Induced almost-additive Gurevich pressure for countable state Markov shifts

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Abstract. In this paper, we study the induced almost-additive Gurevich pressure for countable state Markov shifts and obtain its variational principle.

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1 Introduction and main result

Let (Σ, σ) be a one-sided *topological Markov shift (TMS)* over a countable set of *states* S. This means that there exists a matrix $A = (t_{ij})_{S\times S}$ of zeros and ones(with no row and no column made entirely of zeros) such that

$$\Sigma := \{\omega := (\omega_0, \omega_1, \dots) \in S^{\mathbb{N}_0} : t_{\omega_i\omega_{i+1}=1} \text{ for every } i \in \mathbb{N}_0\}.$$

The *shift map* $\sigma : \Sigma \to \Sigma$ is defined by $(\omega_0, \omega_1, \omega_2 \dots) \mapsto (\omega_1, \omega_2, \dots)$. We denote the set of A-admissible words of $n \in \mathbb{N}$ by

$$\Sigma^n := \{\omega := (\omega_0, \omega_1, \dots, \omega_{n-1}) \in S^n : t_{\omega_i\omega_{i+1}=1} \text{ for every } i \in \{0, 1, \dots, n-2\}\}.$$

and the set of A-admissible words of arbitrary length by $\sum^* := \bigcup_{n\in\mathbb{N}} \Sigma^n$. For $\omega \in \Sigma^*$, we let $|\omega|$ denote the length of ω, which is the unique $n \in \mathbb{N}$ such that $\omega \in \Sigma^n$. For $\omega \in \Sigma^n$ we call $[\omega] := \{\gamma \in \Sigma : \gamma|_n = \omega\}$ the *cylindrical set* of ω. We equip Σ with

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the topology generated by the cylindrical sets. The topology of the TMS is metrizable, it may be given by the metric $d_\alpha(\omega,\omega') := e^{-\alpha|\omega\wedge\omega'|}, \alpha > 0$, where $\omega \wedge \omega'$ denote the longest common initial block of $\omega, \omega' \in \Sigma$. The shift map σ is continuous with respect to this metric. If S is a finite set of states , (Σ,σ) is called a *subshift of finite type.* We call a function $f : \Sigma \to \mathbb{R}$ is α-Hölder continuous, if there exists $\alpha > 0$ and a constant $V_\alpha(f)$ such that for all $\omega, \omega' \in \Sigma$, $|f(\omega) - f(\omega')| \leq V_\alpha(f) d_\alpha(\omega,\omega')$. We say f is Hölder continuous, if there exists $\alpha > 0$ such that f is α-Hölder continuous. Let $H(\Sigma,\mathbb{R})$ be the space of all real-valued Hölder continuous functions of Σ. For $f \in H(\Sigma,\mathbb{R})$ and $n \geq 1$, let $S_n f(\omega) := \sum_{i=0}^{n-1} f(\sigma^i\omega)$. We denote by $\mathcal{M}$ the set of all σ-invariant Borel probability measures on Σ. We will always assume (Σ,σ) to be *topologically mixing*, that is, for every $a, b \in S$ there exists $N_{ab} \in \mathbb{N}$ such that for every $n > N_{ab}$ we have $[a] \cap \sigma^{-n}[b] \neq \emptyset$.

Definition 1.1. *Let (Σ,σ) be a one-sided countable states Markov shift. For each $n \in \mathbb{N}$, let $f_n : \Sigma \to \mathbb{R}^+$ be a continuous function. A sequence $\mathcal{F} := \{\log f_n\}_{n=1}^\infty$ on Σ is called almost-additive if there exists a constant $C \geq 0$ such that for every $n, m \in \mathbb{N}, \omega \in \Sigma$, we have*

$$f_n(\omega) f_m(\sigma^n\omega) e^{-C} \leq f_{n+m}(\omega). \tag{1.1}$$

and

$$f_{n+m}(\omega) \leq f_n(\omega) f_m(\sigma^n\omega) e^{C}. \tag{1.2}$$

Definition 1.2. *Let (Σ,σ) be a one-sided countable states Markov shift. For each $n \in \mathbb{N}$, let $f_n : \Sigma \to \mathbb{R}^+$ be a continuous function. A sequence $\mathcal{F} =: \{\log f_n\}_{n=1}^\infty$ on Σ is called a Bowen sequence if there exists a constant $M \in \mathbb{R}^+$ such that*

$$\sup\{A_n : n \in \mathbb{N}\} \leq M \tag{1.3}$$

where

$$A_n := \sup\{\frac{f_n(\omega)}{f_n(\omega')} : \omega, \omega' \in \Sigma, \omega_i = \omega_i' \textit{ for every } i \in \{0, 1, \ldots, n-1\}\}.$$

