ON IMMUNIZATION, 3-CONVEX ORDERS AND THE MAXIMUM SKEWNESS INCREASE

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Abstract.

The extension of some immunization results to the framework of generalized s-convex orders is considered. A detailed study of the special case s=3 is undertaken. A portfolio strategy, which achieves immunization against 3-convex shift factors, necessarily matches durations and convexities. For some more general shift factors, we derive bounds on the change in portfolio value, which depend on the skewness increase between the liability and asset risks. A linear control of these immunization bounds is examined. For a specific minimax strategy, these bounds can be reduced to a constant independently of the time horizon.

Keywords: asset and liability management, immunization, duration, convexity, s-convex orders, linear control

1. Immunization and the s-convex orders.

In Hürlimann (2002) it has been shown that the usual convex order (=stop-loss order by equal means) provides a natural framework for understanding and unifying the main immunization results by Fong and Vasicek (1983a/b), Shiu (1988/90), Montruchio and Peccati (1991), and Uberti (1997). Also, it has been demonstrated that the "Shiu measure", which is an appropriate measure of the immunization risk that has been introduced by Shiu (1986), can be controlled in a linear way.

We look at similar immunization results in the framework of the generalized s-convex orders considered by Denuit et al. (1998). In the particular case s=3, which corresponds to a duration and convexity matching immunization strategy, we show that the Shiu measure can be reduced to a constant, independently of the time horizon, provided a specific minimax strategy is applied (Theorem 4.1).

Recall the setting of immunization theory (e.g. Panjer et al. (1998), Section 3). Consider a *frictionless*, *competitive* and *discrete trading* economy with trading dates $\{0,1,2,...,T\}$, where T is the *time horizon*. The traded securities in the economy are zero-coupon bonds of all maturities $\{0,1,2,...,T\}$ and a money market account. The *price of a zero-coupon bond* at time t that pays one unit at time $s \ge t$ is denoted $P(t,s)$. Only the current time $t=0$ is of interest, in which case we write $P(s)$ instead of $P(0,s)$. As shown in Hürlimann (2002), Section 2, it suffices to consider a portfolio with non-negative asset inflows $\{A_1,...,A_m\}$, occurring at dates $\{1,...,m\}$, and non-negative liability outflows $\{L_1,...,L_n\}$, due at dates $\{1,...,n\}$ with time $t=0$ portfolio value

$$V = \sum_{k=1}^{m} \alpha_k - \sum_{j=1}^{n} \ell_j = 0\,, \tag{1.1}$$

where $\alpha_k = A_k P(k)$ and $\ell_j = L_j P(j)$ are the current arbitrage-free prices of the asset and liability flows. One is interested in the possible changes of the current arbitrage-free value of a portfolio at a time immediately following the current time $t = 0$, under a change of the *term structure of interest rates* (TSIR) from $P(s)$ to $P'(s)$ such that $f(s) = \frac{P'(s)}{P(s)}$ is the *shift factor*. Immediately following the initial time, the post-shift change in value is then given by

$$\Delta V = V' - V = \sum_{k=1}^{m} \alpha_k f(k) - \sum_{j=1}^{n} \lambda_j f(j)\,, \tag{1.2}$$

The *classical immunization problem* consists to find conditions under which (1.2) is non-negative, and give precise bounds on this change of value in case this change cannot be guaranteed to be non-negative. To establish the connection with the theory of ordering of risks, one uses elementary probability theory.

<u>Definitions 1.1</u>. The random variable A with support $\{1,...,m\}$ and probabilities $\{q_1,...,q_m\}$, where $q_k = \alpha_k \cdot \left(\sum_{i=1}^{m} \alpha_i\right)^{-1}$ is the normalized asset inflow at time k, is called *asset risk*. Similarly, the random variable L with support $\{1,...,n\}$ and probabilities $\{p_1,..., p_n\}$, where $p_j = \lambda_j \cdot \left(\sum_{i=1}^{n} \lambda_i\right)^{-1}$ is the normalized liability outflow at time j, is called *liability risk*.

With these definitions, the normalization assumption

$$\sum_{k=1}^{m} \alpha_k = \sum_{j=1}^{n} \lambda_j = 1, \tag{1.3}$$

which will be made throughout, does not lead to a loss of generality. It follows that the classical immunization measures of *durations*, *M-squared indices*, and *convexities* of assets and liabilities are just the means, variances and second order moments of the asset and liability risks, that is

$$\begin{aligned} &D_A = E[A], \quad D_L = E[L], \quad M_A^2 = Var[A], \quad M_L^2 = Var[L], \\ &C_A = M_A^2 + D_A^2, \quad C_L = M_L^2 + D_L^2. \end{aligned} \tag{1.4}$$

Similarly, the change in portfolio value (1.2) identifies with the mean difference between transformed asset and liability risks

$$\Delta V(f) = E[f(A)] - E[f(L)]. \tag{1.5}$$

Now, the theory of integral stochastic orders, studied among others by Whitt (1986), Marshall (1991) and Müller (1997), describes classes U^S of real functions $f : S \to R$ (S some domain of definition) such that (1.5) is non-negative for all $f \in U^S$ provided L is U^S-smaller than A. For example, the class U_{cx}^S of convex functions with $L \le_{cx} A$ the usual convex order corresponds to the setting discussed in Hürlimann (2002).

