Two dimensional hyperbolic geometry

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Abstract

The geodesics in 2-dimensional hyperbolic geometry should be the same as non-Euclidean geometry since hyperbolic geometry is a non-Euclidean geometry which discards one of the Euclid's axioms, that is, the fifth axiom which is the parallel postulate. Also, our determination is how to replace the fifth axiom of Euclid to obtain hyperbolic geometry. We go ahead and find an axiom called Playfairs axiom which can replace the fifth axiom of Euclid by negating the Playfairs axiom. We talk about Euclidean and hyperbolic geometry structures and how they differ in terms of their distances. We learn that hyperbolic geometry contains only two asymptotic lines called X and Y. We describe the 2-dimensional hyperbolic geometry in a Riemann geometry and also describe its isometries. We discuss its discontinuous group also. We are also going to look at how Riemannian geometry and hyperbolic geometry violate the fifth postulate of the Euclidean geometry.

Keywords: *Riemann geometry, Euclidean geometry, Hyperbolic geometry, Metric space, Isometry, Groups, Manifold, Differential manifold, Riemannian manifold.*

I. Introduction

Riemann Geometry which is also called elliptic geometry or spherical geometry is the study of curved surfaces.
The study of Riemannian Geometry has a direct connection to our daily existence since we live on a curved surface called **Planet Earth**.
What effects does working on a sphere, or a curved space, have on what we think of as geometrical truths?
• In curved space, the sum of the angles of any triangle is always greater than 180^0.
• On a sphere, there are no straight lines. As soon as you start to draw a straight line, it curves on the sphere.
• In curved space, the shortest distance between any two points called **geodesic** is not **unique**.
For example, there are many geodesics between the north and the south poles of the Earth (lines of longitude) that are not parallel since they intersect at the poles.

Riemannian Geometers also study higher dimensional spaces. The universe can be described as a three dimensional space. Near the earth, the universe looks roughly like three dimensional Euclidean space. However, near very heavy stars and black holes, the space is curved and bent. There are pairs of points in the universe which have more than one **minimal geodesic** between them.
The Hubble Telescope has discovered points which have more than one minimal geodesic between them and the point where the telescope is located.
This is called **gravitational lensing**. The amount that space is curved can be estimated by using theorems from Riemannian Geometry

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and measurements taken by astronomers.

Physicists believe that the curvature of space is related to the gravitational field of a star according to a partial differential equation called Einstein's equation. So using the results from the theorems in Riemannian Geometry they can estimate the mass of the star or black hole which causes the gravitational lensing.

One kind of theorem Riemannian Geometers are looking for today is a relationship between the curvature of a space and its shape.

Examples
There are many different shapes that surfaces can take.
They can be:

1. Cylinder.

2. Sphere.

3. Paraboloids or tori, to name a few.

A torus is the surface of a bagel(bread) and it has a hole in it.
You could also stick together two bagels and get a surface with two holes. How many holes can you get?
Certainly,as many as you want. If you string together infinitely many bagels then you will get a surface with infinitely many holes in it.
Now suppose you make a rule about how the surface is allowed to bend. If a surface must always bend in a rounded way (like a sphere) at every point, then we say it has positive curvature.
A paraboloid has positive curvature and so does a sphere. A cylinder doesn't and neither does a torus.
Geodesic which is a shortest line between two points on a curved or flat surfaces can also be identify and noted.
Our research on this paper suggested that:

- Two dimensional hyperbolic geometry which is the Non-Euclidean geometry violate the **fifth postulate(parallel postulate)** of the Euclidean geometry.
- To obtain hyperbolic geometry, the **fifth postulate(parallel postulate)** must be replaced by its negation, that is, the **Playfair's** axiom.

Our research on this paper is to establish the following basic objectives:

- The description of two dimensional hyperbolic geometry in a Riemann geometry and its isometries.

II. Euclidean geometry

Euclidean geometry is an axiomatic system in which all theorems ("the statements") are derived from a small number of axioms. Euclid gives five postulates or axioms for plane geometry, stated in terms of constructions.(as translated by Thomas Heath). It is also the study of flat spaces or surfaces.

Axioms The five axioms mentioned above are:

1. Draw a straight line from any point to any point.

2. Produce a finite straight line continuously in a straight line.

3. Describe a circle with any center and distance [radius].

4. That all right angles are equal to one another.

5. The parallel postulate.

"It states that, if a line segments intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines if extend indefinitely meet on that side on which the angles sum to less than two right angles." [1]

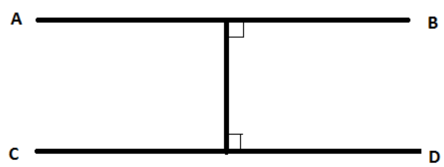

Figure 1: *Euclidean Geometry*

Base on the above axioms given, one of the axioms was violated which made us obtain another form of geometry called hyperbolic geometry.