For $\omega = (\omega_0, \omega_1, \ldots, \omega_{n-1}) \in \Sigma^*$, let

$$\overline{\omega} := (\omega_0, \omega_1, \ldots, \omega_{n-1}, \omega_0, \omega_1, \ldots, \omega_{n-1}, \ldots)$$

denote the *periodic word* with period $n \in \mathbb{N}$ and initial block ω. Let

$$\Sigma^{per} := \{\omega \in \Sigma^* : \overline{\omega} \in \Sigma\}, \Sigma_a^{per} := \{\omega \in \Sigma^{per} : \omega_0 = a\}.$$

Definition 1.3. *Let (Σ, σ) be a one-sided countable states Markov shift, $\psi \in H(\Sigma, \mathbb{R})$ with $\psi \geq 0$, and $\mathcal{F} := \{\log f_n\}_{n=1}^{\infty}$ on Σ is an almost-additive Bowen sequence. We define for $\eta > 0$ the ψ-induced almost-additive Gurevich pressure of $\mathcal{F}$ with respect to Σ_a^{per} by*

$$\mathcal{P}_{\psi}(\mathcal{F}, \Sigma_a^{per}) := \limsup_{T\to\infty} \frac{1}{T} \log \sum_{\substack{\omega\in\Sigma_a^{per} \\ T-\eta<S_{|\omega|}\psi(\overline{\omega})\leq T}} f_{|\omega|}(\overline{\omega}),$$

which takes values in $\overline{\mathbb{R}} := \mathbb{R} \cup \{\mp\infty\}$.

In particular, for $\psi = 1$ our definition coincides with the almost-additive Gurevich pressure[2].

The thermodynamic formalism for countable states Markov shifts has been developed by Mauldin and Urbanski [5,6] and by Sarig [8,9,10,11]. Recently, Gurvich pressure for countable state Markov shifts [8]and the pressure for almost-additive sequences on compact spaces [1,4] were extended to the almost-additive pressure for countable states Markov shifts and a variational principle was set up [2]. In [3], the authors defined the induced pressure for countable states Markov shifts. Inspired by the articles [2] and [3], we define the induced almost-additive Gurevich pressure for countable state Markov shifts and obtain its variational principle as follows:

Theorem 1.1. *Let (Σ, σ) be a topologically mixing countable state Markov shift, $\mathcal{F} := \{\log f_n\}_{n=1}^{\infty}$ on Σ is an almost-additive Bowen sequence on Σ with $\sup f_1 < \infty$ and $\psi \in H(\Sigma, \mathbb{R}), \psi \geq c > 0$ with $\sup \psi < \infty$. Then*

$$\mathcal{P}_{\psi}(\mathcal{F}, \Sigma_a^{per}) = \sup\{\frac{h_v(\sigma)}{\int \psi d\nu} + \frac{\int f_*(\omega) d\nu}{\int \psi d\nu} : \nu \in \mathcal{M} \text{ and } \int f_*(\omega) d\nu \neq -\infty\}, \tag{1.4}$$

where $f_*(\omega) := \lim\limits_{n\to\infty} \frac{1}{n} \log f_n(\omega)$.

2 Preliminaries

In this section, we study the relation between the ψ-induced Gurevich pressure and the almost-additive Gurevich pressure. Making a similar proof as in [3, Theorem 2.1], we can obtain the following statement:

Theorem 2.1. *For $\psi \in H(\Sigma, \mathbb{R})$ with $\psi \geq 0$, $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ on Σ is a Bowen sequence, we have*

$$\mathcal{P}_{\psi}(\mathcal{F}, \Sigma_a^{per}) = \inf\{\beta \in \mathbb{R} : \limsup_{T\to\infty} \sum_{\substack{\omega\in\Sigma_a^{per} \\ T<S_{|\omega|}\psi(\overline{\omega})}} f_{|\omega|}(\overline{\omega}) e^{-\beta S_{|\omega|}\psi(\overline{\omega})} < \infty\}.$$

In particular, the definition of $\mathcal{P}_{\psi}(\mathcal{F}, \Sigma_a^{per})$ is independent of the choice of $\eta > 0$.