As natural generalizations, consider the classes U^S_{s-cx} of all *s-convex functions* and the classes U^S_{s-icx} of all *s-increasing convex functions* described in detail in Denuit et al. (1998). By definition $f: S \to R$ is *s-convex*, *s=1,2,3,...*, if, and only if, for all choices of *s+1* distinct points $x_0 < x_1 < ... < x_s$ in S the determinant

$$\Delta_s(x_0,...,x_s; f) = \begin{vmatrix} 1 & 1 & ... & 1 \\ x_0 & x_1 & ... & x_s \\ & & & \\ x_0^{s-1} & x_1^{s-1} & ... & x_s^{s-1} \\ f(x_0) & f(x_1) & ... & f(x_s) \end{vmatrix} \quad (1.6)$$

is non-negative. The 1-convex functions are the non-decreasing functions and the 2-convex functions are the usual convex functions. Similarly $f: S \to R$ is *s-increasing convex*, *s=1,2,3,...*, if, and only if, for all choices of *k+1* distinct points $x_0 < x_1 < ... < x_k$ in S one has $\Delta_k(x_0,...,x_k; f) \ge 0$, for all *k=1,2,...,s*. The 1-increasing convex functions are the non-decreasing functions and the 2-increasing convex functions are the usual increasing convex functions. These classes of generalized convex functions lead to the following stochastic order relations.

Definitions 1.2. Let X and Y be two random variables taking values in the continuum subset $S \subset R$. Then X is called smaller than Y in the *s-convex* (*s-increasing convex*) *order*, written $X \le^S_{s-cx} Y$ ($X \le^S_{s-icx} Y$), if $E[f(X)] \le E[f(Y)]$ for all $f \in U^S_{cx}$ ($f \in U^S_{icx}$). In the often encountered case $S = [0,\infty)$ the super-script S will be deleted by convention.

In applications, one needs characterizations of the *s*-convex (*s*-increasing convex) functions and orders (see Denuit et al. (1998) for details). For our purpose, the following characterizations, valid for the case $S = [0,\infty)$, will suffice:

$$X \le_{s-cx} Y \quad \Leftrightarrow \quad E[X^k] = E[Y^k], \quad k = 1,...,s-1, \text{ and} \quad (1.7)$$
$$\pi_X^{s-1}(d) = E[(X-d)_+^{s-1}] \le \pi_Y^{s-1}(d) = E[(Y-d)_+^{s-1}] \text{ for all } d \in S,$$

$$X \le_{s-icx} Y \quad \Leftrightarrow \quad E[X^k] \le E[Y^k], \quad k = 1,...,s-1, \text{ and} \quad (1.8)$$
$$\pi_X^{s-1}(d) \le \pi_Y^{s-1}(d) \text{ for all } d \in S.$$

Applied to immunization theory, the above definitions imply through reinterpretation the following straightforward results.

Theorem 1.1. Let A and L be random variables representing asset and liability risks as in Definitions 1.1. Then, a portfolio (A,L) is said to be immunized against *s*-convex (*s*-increasing convex) shift factors $f(t)$, that is $E[f(A)] \ge E[f(L)]$ for all $f \in U_{cx}$ ($f \in U_{icx}$), if and only if one has $L \le_{s-cx} A$ ($L \le_{s-icx} A$).

In the present paper, we specialize to the second important special case after *s=2*, namely *s=3*. In particular, by (1.7) and Theorem 1.1, a portfolio strategy achieving immunization against 3-convex shift factors necessarily matches durations and convexities, that is $D_A = D_L$ and $C_A = C_L$ (see e.g. Bühlmann and Berliner (1992), p.140-41).

It is also possible to derive bounds on the change in portfolio value for more general shift factors, which generalize the bounds by Uberti (1997) (Theorem 2.3 in Hürlimann (2002)). For example, assume that $L \le_{3-cx} A$, and suppose that there exist $\alpha, \beta > 0$ such that $f(t) - \frac{1}{3}\alpha \cdot t^3$ is 3-convex on $[1, m]$ and $\frac{1}{3}\beta \cdot t^3 - f(t)$ is 3-convex on $[1, n]$. Since $L \le_{3-cx} A$ one has the inequalities

$$E[f(L)] - \frac{1}{3}\alpha \cdot E[L^3] \le E[f(A)] - \frac{1}{3}\alpha \cdot E[A^3],$$

$$\frac{1}{3}\beta \cdot E[L^3] - E[f(L)] \le \frac{1}{3}\beta \cdot E[A^3] - E[f(A)],$$

hence

$$\frac{1}{3}\alpha \cdot \left(E[A^3] - E[L^3]\right) \le E[f(A)] - E[f(L)] \le \frac{1}{3}\beta \cdot \left(E[A^3] - E[L^3]\right). \tag{1.9}$$

These bounds depend on the *skewness increase* between the liability and asset risks because by equal means and variances one has the identity

$$\frac{1}{3}\left(E[A^3] - E[L^3]\right) = \frac{1}{3}(\gamma_A - \gamma_L) \cdot M^3, \tag{1.10}$$

where $M^2 = M_A^2 = M_L^2$ and γ_A, γ_L describe the skewness of the asset and liability risks. The rest of the paper is devoted to the *linear control* of the immunization bounds (1.9), which generalizes the linear control of the Shiu measure in Hürlimann (2002).

2. **<u>Maximum skewness increase under the 3-convex orders</u>**.

Let X and Y be random variables with the common finite arithmetic support $\{0, ..., n\}$ and probabilities $p_j = \Pr(X = j), q_j = \Pr(Y = j),\ j = 0, ..., n$. The means, variances and skewnesses are denoted by $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \gamma_X, \gamma_Y$. We are interested in the maximum difference $E[Y^3] - E[X^3]$ under the restriction that X and Y are 3-convex or 3-increasing convex ordered. This optimization problem transforms into a simple linear programming problem as will be shown below. The following relationship is needed.

