? Interestingly, the proof of the existence of invariant means will be non-constructive. Basically, we consider the set $\mathfrak{M} \subseteq V^B$ of all means and we topologize it in such a way that $\mathfrak{M}$ becomes a compact space. To do this we use a special case of Tychonoff's theorem that is equivalent to the *Ultrafilter Theorem*[1]. Then we show that the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of the approximately invariant means has a cluster point μ, which is shift invariant. That μ is the mean-like function we are looking for.

We view $\mathfrak{M}$ as a subset of V^B, by identifying a mean μ with the indexed family $(\mu(x))_{x\in B}$. We then give V^B the product topology and $\mathfrak{M}$ the subspace topology. This way, the evaluation maps (projections) $\mu \mapsto \mu(x),\ x\in B$, are continuous maps from $\mathfrak{M}$ to V for each $x \in B$. Notice also that the definition of a "mean" actually makes $\mathfrak{M}$ a subset of a product of ordered closed intervals[2]. That is, $\mathfrak{M} \subseteq \prod_{x\in B}[m(x), M(x)]$, where $m(x) = \inf_n x_n$ and $M(x) = \sup_n x_n$.

For the proof of the proposition 3.4, we need the next lemma. Its proof is presented in the Appendix and it does not use the general Tychonoff's theorem (and thus it avoids general Axiom of Choice) but instead the strictly weaker *Ultrafilter Theorem*.

Lemma 3.3: Let I be an indexed set and assume that (X_i, T_i), $i \in I$, are compact Hausdorff spaces. Then $\left(\prod_{i\in I} X_i, T_{pr}\right)$ with the product topology is also compact.

Proposition 3.4: The set of all means $\mathfrak{M} \subseteq V^B$ is a compact space.

Proof: We show that $\mathfrak{M}$ is a closed subset of a compact space. Since $[m(x), M(x)] \subseteq (V, T_{ord})$ is an ordered closed and bounded interval that it has the least upper bound property (recall that, (V, T_{ord}) is σ-Dedekind complete space), the Heine-Borel [3] theorem implies that it is also compact (see [7] Thm. 27.1). Moreover, $[m(x), M(x)]$ with the subspace topology induced by the T_{ord} on V is a Hausdorff space. Hence, from Lemma 3.3, $X = \prod_{x\in B}[m(x), M(x)]$ is a compact space. We show that $\mathfrak{M} = \bigcap f^{-1}(0)$, for various continuous maps $f : X \to V$. Recall also that the evaluation maps $\mu \mapsto \mu(x),\ x \in B$, are continuous. Recall (see definition 2.1) that an element $\mu \in \mathfrak{M}$ is a "mean", if and only if is a bounded linear map. That is, $\forall x, y \in B, \forall \lambda \in \mathbb{R}$,

[1] In contrast to the general version of Tychonoff's Theorem that is equivalent to the Axiom of Choice (see [6]), the version that we use here is equivalent to the Ultrafilter Theorem (See the Appendix) . It is also well know that Ultrafilter theorem is strictly weaker that the Axiom of choice (see [4]).

[2] which by Lemma 1.3 are also normed closed.

[3] Note that the Heine-Borel theorem can be proved in ZF without using any version of the Axiom of Choice. The proof of this is given in 4.6 of [4].

$$\mu(x+y)-\mu(x)-\mu(y)=0$$
$$\mu(\lambda x)-\lambda\mu(x)=0$$

where, for every $x=\{x_n\}_{n\in\mathbb{Z}}\in B$ we have, $\inf_{n\in\mathbb{Z}} x_n \preceq \mu(x) \preceq \sup_{n\in\mathbb{Z}} x_n$.

Hence, $\mathfrak{M}=\bigcap f^{-1}(0)$, where f 's are the above evaluation maps $\mu\mapsto\mu(x)$. Since f is continuous, then $f^{-1}(0)$ is closed (because $\{0\}$ is closed in (V,T_{ord})). Thus, since $\mathfrak{M}$ is the intersection of closed sets, it is closed. □

The next lemma is a version of the Bolzano-Weierstrass property for compact spaces. A concise proof is presented in the appendix.

Lemma 3.5: If X is a compact space then every sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in X has a cluster point. That is, there is a point $x\in X$ such that for every open neighborhood U_x of $x\in X$, $U_x\cap\langle x_n\rangle_{n\in\mathbb{N}}\in X$ contains infinitely many points.

Now it follows the proof of the main result of this article.

Theorem 3.6: There exists an invariant mean $\mu\in\mathfrak{M}$.