Lemma 2.1. *Let (Σ, σ) be a topologically mixing countable state Markov shift, $\mathcal{F} = \{\log f_n\}_{n=1}^{\infty}$ is an almost-additive Bowen sequence on Σ with $\sup f_1 < \infty$ and $\psi \in H(\Sigma, \mathbb{R}), \psi > 0$ with $\sup \psi < \infty$. Then for every $\beta \in \mathbb{R}$, $\mathcal{F}^{\beta} := \{\log f_n(\omega) e^{-\beta S_n \psi(\omega)}\}_{n=1}^{\infty}$ is an almost-additive Bowen sequence on Σ and*

$$\mathcal{P}_1(\mathcal{F}^{\beta}, \Sigma_a^{per}) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\substack{\omega \in \Sigma_a^{per} \\ |\omega| = n}} f_{|\omega|}(\overline{\omega}) e^{-\beta S_{|\omega|}\psi(\overline{\omega})}.$$

exists and it is not minus infinity and

$$\begin{aligned}
&\mathcal{P}_1(\mathcal{F}^{\beta}, \Sigma_a^{per}) \\
&= \sup\{h_{\nu}(\sigma) + \int \mathcal{F}_*^{\beta}(\omega) d\nu : \nu \in \mathcal{M} \text{ and } \int \mathcal{F}_*^{\beta}(\omega) d\nu \neq -\infty\} \\
&= \sup\{h_{\nu}(\sigma) + \int f_*(\omega) d\nu - \beta \int \psi d\nu \ : \nu \in \mathcal{M} \text{ and } \int f_*(\omega) d\nu \neq -\infty\}.
\end{aligned}$$

where $\mathcal{F}_*^{\beta}(\omega) := \lim_{n\to\infty} \frac{1}{n} \log f_n(\omega) e^{-\beta S_n \psi(\omega)}$, $f_*(\omega) := \lim_{n\to\infty} \frac{1}{n} \log f_n(\omega)$.

Proof. Since $\mathcal{F}$ is a Bowen sequence on Σ, the *bounded distortion property* (see [3]) shows for every $\psi \in H(\Sigma, \mathbb{R})$ there exists a constant $C_{\psi} > 0$ such that $|S_{|\omega|}\psi(\gamma) - S_{|\omega|}\psi(\gamma^{'})| \le C_{\psi}$ for all $\omega \in \Sigma^*$ and $\gamma, \gamma^{'} \in [\omega]$. Let $g_n(\omega) := f_n(\omega) e^{-\beta S_n \psi(\omega)}$. For each $n \in \mathbb{N}, \beta \in \mathbb{R}$, we have

$$B_n := \sup\{\frac{g_n(\omega)}{g_n(\omega^{'})} : \omega, \omega^{'} \in \Sigma, \omega_i = \omega_i^{'} \text{ for } i \in \{0, 1, \dots n-1\}\} \le e^{|\beta| C_{\psi}} A_n$$

and

$$\sup\{B_n : n \in \mathbb{N}\} \le M e^{|\beta| C_{\psi}}.$$

So $\mathcal{F}^{\beta}$ is a Bowen sequence on Σ.

Since $\mathcal{F}$ is an almost-additive sequence on Σ we have

$$f_n(\omega) f_m(\sigma^n \omega) e^{-C} e^{-\beta S_n \psi(\omega)} e^{-\beta S_m \psi(\sigma^n \omega)} \le f_{n+m}(\omega) e^{-\beta S_{n+m}\psi(\omega)}$$

and

$$f_{n+m}(\omega) e^{-\beta S_{n+m}\psi(\omega)} \le f_n(\omega) f_m(\sigma^n \omega) e^{C} e^{-\beta S_n \psi(\omega)} e^{-\beta S_m \psi(\sigma^n \omega)}.$$

Then $\mathcal{F}^{\beta}$ is an almost-additive Bowen sequence on Σ with $\sup f_1(\omega) e^{-\beta\psi(\omega)} < \infty$.