<u>Lemma 2.1</u>. The difference in third moment of two finite arithmetic random variables X and Y with support $\{0, ..., n\}$ is given by

$$\frac{1}{3}(E[Y^3] - E[X^3]) = \frac{1}{2}(E[Y^2] - E[X^2]) + \frac{1}{6}(E[Y] - E[X]) + \sum_{j=0}^{n-2} c_j, \tag{2.1}$$

where $c_j = \pi_Y^2(j+1) - \pi_X^2(j+1),\quad j = 0, ..., n-2,$ is the finite sequence of difference in degree two stop-loss values evaluated at integer points.

Proof. Recall the recursive relationships between higher-degree stop-loss transforms (e.g. Hürlimann (2000)) :

$$\pi_X^n(x) = n \cdot \int_x^\infty \pi_X^{n-1}(t)dt, \quad n = 1,2,\ldots \ .$$

Through partial integration (e.g. Kaas et al. (1994), p.110) or analytically applying the method of generating functions, one obtains in particular

$$\frac{1}{3}E\left[X^3\right] = \int_0^\infty \pi_X^2(t)dt \ .$$

To express this as a function of the degree two stop-loss transform values $\pi_X^2(j+1), \quad j = 0,\ldots,n-2,$ consider the integral summation

$$\frac{1}{3}E\left[X^3\right] = \sum_{k=0}^{n-1} \int_k^{k+1} \pi_X^2(t)dt \ .$$

Since the usual (degree one) stop-loss transform is piecewise linear, more precisely one has

$$\pi_X(t) = \pi_X(k) + (t-k)\cdot \nabla \pi_X(k+1), \quad t \in [k,k+1],$$

the integrand can be rewritten for $x \in [k,k+1]$ as

$$\begin{aligned}
&\pi_X^2(t) = 2\cdot \int_t^\infty \pi_X(u)du = \pi_X^2(k+1) + 2\cdot \int_k^x \pi_X(u)du \\
&\pi_X^2(k+1) + 2(t-k)\pi_X(k) + (t-k)^2 \nabla \pi_X(k+1).
\end{aligned}$$

An elementary integration shows that

$$\int_k^{k+1} \pi_X^2(t)dt = \pi_X^2(k+1) + \frac{1}{3}\pi_X(k+1) + \frac{2}{3}\pi_X(k) \ .$$

It follows that

$$\begin{aligned}
&\frac{1}{3}E\left[X^3\right] = \sum_{k=0}^{n-2} \pi_X^2(k+1) + \sum_{k=0}^{n-2} \pi_X(k+1) + \frac{2}{3}\pi_X(0) \\
&= \sum_{k=0}^{n-2} \pi_X^2(k+1) + \frac{1}{2}E\left[X^2\right] + \frac{1}{6}E[X],
\end{aligned}$$

where use has been made of Lemma A2.1 in Hürlimann (2002). ◊

A further auxiliary result is required.

Lemma 2.2. The degree two stop-loss transform of a finite arithmetic random variable X with support $\{0,\ldots,n\}$ satisfies the recursive relationship

$$\begin{aligned}
&\pi_X^2(j) = \mu_X^2 + \sigma_X^2 - 2\mu_X \cdot j + j^2, \quad j = 0,-1,-2,\ldots, \\
&\nabla^3 \pi_X^2(j+1) = -\left(p_j + p_{j-1}\right), \quad j = 0,1,2,\ldots,
\end{aligned} \tag{2.2}$$

where $\nabla^3 x_{n+1} = x_{n+1} - 3x_n + 3x_{n-1} - x_{n-2}$ is the third order backward difference operator acting on sequences of real numbers, and $p_j = 0$ for $j \notin \{0,...,n\}$.

Proof. Recall that $\pi_X^2(0) = \mu_X^2 + \sigma_X^2$ (e.g. Kaas et al. (1994), Exercise III.1). Since $\pi_X(t) = \mu_X - t$ for $t \le 0$, one obtains the first formula in (2.2) through integration from

$$\pi_X^2(j) = 2 \cdot \int_j^0 \pi_X(t)dt + \mu_X^2 + \sigma_X^2, \quad j = -1,.2,... \ .$$

Similarly, using that $\pi_X(t)$ is linear in each interval $[j, j+1]$, $j = 0,1,2,...$, one obtains

$$\nabla \pi_X^2(j+1) = -2 \cdot \int_j^{j+1} \pi_X(t)dt = -[\pi_X(j) + \pi_X(j+1)],$$

where $\nabla x_{n+1} = x_{n+1} - x_n$. Applying the well-known relationships

$$\pi_X(j) = \mu_X - j, \quad j = 0,-1,-2,..., \quad \nabla^2 \pi_X(j+1) = p_j, \quad j = 0,1,2,...,$$

one obtains without difficulty the second formula in (2.2). ◊

Given $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2,$ it follows from Lemma 2.1 and 2.2 that the maximum difference in third order moment under the restriction $X \le_{3-icx} Y$ is given by the linear programming problem

$$\frac{1}{3}(E[Y^3] - E[X^3]) = \frac{1}{2}(\sigma_Y^2 - \sigma_X^2) + \frac{1}{6}(\mu_Y - \mu_X)[3(\mu_Y + \mu_X) + 1] + \sum_{j=0}^{n-2} c_j = \max. \quad (2.3)$$