III. Hyperbolic Geometry

Hyperbolic geometry is a non-Euclidean geometry, that is, a geometry that discards one of Euclids axioms, that is, **parallel postulate** of Euclidean geometry. You can see that, the main difference between Euclidean and the hyperbolic geometry is the nature of their parallel lines.

In Euclidean geometry, parallel lines are defined as lines that remains at a constant distance from each other even if extend to infinity. But the hyperbolic geometry violates the parallel axiom and replace with their own parallel axiom. So hyperbolic geometry has a special kind of parallel lines called **Ultraparallel** which curves away from each other gradually increasing in distance as they move further away from the points of intersection with common perpendicular.[2]

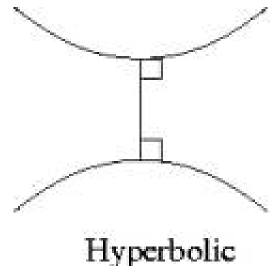

Figure 2: *Hyperbolic*

Theorem 1. *The ultra-parallel theorem states that, every pair of ultraparallel lines in the hyperbolic plane has a* ***unique*** *common perpendicular hyperbolic line.*

The fifth postulate can be replaced by the **Playfair's** axiom which state that, *in a plane, given a line and a point not on the line, at most one line parallel to the given line can be drawn through the point*.

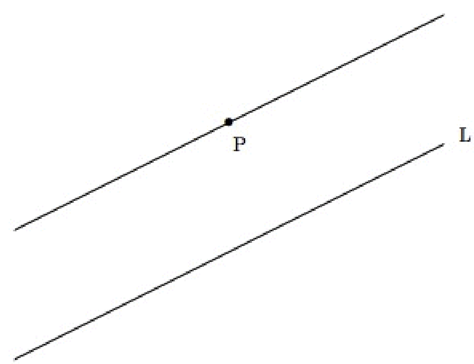

Figure 3: *Playfair's diagram*

In this studies ,To obtain the hyperbolic geometry, the parallel postulate (or its equivalent) must be replaced by its negation. That is negating, the **Playfair's** axiom form, since it is a compound statement. This can be done in **two** ways.
• Either there will exist more than one line through the point parallel to the given line. Or • There will exist no lines through the point parallel to the given line.[3] In the first case,replacing the Fifth postulate (parallel postulate) with a different axiom or its equivalent

with the statement: *"through any point P not on line R, there exists at least two lines that do not intersect R"* and keeping the first four axioms of Euclid, yield **hyperbolic geometry**.

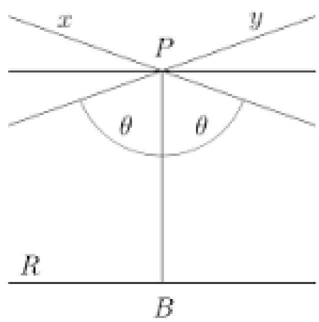

Figure 4: *Hyperbolic geometry*

The only two lines **asymptotic** to R through P is x and y. There are many lines that can be drawn to pass through P but will never intersect R. In **Fig. 2.4**, lines x and y intersect perpendicular through P at angle θ, but do not intersect R itself (the lines actually curve in hyperbolic space). If θ is the minimum such angle, then the lines x and y are said to be **asymptotic** to R, meaning that they intersect at **infinity**. At any larger angle, the lines are called **ultraparallel**, and never intersect R, even at **infinity**. We realized that, there are an infinite number of **ultraparallel lines**, but only two asymptotic lines to a given line through a point not on it.

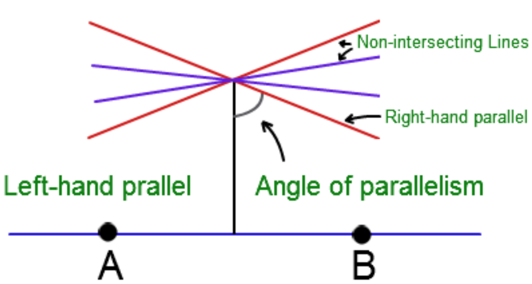

Figure 5: *Describing angel of parallelism, non-intersecting lines and parallel lines*

I. Examples

Examples of hyperbolic geometry specifically, 2-dimensional.
- Poincaré half-plane model.
- Poincaré disc model

Definition 2. Poincaré half-plane model is the upper half-plane, denoted as H together with a metric, the poincaré metric, that makes it a model of two-dimensional hyperbolic geometry.

II. History

It is named after Henri Poincaré, but originated with Eugenio Beltrami, who used it along with the Klein model and the Poincaré disk model (due to Riemann), to show that hyperbolic geometry was equiconsistent with Euclidean geometry. The disk model and the half-plane model are isomorphic under a conformal mapping.