By [2,Theorem 3.1] and Birkhoff Ergodic Theorem [7], we have $\mathcal{P}_1(\mathcal{F}^{\beta}, \Sigma_a^{per})$ exists and it is not minus infinity and

$$\begin{aligned}
&\mathcal{P}_1(\mathcal{F}^{\beta}, \Sigma_a^{per}) \\
&= \sup\{h_{\nu}(\sigma) + \int \mathcal{F}_*^{\beta}(\omega) d\nu : \nu \in \mathcal{M}\text{and} \int \mathcal{F}_*^{\beta}(\omega) d\nu \neq -\infty\} \\
&= \sup\{h_{\nu}(\sigma) + \int f_*(\omega) d\nu - \beta \int \psi(\omega) d\nu \ : \nu \in \mathcal{M} \text{ and } \int f_*(\omega) d\nu \neq -\infty\}.
\end{aligned}$$

Corollary 2.1. *For $\psi \in H(\Sigma, \mathbb{R})$ with $\psi \geq c > 0$, $\mathcal{F} := \{\log f_n\}_{n=1}^{\infty}$ on Σ is a Bowen sequence, we always have*

$$\mathcal{P}_{\psi}(\mathcal{F}, \Sigma_a^{per}) \geq \inf\{\beta \in \mathbb{R} : \mathcal{P}_1(\mathcal{F}^{\beta}, \Sigma_a^{per}) \leq 0\}. \tag{2.5}$$

Proof. Let

$$B := \{\beta \in \mathbb{R} : \limsup_{T\to\infty} \sum_{\substack{\omega\in\Sigma_a^{per} \\ T<S_{|\omega|}\psi(\overline{\omega})}} f_{|\omega|}(\overline{\omega}) e^{-\beta S_{|\omega|}\psi(\overline{\omega})} < \infty\}.$$

For each $\beta \in B$, it is easy to see that

$$\limsup_{T\to\infty} \frac{1}{T} \log \sum_{\substack{\omega\in\Sigma_a^{per} \\ T<S_{|\omega|}\psi(\overline{\omega})}} f_{|\omega|}(\overline{\omega}) e^{-\beta S_{|\omega|}\psi(\overline{\omega})} \leq 0$$

and we conclude that

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{\substack{\omega\in\Sigma_a^{per} \\ |\omega|=n}} f_{|\omega|}(\overline{\omega}) e^{-\beta S_{|\omega|}\psi(\overline{\omega})} \leq 0.$$

Otherwise, we assume that

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{\substack{\omega\in\Sigma_a^{per} \\ |\omega|=n}} f_{|\omega|}(\overline{\omega}) e^{-\beta S_{|\omega|}\psi(\overline{\omega})} = 2a > 0.$$

There exists a sequence $\{n_j\}_{j\in\mathbb{N}}$ such that for each $j \in \mathbb{N}$

$$\sum_{\substack{\omega\in\Sigma_a^{per} \\ |\omega|=n_j}} f_{|\omega|}(\overline{\omega}) e^{-\beta S_{|\omega|}\psi(\overline{\omega})} > e^{an_j}.$$

For sufficiently large $T > 0$, let $\{n_{k_i}\} \subset \{n_j\}$ with $S_{n_{k_i}}\psi(\overline{\omega}) > T$. We have

$$\sum_{\substack{\omega\in\Sigma_a^{per} \\ T<S_{|\omega|}\psi(\overline{\omega})}} f_{|\omega|}(\overline{\omega}) e^{-\beta S_{|\omega|}\psi(\overline{\omega})} \geq \sum_{\substack{\omega\in\Sigma_a^{per} \\ |\omega|=n_{k_i}}} f_{|\omega|}(\overline{\omega}) e^{-\beta S_{|\omega|}\psi(\overline{\omega})} > e^{an_{k_i}}.$$

Since $i \to \infty$ when $T \to \infty$, we conclude that

$$\limsup_{T\to\infty} \sum_{\substack{\omega\in\Sigma_a^{per} \\ T<S_{|\omega|}\psi(\overline{\omega})}} f_{|\omega|}(\overline{\omega}) e^{-\beta S_{|\omega|}\psi(\overline{\omega})} = \infty.$$

This shows that (2.5) holds.