under the linear constraints

$$\begin{aligned} &c_j \ge 0, \quad 0 \le q_j = -q_{j-1} + p_j + p_{j-1} - \nabla^3 c_j \le 1, \quad j = 0,...,n, \\ &\sum_{j=1}^{n+1} j \cdot \nabla^3 c_j = 2(\mu_X - \mu_Y), \\ &\sum_{j=1}^{n+1} j^2 \cdot \nabla^3 c_j = 2(\sigma_X^2 - \sigma_Y^2) + (\mu_X - \mu_Y)[2(\mu_X + \mu_Y) + 1], \end{aligned} \quad (2.4)$$

where use has been made of the following "dummy" variables

$$\begin{aligned} &c_{-3} = \pi_Y^2(-2) - \pi_X^2(-2) = (\mu_Y - \mu_X)(\mu_Y + \mu_X + 4) + \sigma_Y^2 - \sigma_X^2, \\ &c_{-2} = \pi_Y^2(-1) - \pi_X^2(-1) = (\mu_Y - \mu_X)(\mu_Y + \mu_X + 2) + \sigma_Y^2 - \sigma_X^2, \\ &c_{-1} = \pi_Y^2(0) - \pi_X^2(0) = (\mu_Y - \mu_X)(\mu_Y + \mu_X) + \sigma_Y^2 - \sigma_X^2, \\ &c_{n-1} = c_n = c_{n+1} = 0. \end{aligned} \quad (2.5)$$

As explained in Section 1, one is especially interested in the case of equal means and variances, for which the above linear program simplifies considerably and can be solved analytically. Under the assumption $\mu_X = \mu_Y = \mu, \sigma_X^2 = \sigma_Y^2 = \sigma^2$, and $p_0 = q_0 = 0$, that is

$c_0 = 0$ (which is no essential loss of generality), one shows that the maximum skewness increase when $X \le_{3-cx} Y$ is given by the linear program

(LP) $$\frac{1}{3}(\gamma_Y - \gamma_X)\sigma^2 = \sum_{j=0}^{n-2} c_j = \max.$$

under the linear constraints

(LC) $$\begin{aligned} &c_j \ge 0, \quad 0 \le q_j = -q_{j-1} + p_j + p_{j-1} - \nabla^3 c_j \le 1, \quad j = 1,...,n, \\ &\sum_{j=1}^{n-2} (-1)^{j-1} c_j = 0, \quad c_{-2} = c_{-1} = c_0 = c_{n-1} = c_n = 0, \end{aligned}$$

where the vanishing of the alternating sum follows from the equality of the mean and variance of X and Y. Since the special case $n=3$ has the trivial solution $c_1 = 0$, hence $Y =_d X$ (equality in distribution), one assumes $n \ge 4$ from now on. To solve analytically this linear program, one requires the following auxiliary result.

Lemma 2.3. The linear constraints (LC) imply the formulas

$$\begin{aligned} &q_n = p_n - \sum_{i=1}^{n-3} (-1)^i c_{n-2-i}, \\ &q_{n-1} = p_{n-1} - 3c_{n-3} - 4 \cdot \sum_{i=1}^{n-4} (-1)^i c_{n-3-i}, \\ &q_{n-2} = p_{n-2} + 3c_{n-3} - 6c_{n-4} - 7 \cdot \sum_{i=1}^{n-5} (-1)^i c_{n-4-i}, \\ &q_{n-j} = p_{n-j} - c_{n-j} + 4c_{n-j-1} - 7c_{n-j-2} - 8 \cdot \sum_{i=1}^{n-j-3} (-1)^i c_{n-j-i+1}, \\ &j = 3,4,...,n-1. \end{aligned} \tag{2.6}$$

Proof. Use induction on n. ◊

To describe our main result, we set $R(c) = \sum_{j=1}^{n-2} c_j$, where $\mathbf{c} = (c_1,...,c_{n-2})$.

Theorem 2.1. The maximum skewness increase for finite arithmetic random variables with support $\{1,...,n\}$, $n \ge 5$, under the restriction $X \le_{3-cx} Y$ is given and attained at 3-atomic random variables Y_n^* as follows :

Case 1: $q_1^* = q_2^* = ... = q_{n-3}^* = 0$

The maximum equals $R(c^*) = \frac{1}{3} \cdot \sum_{k=1}^{n-3} (n-k)(n-k-1)(n-k-2) p_k$, where one has $c_j^* = \sum_{k=1}^{j} (j-k+1)^2 p_k$, $j = 1,...,n-3$, $c_{n-2}^* = \frac{1}{2} \cdot \sum_{k=1}^{n-3} (n-k-1)(n-k-2) p_k$, and is attained at Y_n^* under the following precise conditions on X :

$$q_{n-2}^{*}=\frac{1}{2}\cdot\sum_{i=3}^{n}(i-1)(i-2)\,p_{n-i+1},$$
$$q_{n-1}^{*}=p_{n-1}-\sum_{i=4}^{n}(i-1)(i-3)\,p_{n-i+1}\geq 0,$$
$$q_{n}^{*}=\frac{1}{2}\cdot\sum_{i=4}^{n}(i-2)(i-3)\,p_{n-i+1}.$$