Proof: Applying Lemma 3.5 on $\mathfrak{M}$ (which is compact by Proposition.3.4) and on the sequence of approximately invariant means $\{\mu_n\}_{n\in\mathbb{N}}$ (see definition 3.1), we conclude that $\{\mu_n\}_{n\in\mathbb{N}}$ has a cluster point μ. We show that this μ is *shift invariant*. Let $x\in B$ arbitrary, and consider the neighborhood U_μ of μ:

$$U_\mu=\{\nu\in\mathfrak{M}:\ \forall x\in B,\ |\nu(x)-\mu(x)|<\varepsilon \text{ and } |\nu(S(x))-\mu(S(x))|<\varepsilon\ \}$$

Actually this is a basic neighborhood defined by imposing conditions on two coordinates of a general element $\nu\in\mathfrak{M}\subseteq V^B$. Since μ is a cluster point of $\{\mu_n\}_{n\in\mathbb{N}}$, then $\mu_n\in U_\mu$ for infinitely many n. Let $\varepsilon>0$ (arbitrary), then $\forall n'\in\mathbb{N}\ \exists n\geq n'$ so that $\mu_n\in U_\mu$. We can take n' large enough such that we have (see also Lemma 3.2) $|\mu_n(S(x))-\mu_n(x)|<\varepsilon$, $\mu_n\in U_\mu$. Thus,

$$|\mu(S(x))-\mu(x)|\leq|\mu(S(x))-\mu_n(S(x))|+|\mu_n(S(x))-\mu(x)|+|\mu_n(x)-\mu(x)|<3\varepsilon$$

Since $\varepsilon>0$ is arbitrary, $\mu(S(x))=\mu(x)$. □

4. Conclusion

In this paper we have shown how one can derive the existence of invariant "mean-like" function for any countable sequence of elements in a σ-Dedekind-complete totally ordered Riesz Space. For this we have used a weaker version of the Axiom of Choice, namely the *Ultrafilter Theorem*. At first, we provided an example on $\mathbb{R}$ in which the interpretation of the "mean" seemed quite intuitive and natural. It would be interesting to provide an interpretation, if any, of the "mean" in Riesz Spaces. Also, it would be interesting to extend these results in more general topologically ordered spaces.

As we have seen, the proof of Proposition 3.4 relies on the Heine-Borel Theorem. Recall that a topological vector space has the Heine–Borel property, if and only if, every closed and bounded subset of it is compact. For example, this theorem does not hold as stated for general topological vector spaces. Many metric spaces fail to have the Heine–Borel property. For instance, the metric space of rational numbers (or indeed any incomplete metric space) fails to have the Heine–Borel property. Complete metric spaces may also fail to have the property. For example, no infinite-dimensional Banach space has the Heine–Borel property. On the other hand, some infinite-dimensional Fréchet spaces do have the Heine–Borel property. The space $C^{\infty}(K)$ of smooth functions on a compact set $K \subset \mathbb{R}^n$, considered as a Fréchet space, has the Heine–Borel property, as can be shown by using the Arzelà–Ascoli theorem. We ultimately aim to investigate how the concept of invariant mean or an analogous one can be defined whenever we have an ordered structure on a topological vector space that satisfies the Heine-Borel property. Since any nuclear Fréchet space (see [3]) has the Heine–Borel property it is natural to consider at first these spaces.

Conflict of Interests
The authors declare that there is no conflict of interest regarding the publication of this paper.

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Appendix

In this appendix we prove Lemmas 3.3 and 3.5 that are crucial for the proof of our main result.

For the proof of Lemma 3.3 we use the *Ultrafilter theorem*, that is: "Every filter on a set can be extended to an ultrafilter".

It should be mentioned that the *Ultrafilter Theorem* is equivalent to many useful theorems of ZFC: Compactness theorem in logic; completeness theorem in logic; Tychonoff's theorem for Hausdorff spaces; and more. However, even if the general version of Tychonoff's Theorem is equivalent to Axiom of Choice (see [6]), the *Ultrafilter Theorem* is strictly weaker than the Axiom of Choice. That is, the Axiom of Choice implies the Ultrafilter Theorem; however the implication cannot be reversed. Specifically, in Cohen's first model (see [4]) the Axiom of Choice fails badly: The real numbers cannot be well-ordered, and the Axiom of Countable Choice fails (and consequently the Axiom of Choice). However, in that model the *Ultrafilter Theorem* holds.

Next, we show that the *Ultrafilter theorem* implies the Tychonoff Theorem in the case of Hausdorff spaces[4]. (That is, "The product of compact Hausdorff spaces is compact"). To do this we need the following results from the general theory of filters (see [1]).

<u>Useful Results from the theory of Filters</u>

A: Let $\mathfrak{I}$ be an ultrafilter on the set X and let $f: X \to Y$ be a map of sets. Then, $f^*\mathfrak{I} = \left\{ A \subseteq Y : f^{-1}(A) \in \mathfrak{I} \right\}$ is an ultrafilter on Y.

B: A filter $\mathfrak{I}$ on a topological space Y converges to a point $y \in Y$ if for all open sets U that contain y, $U \in \mathfrak{I}$.

[4] Actually it can be proven that the reverse implication also holds (see 4.37, p 61 in [4]).