Corollary 2.2. *Let (Σ, σ) be a topologically mixing countable state Markov shift and $\mathcal{F} := \{\log f_n\}_{n=1}^{\infty}$ is an almost-additive Bowen sequence on Σ with $\sup f_1 < \infty$ and $\psi \in H(\Sigma, \mathbb{R}), \psi \geq c > 0$ with $\sup \psi < \infty$. We have the map $\beta \mapsto \mathcal{P}_1(\mathcal{F}^\beta, \Sigma_a^{per})$ is strictly decreasing on $\operatorname{int}\{\beta \in \mathbb{R} : \mathcal{P}_1(\mathcal{F}^\beta, \Sigma_a^{per}) < \infty\}$ and*

$$\mathcal{P}_\psi(\mathcal{F}, \Sigma_a^{per}) = \inf\{\beta \in \mathbb{R} : \mathcal{P}_1(\mathcal{F}^\beta, \Sigma_a^{per}) \leq 0\} = \sup\{\beta \in \mathbb{R} : \mathcal{P}_1(\mathcal{F}^\beta, \Sigma_a^{per}) \geq 0\}. \tag{2.6}$$

In particular , if (Σ, σ) is a subshift of finite type, then the map $\beta \mapsto \mathcal{P}_1(\mathcal{F}^\beta, \Sigma_a^{per})$ is a strictly decreasing continuous map on $\mathbb{R}$. Hence we conclude that $\mathcal{P}_\psi(\mathcal{F}, \Sigma_a^{per})$ is its unique zero , i.e.

$$\mathcal{P}_1(\mathcal{F}^{\mathcal{P}_\psi(\mathcal{F}, \Sigma_a^{per})}, \Sigma_a^{per}) = 0.$$

Proof. By the Lemma 2.1 we have

$$\mathcal{P}_1(\mathcal{F}^\beta, \Sigma_a^{per}) = \sup\{h_\nu(\sigma) + \int f_*(\omega) d\nu - \beta \int \psi(\omega) d\nu \ : \nu \in \mathcal{M} \text{and} \int f_*(\omega) d\nu \neq -\infty\}.$$

For any $\beta_1, \beta_2 \in \operatorname{int}\{\beta \in \mathbb{R} : \mathcal{P}_1(\mathcal{F}^\beta, \Sigma_a^{per}) < \infty\}, \beta_1 < \beta_2$ and $0 < \epsilon < \frac{c(\beta_2 - \beta_1)}{2}$, there exists $\mu \in \mathcal{M}$ such that

$$\begin{aligned}
&\sup\{h_\nu(\sigma) + \int f_*(\omega) d\nu - \beta_2 \int \psi(\omega) d\nu \ : \nu \in \mathcal{M} \text{and} \int f_*(\omega) d\nu \neq -\infty\} \\
<& h_\mu(\sigma) + \int f_*(\omega) d\mu - \beta_2 \int \psi(\omega) d\mu + \epsilon \\
=& h_\mu(\sigma) + \int f_*(\omega) d\mu - \beta_1 \int \psi(\omega) d\mu + \epsilon - (\beta_2 - \beta_1) \int \psi(\omega) d\mu \\
\leq& h_\mu(\sigma) + \int f_*(\omega) d\mu - \beta_1 \int \psi(\omega) d\mu - (\beta_2 - \beta_1)(\int \psi(\omega) d\mu - \frac{c}{2}) \\
\leq& \sup\{h_\nu(\sigma) + \int f_*(\omega) d\nu - \beta_1 \int \psi(\omega) d\nu \ : \nu \in \mathcal{M} \text{and} \int f_*(\omega) d\nu \neq -\infty\} \\
-& (\beta_2 - \beta_1)(\int \psi(\omega) d\mu - \frac{c}{2}).
\end{aligned}$$