Case 2: $\quad q_1^*=q_2^*=...=q_{n-4}^*=0,\quad q_{n-1}^*=0$

The maximum equals $R(c^*)=\frac{2}{3}p_{n-1}+\frac{1}{3}\cdot\sum_{k=1}^{n-4}(n-k)(n-k-2)(n-k-3)\,p_k$, where one has $c_j^*=\sum_{k=1}^{j}(j-k+1)^2 p_k,\quad j=1,...,n-4,\qquad 3c_{n-3}^*=p_{n-1}+2\cdot\sum_{k=1}^{n-4}(n-k-2)(n-k-3)\,p_k$, $3c_{n-2}^*=p_{n-1}+\frac{1}{2}\cdot\sum_{k=1}^{n-4}(n-k-2)(n-k-3)\,p_k$ and is attained at Y_n^* under the following precise conditions on X:

$$q_{n-3}^{*}=\frac{1}{3}\cdot\left(\sum_{i=4}^{n}(i-1)(i-3)\,p_{n-i+1}-p_{n-1}\right)\geq 0,$$
$$q_{n-2}^{*}=p_{n-1}+p_{n-2}-\frac{1}{2}\cdot\sum_{i=5}^{n}(i-1)(i-4)\,p_{n-i+1}\geq 0,$$
$$q_{n}^{*}=p_n+\frac{1}{3}p_{n-1}+\frac{1}{6}\cdot\sum_{i=5}^{n}(i-3)(i-4)\,p_{n-i+1}.$$

Case 3: $\quad q_1^*=q_2^*=...=q_{n-5}^*=0,\quad q_{n-2}^*=q_{n-1}^*=0$

The maximum equals $R(c^*)=\frac{1}{3}\cdot\left(6p_{n-1}+4p_{n-2}+\sum_{k=1}^{n-5}(n-k)(n-k-3)(n-k-4)\,p_k\right)$, with $c_j^*=\sum_{k=1}^{j}(j-k+1)^2 p_k,\quad j=1,...,n-5$,
$2c_{n-4}^*=p_{n-1}+p_{n-2}+\frac{3}{2}\cdot\sum_{k=1}^{n-5}(n-k-3)(n-k-4)\,p_k$,
$3c_{n-3}^*=3p_{n-1}+2p_{n-2}+\sum_{k=1}^{n-5}(n-k-3)(n-k-4)\,p_k$,
$6c_{n-2}^*=3p_{n-1}+p_{n-2}+\frac{1}{2}\cdot\sum_{k=1}^{n-5}(n-k-3)(n-k-4)\,p_k$, and is attained at Y_n^* under the following precise conditions on X:

$$q_{n-4}^{*}=\frac{1}{2}\cdot\left(\frac{1}{2}\cdot\sum_{i=5}^{n}(i-1)(i-4)\,p_{n-i+1}-p_{n-1}-p_{n-2}\right)\geq 0,$$
$$q_{n-3}^{*}=\frac{1}{3}\cdot\left(3p_{n-1}+4p_{n-2}+3p_{n-3}-\sum_{i=6}^{n}(i-1)(i-5)\,p_{n-i+1}\right)\geq 0,$$
$$q_{n}^{*}=\frac{1}{6}\cdot\left(6p_n+3p_{n-1}+p_{n-2}+\frac{1}{2}\cdot\sum_{i=6}^{n}(i-4)(i-5)\,p_{n-i+1}\right).$$

Proof. Using the constraint $\sum_{j=1}^{n-2}(-1)^{j-1}c_j=0$, one gets the formula

$$R(c)=\begin{cases} 2\cdot\sum_{j=1}^{\frac{n-2}{2}} c_{2j-1}, & n \quad even, \\ 2\cdot\sum_{j=1}^{\frac{n-3}{2}} c_{2j}, & n \quad odd. \end{cases} \tag{2.7}$$

Through backward analysis, this can be rewritten in three different ways (proof through induction).

<u>Case 1</u>:

Insert successively the expressions for c_{n-j} obtained from the equations for q_{n-j} in (2.6) into $R(c)$ for $j=3,4,...,n-1$ to get $R(c)=\frac{1}{3}\cdot\sum_{k=1}^{n-3}(n-k)(n-k-1)(n-k-2)(p_k-q_k)$. Given X, that is the p_k's, this is maximal exactly when $q_1=q_2=...=q_{n-3}=0$, which yields the probability conditions stated under Case 1 as well as the maximum $R(c^*)$. The expressions for $c_j^*,\ \ j=1,...,n-3,$ are obtained from the equations (2.6) setting successively $q_j=0,\ \ j=1,...,n-3$. The remaining c_{n-2}^* is obtained from the vanishing alternating sum in (LC). Inserting the c_j^*'s into the remaining equations of (2.6), the maximizing probabilities q_{n-2}^*, q_{n-1}^* and q_n^* are obtained.

<u>Case 2</u>:

Insert successively c_{n-1}, c_{n-j}, $j=4,5,...,n-1$, from the corrresponding equations for q_{n-1}, q_{n-j} into $R(c)$ to get $R(c)=\frac{2}{3}(p_{n-1}-q_{n-1})+\frac{1}{3}\cdot\sum_{k=1}^{n-4}(n-k)(n-k-2)(n-k-3)(p_k-q_k)$, which is maximal when $q_1=q_2=...=q_{n-4}=0, q_{n-1}=0$. Setting successively $q_j=0,\ j=1,...,n-4$, $q_{n-1}=0$ in (2.6), one determines $c_j^*,\ \ j=1,...,n-3$. The remaining c_{n-2}^* as well as q_{n-2}^*, q_{n-1}^* and q_n^* follow as in Case 1.