The Ultrafilter Theorem is essential for the proof of the next two results (see [1]).

C: A topological space Y is compact, if and only if every ultrafilter $\Im$ on Y converges to at least one point.

D: A topological space Y is Hausdorff, if and only if every ultrafilter $\Im$ on Y converges to at most one point.

From C and D we have:

E: A topological space Y is compact and Hausdorff, if and only if every ultrafilter $\Im$ on Y converges to exactly one point.

Lemma 3.3: Let I be an indexed set and assume that (X_i, T_i), $i \in I$, are compact Hausdorff spaces. Then $\left(\prod_{i\in I} X_i, T_{pr}\right)$ with the product topology is also compact.

Proof: Let $\langle X_i \rangle_{i\in I}$ be the collection of compact Hausdorff spaces. To show that $\left(\prod_{i\in I} X_i, T_{pr}\right)$ is compact it is enough to show (see result **C**) that every Ultrafilter $\Im$ on $X = \prod_{i\in I} X_i$ has a limit point. Let $p_i : X \mapsto X_i$ be the i^{th} projection map. Note that the product topology T_{pr} on $X = \prod_{i\in I} X_i$ is generated by the sets $\left\{p_i^{-1}(U) : i \in I, U \in T_i\right\}$. In other words, $\left\{p_i^{-1}(U) : i \in I, U \in T_i\right\}$ is a sub-basis for T_{pr}. By result **A**, $p_i * \Im = \left\{G \subseteq X_i : p_i^{-1}(G) \in \Im\right\}$ is an ultrafilter on X_i. Since X_i is compact and Hausdorff, by result **E**, it has exactly one limit point, $x_i \in X_i$. Thus, $x = \langle x_i \rangle_{i\in I} \in \prod_{i\in I} X_i$ [5].

The following claim concludes the proof.

We claim that the above point $x = \langle x_i \rangle_{i\in I} \in X$, is a limit point for the Ultrafilter $\Im$.

[5] Note that here we do not use the Axiom of Choice (or any its versions) because the $x_i \in X_i$, $i \in I$, are unique, and thus, there is no need for any selection principle.

Indeed, since $\left\{p_i^{-1}(U): i\in I, U\in T_i\right\}$ is a sub-basis of T_{pr} on $X=\prod_{i\in I} X_i$, any open set G of X containing $x=\langle x_i\rangle_{i\in I}$, it contains also a finite intersection of sets of the form $\left\{p_i^{-1}(U): i\in I, U\in T_i\right\}$ which contain $x=\langle x_i\rangle_{i\in I}$. That is, $x\in \bigcap_{i\in J\subseteq I} p_i^{-1}(U)\subseteq G$ for some finite subset J of I and any open set G in T_{pr} with $x\in G$. To show that $x=\langle x_i\rangle_{i\in I}$ is a limit point of $\mathfrak{I}$, it is enough to show that $\mathfrak{I}$ contains every open neighborhood of $x=\langle x_i\rangle_{i\in I}$.

Since filters are closed under supersets and finite intersections, according to the above it is enough to show that if $x\in V=p_i^{-1}(U)$, where $i\in I, U\in T_i$, then $V\in\mathfrak{I}$.

Indeed, suppose $x\in V$. Since $p_i(x)=x_i\in U$ is a limit point of $p_i*\mathfrak{I}$, we have by result **B** that $U\in p_i*\mathfrak{I}$ for all $i\in I$. Hence, $\forall i\in I$, $V=p_i^{-1}(U)\in\mathfrak{I}$.

This concludes the proof of the theorem. □

The following result is a version of Bolzano-Weierstrass property for general topological compact spaces.

Lemma 3.5: If X is a compact space then every sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ of X has a cluster point. That is, there is a point $x\in X$ such that for every open neighborhood U_x of $x\in X$, $U_x\cap\langle x_n\rangle_{n\in\mathbb{N}}\in X$ contains infinitely many points.

Proof: For each x that is not a cluster point, there is a neighborhood U_x of $x\in X$, that contains only finitely many point from the sequence $\langle x_n\rangle_{n\in\mathbb{N}}$. Clearly no finite subcollection of such neighborhoods of x could cover X. (Any finite union of such neighborhoods of x contains only finitely many points from $\langle x_n\rangle_{n\in\mathbb{N}}\in X$ and hence cannot cover $\langle x_n\rangle_{n\in\mathbb{N}}$ and thus X). Hence, the family $\{U_x\}_{x\in X}$ does not cover X [6]. Since $\bigcup_{x\in X} U_x$ contains all the points of X that are not cluster points, and since $\bigcup_{x\in X} U_x$ does not cover X, then X must contain at least one cluster point of $\langle x_n\rangle_{n\in\mathbb{N}}$. Thus, $\langle x_n\rangle_{n\in\mathbb{N}}$ has a cluster point in X. □

[6] Recall the definition of Compactness.