We thus obtain the map $\beta \mapsto \mathcal{P}_1(\mathcal{F}^\beta, \Sigma_a^{per})$ is strictly decreasing. Since

$$\begin{aligned}
&\inf\{\beta \in \mathbb{R} : \mathcal{P}_1(\mathcal{F}^\beta, \Sigma_a^{per}) < 0\} \\
\geq& \inf\{\beta \in \mathbb{R} : \limsup_{T \to \infty} \sum_{\substack{\omega \in \Sigma_a^{per} \\ T < S_{|\omega|}\psi(\overline{\omega})}} f_{|\omega|}(\overline{\omega}) e^{-\beta S_{|\omega|}\psi(\overline{\omega})} < \infty\},
\end{aligned} \tag{2.7}$$

we have

$$\begin{aligned}
\inf\{\beta \in \mathbb{R} : \mathcal{P}_1(\mathcal{F}^\beta, \Sigma_a^{per}) \leq 0\} &= \inf\{\beta \in \mathbb{R} : \mathcal{P}_1(\mathcal{F}^\beta, \Sigma_a^{per}) < 0\} \\
&= \sup\{\beta \in \mathbb{R} : \mathcal{P}_1(\mathcal{F}^\beta, \Sigma_a^{per}) \geq 0\}.
\end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8), we obtain (2.6).

If (Σ, σ) is a subshift of finite type, obviously, for each $\beta \in \mathbb{R}$, $\mathcal{P}_1(\mathcal{F}^\beta, \Sigma_a^{per}) < \infty$, we easily obtain the conclusion.

We denote by $C_{\Sigma,\sigma} := \{K \subset \Sigma : K \text{ is compact and } \sigma^{-1}(K) = K\}$ the set of compact σ-invariant subsets on Σ. we say that the *exhaustion principle* holds for $\mathcal{P}_\psi(\mathcal{F}, \Sigma_a^{per})$, if there exists a sequence $\{K_n\}_{n\in\mathbb{N}} \subset C_{\Sigma,\sigma}$ such that

$$\lim_{n\to\infty} \mathcal{P}_{\psi,K_n}(\mathcal{F}|_{K_n}, \Sigma_a^{per} \cap K_n^*) = \mathcal{P}_\psi(\mathcal{F}, \Sigma_a^{per})$$

where

$$\mathcal{P}_{\psi,K}(\mathcal{F}|_K, \Sigma_a^{per} \cap K^*) := \limsup_{T\to\infty} \frac{1}{T} \log \sum_{\substack{\omega\in\Sigma_a^{per}\cap K^*, \\ T-\eta<S_{|\omega|}\psi|_K(\overline{\omega})\le T}} f_{|\omega|}(\overline{\omega}) 1_K(\overline{\omega})$$

and $K^* := \{\omega \in \Sigma^* : [\omega] \cap K \neq \emptyset\}$. Obviously, the conclusions of Theorem 2.1 and Corollary 2.1 are valid for $\mathcal{P}_{\psi,K_n}(\mathcal{F}|_{K_n}, \Sigma_a^{per} \cap K_n^*)$.

Corollary 2.3. *Let (Σ, σ) be a topologically mixing countable state Markov shift and $\mathcal{F} = \{\log f_n\}_{n=1}^\infty$ is an almost-additive Bowen sequence on Σ with $\sup f_1 < \infty$, $\psi \in H(\Sigma, \mathbb{R}), \psi \ge c > 0$ with $\sup \psi < \infty$. We have the exhaustion principle holds for $\mathcal{P}_\psi(\mathcal{F}, \Sigma_a^{per})$.*

Proof. Let $\delta > 0$, it follows from Corollary 2.2 that $\mathcal{P}_1(\mathcal{F}^{\mathcal{P}_\psi(\mathcal{F},\Sigma_a^{per})-\delta}, \Sigma_a^{per}) > 0$. By [2, Proposition 3.1], we have the exhaustion principle holds for $\mathcal{P}_1(\mathcal{F}^\beta, \Sigma_a^{per})$ with $\beta \in \mathbb{R}$. There exists a subset $K \in C_{\Sigma,\sigma}$ such that

$$\begin{aligned} &\mathcal{P}_{1,K}(\mathcal{F}^{(\mathcal{P}_\psi(\mathcal{F},\Sigma_a^{per})-\delta)}|_K, \Sigma_a^{per} \cap K^*) \\ := &\lim_{n\to\infty} \frac{1}{n} \log \sum_{\substack{\omega\in\Sigma_a^{per}\cap K^* \\ |\omega|=n}} f_n(\overline{\omega}) e^{-(\mathcal{P}_\psi(\mathcal{F},\Sigma_a^{per})-\delta) S_n\psi(\overline{\omega})} 1_K(\overline{\omega}) > 0. \end{aligned}$$