<u>Case 3</u>:

First, insert c_{n-1} from the equation for q_{n-1} into $R(c)$. Second, add the equations for q_{n-1}, q_{n-2} to get an expression for c_{n-4}, which is inserted into $R(c)$. Third, insert successively c_{n-j}, $j=5,...,n-1$, from the corrresponding equations for q_{n-j} into $R(c)$ to get $R(c)=\frac{1}{3}\cdot(6(p_{n-1}-q_{n-1})+4(p_{n-2}-q_{n-2})+\sum_{k=1}^{n-5}(n-k)(n-k-3)(n-k-4)(p_k-q_k))$, which is maximal when $q_1=q_2=...=q_{n-5}=0, q_{n-2}=q_{n-1}=0$. The rest is shown similarly to Case 2 and Case 3.

It remains to show that the above three cases exhaust all possible random variables X. Let C be the space of all probability vectors $\boldsymbol{p}=(p_1,...,p_n)$ such that $p_j\ge 0,\ \ j=1,...,n$, $\sum p_j=1$, $\sum jp_j=\mu$, and $\sum j^2 p_j=\mu^2+\sigma^2$, which describes the set of all possible X's. Corresponding to the constraints imposed on the maximizing probability vector q^*, define the subsets

Case 1: $C_1 = \left\{ p \in C : p_{n-1} \geq \sum_{i=4}^{n} (i-1)(i-3) p_{n-i+1} \right\}$

Case 2: $C_2 = \left\{ p \in C : p_{n-1} \leq \sum_{i=4}^{n} (i-1)(i-3) p_{n-i+1},\ p_{n-1} + p_{n-2} \geq \frac{1}{2} \cdot \sum_{i=5}^{n} (i-1)(i-4) p_{n-i+1} \right\}$

Case 3: $C_3 = \left\{ \begin{array}{l} p \in C : p_{n-1} + p_{n-2} \leq \sum_{i=5}^{n} (i-1)(i-4) p_{n-i+1}, \\ 3p_{n-1} + 4p_{n-2} + 3p_{n-3} \geq \sum_{i=6}^{n} (i-1)(i-5) p_{n-i+1} \end{array} \right\}$

But one has $C_1 \cup C_2 \cup C_3 = C$. The complete solution has been found. ◊

Examples 2.1.

For illustration and to better grasp the regular pattern of the solution, let us rewrite the lower dimensional cases $n = 4,5,6$ explicitly. The special case $n = 4$ is obtained immediately applying the method of proof in Theorem 2.1.

n=4:

Case 1: $p_3 \geq 3p_1$

$$R(c^*) = 2p_1, \quad c^* = (p_1, p_1), \quad q^* = (0, p_2 + 3p_1, p_3 - 3p_1, p_1 + p_4)$$

Case 2: $p_3 \leq 3p_1$

$$R(c^*) = \frac{2}{3} p_3, \quad c^* = (\frac{1}{3} p_3, \frac{1}{3} p_3), \quad q^* = (\frac{1}{3}(3p_1 - p_3), p_2 + p_3, 0, \frac{1}{3}(p_3 + 3p_4))$$

n=5:

Case 1: $p_4 \geq 3p_2 + 8p_1$

$$R(c^*) = 2(p_2 + 4p_1), \quad c^* = (p_1, p_2 + 4p_1, p_2 + 3p_1),$$
$$q^* = (0,0, p_3 + 3p_2 + 6p_1, p_4 - 3p_2 - 8p_1, 3p_1 + p_2 + p_5)$$

Case 2: $p_4 \leq 3p_2 + 8p_1, \quad p_3 + p_4 \geq 2p_1$

$$R(c^*) = \frac{2}{3}(p_4 + 4p_1), \quad c^* = (p_1, \frac{1}{3}(p_4 + 4p_1), \frac{1}{3}(p_4 + p_1)),$$
$$q^* = (0, \frac{1}{3}(3p_2 + 8p_1 - p_4), p_4 + p_3 - 2p_1, 0, \frac{1}{3}(3p_5 + p_4 + p_1))$$

Case 3: $p_3 + p_4 \leq 2p_1$

$$R(c^*) = \frac{2}{3}(3p_4 + 2p_3), \quad c^* = (\frac{1}{2}(p_4 + p_3), \frac{1}{3}(3p_4 + 2p_3), \frac{1}{6}(3p_4 + p_3)),$$
$$q^* = (\frac{1}{2}(2p_1 - p_3 - p_4), \frac{1}{3}(3p_4 + 4p_3 + 3p_2), 0, 0, \frac{1}{6}(6p_5 + 3p_4 + p_3))$$

n=6:

Case 1: $p_5 \geq 3p_3 + 8p_2 + 15p_1$

$$R(c^*) = 2(p_3 + 4p_2 + 10p_1), \quad c^* = (p_1, p_2 + 4p_1, p_3 + 4p_2 + 9p_1, p_3 + 3p_2 + 6p_1),$$
$$q^* = (0,0,0, p_4 + 3p_3 + 6p_2 + 10p_1, p_5 - 3p_3 - 8p_2 - 15p_1, p_6 + p_3 + 3p_2 + 6p_1)$$

Case 2: $p_5 \leq 3p_3 + 8p_2 + 15p_1, \quad p_4 + p_5 \geq 2p_2 + 5p_1$

$$R(c^*) = \frac{2}{3}(p_5 + 4p_2 + 15p_1), \quad c^* = (p_1, p_2 + 4p_1, \frac{1}{3}(p_5 + 4p_2 + 12p_1), \frac{1}{3}(p_5 + p_2 + 3p_1)),$$
$$q^* = (0,0, \frac{1}{3}(3p_3 + 8p_2 + 15p_1 - p_5), p_4 + p_5 - 2p_2 - 5p_1, 0, \frac{1}{3}(3p_6 + p_5 + p_2 + 3p_1))$$