By Corollary 2.1 we have

$$\mathcal{P}_{\psi,K}(\mathcal{F}|_K, \Sigma_a^{per} \cap K^*) \ge \inf\{\beta \in \mathbb{R} : \mathcal{P}_{1,K}(\mathcal{F}^\beta|_K, \Sigma_a^{per} \cap K^*) \le 0\} > \mathcal{P}_\psi(\mathcal{F}, \Sigma_a^{per}) - \delta.$$

Hence, for $\mathcal{P}_\psi(\mathcal{F}, \Sigma_a^{per})$ the exhaustion principle holds.

Proposition 2.1. *Let (Σ, σ) be a topologically mixing countable state Markov shift, $\mathcal{F} = \{\log f_n\}_{n=1}^\infty$ is an almost-additive Bowen sequence on Σ, and $\psi \in H(\Sigma, \mathbb{R}), \psi \ge c > 0$. Then $\mathcal{P}_\psi(\mathcal{F}, \Sigma_a^{per})$ is independent of the choice of $a \in S$.*

Proof. It is sufficient to prove that

$$\mathcal{P}_\psi(\mathcal{F}, \Sigma_a^{per}) \le \mathcal{P}_\psi(\mathcal{F}, \Sigma_b^{per}). \tag{2.9}$$

Let $T>0$ be large. For each $a,b\in S$, and $\omega\in\Sigma_a^{per}$ with $T<S_{|\omega|}\psi(\overline{\omega})$. Since (Σ,σ) is topologically mixing, there exists

$$\omega^1:=(b,\omega_1,\ldots,\omega_{k-1})\ ,\omega^2:=(a,\omega_1^{'},\ldots,\omega_{k-1}^{'})\in\Sigma^k$$

such that $\omega^1\omega\omega^2\in\Sigma_b^{per}$. Let $x:=\ \overline{\omega^1\omega\omega^2}$. Making a similar calculation of [2], we find a constant $C_f>0$, such that $f_{|\omega|}(\overline{\omega})\leq C_f f_{|\omega|+2k}(x)$. Since ψ is Hölder continuous , the *bounded distortion property* (see [3]) shows there exists a constant $C_\psi>0$ such that

$$|S_{|\omega|}\psi(\overline{\omega})-S_{|\omega|}\psi(\sigma^kx)|\leq C_\psi,|\beta S_{|\omega|}\psi(\overline{\omega})-\beta S_{|\omega|}\psi(\sigma^kx)|\leq|\beta|C_\psi$$

for $\overline{\omega},\sigma^kx\in[\omega],\beta\in\mathbb{R}$.
Let $C_1=\inf\limits_{\gamma\in[\omega^1]}S_k\psi(\gamma),C_1=\inf\limits_{\gamma\in[\omega^2]}S_k\psi(\gamma)$, we have

$$T+C_1+C_2-C_\psi<S_{2k+|\omega|}\psi(x)$$

and

$$e^{-\beta S_{|\omega|}\psi(\overline{\omega})}<e^{|\beta|(C_1+C_2+C_\psi)}e^{-\beta S_{2k+|\omega|}\psi(x)}.$$

Thus there exists a constant $C^{'}>0$ such that

$$\sum_{\substack{\omega\in\Sigma_a^{per}\\ T<S_{|\omega|}\psi(\overline{\omega})}}f_{|\omega|}(\overline{\omega})e^{-\beta S_{|\omega|}\psi(\overline{\omega})}\leq C^{'}\sum_{\substack{\omega\in\Sigma_b^{per}\\ T+C_1+C_2-C_\psi<S_{|\omega|}\psi(\overline{\omega})}}f_{|\omega|}(\overline{\omega})e^{-\beta S_{|\omega|}\psi(\overline{\omega})}.$$

Then we obtain (2.9).

3 Proof of Theorem 1.1

In this section, we will prove the variational principle of the induced almost-additive Gurevich pressure for countable states Markov shifts. Our result is a generalization of the almost-additive Gurevich pressure.