Case 3: $p_4 + p_5 \leq 2p_2 + 5p_1$

$$R(c^*) = \frac{2}{3}(3p_5 + 2p_4 + 5p_1), \quad c^* = (p_1, \frac{1}{2}(p_5 + p_4 + 3p_1), \frac{1}{3}(3p_5 + 2p_4 + 2p_1), \frac{1}{6}(3p_5 + p_4 + p_1)),$$
$$q^* = (0, \frac{1}{2}(2p_2 + 5p_1 - p_4 - p_5), \frac{1}{3}(3p_5 + 4p_4 + 3p_3 - 5p_1), 0, 0, \frac{1}{6}(6p_6 + 3p_5 + p_4 + p_1))$$

3. The absolute maximum skewness increase.

The maximum skewness increase $R_{n,\max}(p) := R(c^*) = R(c^*(p))$ (note that $c^* = c^*(p)$ depends on $\boldsymbol{p}$) has been determined in Theorem 2.1. First, we ask for the *absolute maximum* skewness increase when $X \leq_{3-cx} Y$ and both X and Y may vary with the same support $\{1,...,n\}$, that is we determine the quantity

$$R_{n,\max} := \max_p \{R_{n,\max}(p)\}. \tag{3.1}$$

Theorem 3.1. The absolute maximum skewness increase for finite arithmetic random variables with support $\{1,...,n\}$, $n \geq 4$, under the restriction $X \leq_{3-cx} Y$, but X and Y are arbitrary, is given and attained at biatomic random variables X_n^*, Y_n^* as follows :

Case 1: n=4

$$R_{n,\max} = R(c^*(p^*)) = \frac{1}{2}, \quad c^*(p^*) = (\frac{1}{4}, \frac{1}{4}), \quad p^* = (\frac{1}{4}, 0, \frac{3}{4}, 0), \quad q^* = (0, \frac{3}{4}, 0, \frac{1}{4})$$

Case 2: $n \geq 5$

$$R_{n,\max} = R(c^*(p^*)) = \frac{2}{3} \cdot \frac{(n-1)(n-4)}{(n-3)}$$
$$p_1^* = \frac{2}{(n-2)(n-3)}, \quad p_{n-1}^* = \frac{(n-1)(n-4)}{(n-2)(n-3)}, \quad p_j^* = 0, \quad j \neq 1, n-1,$$
$$c_j^*(p^*) = \frac{2}{(n-2)(n-3)}, \quad j = 1, \ldots, n-5,$$
$$c_{n-4}^*(p^*) = \frac{2(n-4)^2}{(n-2)(n-3)}, \quad c_{n-3}^*(p^*) = \frac{(n-4)(5n-13)}{3(n-2)(n-3)}, \quad c_{n-2}^*(p^*) = \frac{2(n-4)}{3(n-3)},$$
$$q_{n-3}^* = \frac{(n-1)}{3(n-3)}, \quad q_n^* = \frac{2(n-4)}{3(n-3)}, \quad q_j^* = 0, \quad j \neq n-3, n.$$

Proof. The special case n=4 is derived from the Examples 2.1. In Case 1 one has $R(c^*) \leq \frac{2}{3} p_3$ and in Case 2 one has $R(c^*) \leq 2 p_1$. In both cases the upper bound is attained when $p_3 = 3 p_1$. Setting $p_2 = p_4 = 0$ the absolute maximum is obtained. Let now $n \geq 5$ and consider Case 3. From the constraint $q_{n-4}^* \geq 0$ one gets

$$R_{n,\max}(p) \leq \frac{1}{3} \cdot (2 p_{n-1} + 2 \cdot \sum\nolimits_{k=1}^{n-4} (n-k)(n-k-3) p_k + \sum\nolimits_{k=1}^{n-5} (n-k)(n-k-3)(n-k-4) p_k)$$
$$= \frac{1}{3} \cdot (2 p_{n-1} + 8 \cdot p_{n-4} + \sum\nolimits_{k=1}^{n-5} (n-k)(n-k-2)(n-k-3) p_k).$$

This upper bound is attained when $q_{n-4}^* = 0$ and is maximum in case one has $p_2 = p_3 = \ldots = p_{n-2} = 0,\ p_n = 0$, hence $2 p_{n-1} = (n-1)(n-4) p_1$, which yields the maximizing probability vector p^*. The corresponding upper bound equals $R_{n,\max}(p^*) = \frac{2}{3} \cdot \frac{(n-1)(n-4)}{(n-3)}$. It is straightforward to see that the same maximizing upper bound holds in Case 2. In Case 1 one proceeds similarly to Case 3. An upper bound is obtained from the constraint $q_{n-1}^* \geq 0$ and attained when $q_{n-1}^* = 0$ and $p_2 = p_3 = \ldots = p_{n-2} = 0,\ p_n = 0$, hence $2 p_{n-1} = (n-1)(n-3) p_1$, which yields the maximizing probability vector p^*. The corresponding upper bound is $R_{n,\max}(p^*) = \frac{(n-1)(n-3)}{3(n-2)}$. Since this is strictly less than the upper bound in Case 2 and 3, the absolute maximum is attained in Case 3. The remaining quantities are obtained through calculation using Theorem 2.1. ◊

4. <u>The minimax skewness increase</u>.

From the Examples 2.1 and Theorem 2.1, it is not difficult to see when $R_{n,\max}(p) = 0$, which implies in particular that $X =_d Y$ (the distribution functions of two 3-convex ordered random variables are identical if both have equal mean, variance and skewness). It is interesting and useful to consider the minimum possible values of the maximum skewness increase when $X \le_{3-cx} Y$ and X and Y vary, given the maximum skewness increase is strictly positive. As will be seen in the proof below, there are three continuous sets of possible minimum values, which correspond to the three cases distinguished in Theorem 2.1 (note that for n=4 there are only two such sets). The minimum of the infimum superior of these sets is called *minimax skewness increase* and is defined by

$$R_{n,\min} = \min_{i=1,2,3}\left[\overline{\inf_{p\in C_i}}\{R_{n,\max}(p) > 0\}\right], \quad n \ge 5, \tag{4.1}$$

where C_i is defined in the proof of Theorem 2.1. A similar definition applies in the special case n=4. It is remarkable that this quantity is a constant, which does not depend on n.

<u>Theorem 4.1</u>. The minimax skewness increase for finite arithmetic random variables with support $\{1,...,n\}$, $n \ge 4$, under the restriction $X \le_{3-cx} Y$, but X and Y are arbitrary, equals $R_{n,\min} = \frac{1}{2}$ and is attained at biatomic random variables X_n^*, Y_n^* with probabilities $p_{n-3}^* = \frac{1}{4}$, $p_{n-1}^* = \frac{3}{4}$, $p_j^* = 0$ else, $q_{n-2}^* = \frac{3}{4}$, $q_n^* = \frac{1}{4}$, $q_j^* = 0$ else.

<u>Proof</u>. First, consider the special case $n = 4$. From the Examples 2.1, we distinguish between two cases:

<u>Case 1</u>: $p_2 + p_3 + p_4 = 1-\varepsilon, \quad 0 < p_1 = \varepsilon \le 1$

The condition $p_3 \ge 3p_1$ implies that $\varepsilon \le \frac{1}{4}$, and thus $R_{n,\max}(p) = 2\varepsilon \le \frac{1}{2}$, where equality is attained for $p_1^* = \frac{1}{4}, \quad p_3^* = \frac{3}{4}$.

<u>Case 2</u>: $p_1 + p_2 + p_4 = 1-\varepsilon, \quad 0 < p_3 = \varepsilon \le 1$

The condition $p_3 \le 3p_1$ implies that $\varepsilon \le \frac{3}{4}$, and thus $R_{n,\max}(p) = \frac{2}{3}\varepsilon \le \frac{1}{2}$, where equality is attained for $p_1^* = \frac{1}{4}, \quad p_3^* = \frac{3}{4}$.

Let now $n \ge 5$. One proceeds similarly according to the cases distinguished in Theorem 2.1:

<u>Case 1</u>: $p_{n-2} + p_{n-1} + p_n = 1-\varepsilon$

It is straightforward to see that $\min_{p\in C_1}\{R_{n,\max}(p)\} = 2\varepsilon$ for $p_{n-3} = \varepsilon$. The restriction on the probabilities $p_{n-1} = 1-\varepsilon - p_{n-2} - p_n \ge 3p_{n-3} = 3\varepsilon$ implies that $\varepsilon \le \frac{1}{4}$. The result follows.

<u>Case 2</u>: $p_{n-3} + p_{n-2} + p_n = 1-\varepsilon$

One has $\min_{p\in C_2}\{R_{n,\max}(p)\}=\frac{2}{3}\varepsilon$ for $p_{n-1}=\varepsilon$. The first restriction on the probabilities $p_{n-1}=\varepsilon\le 3p_{n-3}=3(1-\varepsilon-p_{n-2}-p_n)$ implies that $\varepsilon\le\frac{3}{4}$, and the second restriction is always fulfilled. The result follows.

<u>Case 3</u>: $p_{n-4}+p_{n-3}+p_n=1-\varepsilon$

One has $\min_{p\in C_3}\{R_{n,\max}(p)\}=\frac{4}{3}\varepsilon$ for $p_{n-2}=\varepsilon$. The restriction on the probabilities $p_{n-2}=\varepsilon\le 2p_{n-4}=2(1-\varepsilon-p_{n-3}-p_n)$ implies that $\varepsilon\le\frac{2}{3}$. Since $\overline{\inf_{p\in C_3}}\{R_{n,\max}(p)>0\}=\frac{8}{9}>\frac{1}{2}$, this concludes the proof of Theorem 4.1. ◊

<u>Remarks 4.1</u>.

The long term growth of the absolute maximum skewness increase when $X\le_{3-cx}Y$ is linear in n with the following sample values :

n	4	5	6	10	20	30
$R_{n,\max}$	$\frac{1}{2}=0.5$	$\frac{4}{3}=1.33$	$\frac{20}{9}=2.22$	$\frac{36}{7}=5.14$	$\frac{608}{51}=11.92$	$\frac{1508}{81}=18.62$

This is to be compared with the absolute maximum variance increase when $X\le_{2-cx}Y$, which is also linear in n, but with a higher slope (see Hürlimann (2002), Example 5.1). In contrast to the constant minimax skewness increase when $X\le_{3-cx}Y$, the minimax variance increase when $X\le_{2-cx}Y$ is also linear in n. These observations are of significance applications to financial immunization for fixed income securities. Two further related papers on this topic are Hürlimann (2003/2013).

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