Firstly, we show

$$\mathcal{P}_\psi(\mathcal{F},\Sigma_a^{per})\geq\sup\{\frac{h_v(\sigma)}{\int\psi d\nu}+\frac{\int f_*(\omega)d\nu}{\int\psi d\nu}:\nu\in\mathcal{M}\text{and}\int f_*(\omega)d\nu\neq-\infty\}.\qquad(3.10)$$

By Corollary 2.1, for $\beta>\mathcal{P}_\psi(\mathcal{F},\Sigma_a^{per})$, we have $\mathcal{P}_1(\mathcal{F}^\beta,\Sigma_a^{per})\leq 0$. By Lemma 2.1 we have

$$\begin{aligned}0\geq&\mathcal{P}_1(\mathcal{F}^\beta,\Sigma_a^{per})\\ \geq&\sup\{h_\nu(\sigma)+\int f_*(\omega)d\nu-\beta\int\psi d\nu\ :\nu\in\mathcal{M}\text{and}\int f_*(\omega)d\nu\neq-\infty\}\\ =&\sup\{\int\psi(\omega)d\nu(\frac{h_v(\sigma)}{\int\psi d\nu}+\frac{\int f_*(\omega)d\nu}{\int\psi d\nu}-\beta):\nu\in\mathcal{M}\text{and}\int f_*(\omega)d\nu\neq-\infty\},\end{aligned}$$

and we obtain (3.10).

Nextly, we prove

$$\mathcal{P}_{\psi}(\mathcal{F},\Sigma_a^{per}) \leq \sup\{\frac{h_{\upsilon}(\sigma)}{\int \psi d\nu}+\frac{\int f_*(\omega)d\nu}{\int \psi d\nu}:\nu\in\mathcal{M}\text{and}\int f_*(\omega)d\nu\neq-\infty\}. \qquad (3.11)$$

By Corollary 2.3, there exists a sequence $\{K_n\}_{n\in\mathbb{N}}\subset C_{\Sigma,\sigma}$ such that $\lim\limits_{n\to\infty}\mathcal{P}_{\psi,K_n}(\mathcal{F}|_{K_n},\Sigma_a^{per}\cap K_n^*)=\mathcal{P}_{\psi}(\mathcal{F},\Sigma_a^{per})$. For each $n\in\mathbb{N}$, K_n is the finite alphabet case. Combining Corollary 2.2 and [2,Theorem 3.1], we have

$$\begin{aligned}
0 =&\mathcal{P}_{1,K_n}(\mathcal{F}|_{K_n}^{\mathcal{P}_{\psi}|_{K_n}(\mathcal{F}|_{K_n},\Sigma_a^{per}\cap K_n^*)},\Sigma_a^{per}\cap K_n^*)\\
=&\sup\{h_{\nu}(\sigma)+\int f_*(\omega)d\nu-\mathcal{P}_{\psi}|_{K_n}(\mathcal{F}|_{K_n},\Sigma_a^{per}\cap K_n^*)\int\psi d\nu\ :\nu\in\mathcal{M}_{K_n}\text{and}\int f_*(\omega)d\nu\neq-\infty\}\\
\leq&\sup\{\int\psi(\omega)d\nu(\frac{h_{\upsilon}(\sigma)}{\int \psi d\nu}+\frac{\int f_*(\omega)d\nu}{\int \psi d\nu}):\nu\in\mathcal{M}\text{and}\int f_*(\omega)d\nu\neq-\infty\}\\
-&\mathcal{P}_{\psi}|_{K_n}(\mathcal{F}|_{K_n},\Sigma_a^{per}\cap K_n^*).
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{P}_{\psi}(\mathcal{F},\Sigma_a^{per}) =& \lim_{n\to\infty}\mathcal{P}_{\psi,K_n}(\mathcal{F}|_{K_n},\Sigma_a^{per}\cap K_n^*)\\
\leq&\sup\{\frac{h_{\upsilon}(\sigma)}{\int \psi d\nu}+\frac{\int f_*(\omega)d\nu}{\int \psi d\nu}:\nu\in\mathcal{M}\text{and}\int f_*(\omega)d\nu\neq-\infty\}.
\end{aligned}$$

Combining (3.10) and (3.11), we obtain (1.4).

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