III. Upper Half-plane

The hyperbolic geometry can be modeled in the upper half-plane H of all complex numbers with positive imaginary part.
The upper half-plane model in two-dimension is possibly the most popular model for working with two-dimensional hyperbolic geometry.
It is define as follows:
H^2=$\{f = x + iy \in R : Im(f) = y > 0\}$ and its associated given metric is
$h = \frac{dx^2+dy^2}{y^2}$.

IV. What is Upper half-plane?

Upper half-plane H is a set of complex numbers with positive imaginary part.
H=$\{x + iy|y > 0; x, y \in R\}$.
It also play an important role in hyperbolic geometry, where the Poincaré half-plane model provides a way at examine hyperbolic motions. The Poincaré metric provide a hyperbolic metric on the space.

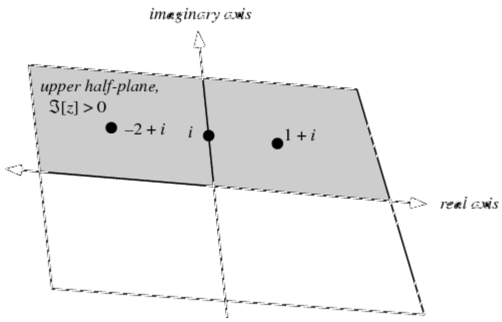

Figure 6: *Upper half-plane*

The upper half-plane is the portion of the complex plane $\{x+iy : x, y \in (-\infty, \infty)\}$ satisfying $y = I[Z] > 0$ i.e.
$\{x+iy : x \in (-\infty, \infty), y \in (0, \infty)\}$.

V. Example 1

Consider the straight line graph with equation $y = x$. When $x = 1$,$y = 1$ and when $x = 2$,$y = 2$ and so on. The line is a set of an infinite number of points. The point $A(2,2)$ is a particular element of this set. The line divides the cartesian plane into two half-planes.

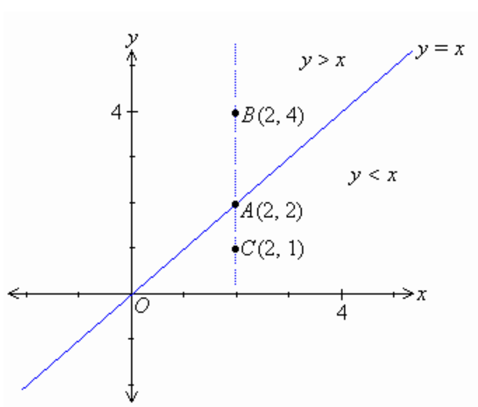

Figure 7: *Example of upper half-plane graph*

The set of points in the cartesian plane which lie above the line $(y = x)$ form the **Upper half-plane**,and the set of points which lie below the line form the **lower half-plane**.
$B(2,4)$ is the **upper half-plane**, and $C(2,1)$ is the **lower half-plane**.
We can see from the graph (Figure 2.7) above that the $y - coordinate$ of B is greater than its $x - coordinate$.
Therefore, the equation of the upper half-plane is $y > x$.
Similarly, the $y - coordinate$ of C is less than its $x - coordinate$. Therefore,the equation of the lower half-plane is $y < x$.

$\{(x,y) : y = x\}$ is the set of points which lie on the line.
$\{(x,y) : y > x\}$ is the set of points which lie above the line (upper half-plane) and
$\{(x,y) : y < x\}$ is the set of points which lie below the line (lower half-plane).

VI. Poincaré Metric

The metric of the model on the half-plane $\{\langle x,y\rangle | y > 0\}$ is given by
$(ds)^2 = \frac{(dx)^2+(dy)^2}{y^2}$
where S measures length along a possibly curved line. The straight lines in the hyperbolic plane(**geodesics** for this metric tensor, i.e. curves which minimize the distance) are represented in this model by circular arcs perpendicular to the X-axis and straight vertical lines ending on the X-axis.
The distance between two points measured in this metric along such a geodesic is
$dist(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) = arccosh(1 + \frac{(x_2-x_1)^2+(y_2-y_1)^2}{2y_1y_2})$.
This model is conformal which means that the angles measured at a point are the same in the model as they are in the actual hyperbolic plane.[4] [5]

IV. Metric Space

Metric space is a space with a defined distance.
A metric space is given by a set X and a metric or distance function
$d : X \times X \longrightarrow R$ such that the following properties holds.
Properties of a metric space

(i) (Positivity)
$d(x,y) \geq 0, \quad \forall x, y \in X,$

equality holds $\iff x = y$.

(ii) (Non-degenerated)
$d(x,y) = 0, \iff x = y, \quad \forall x,y \in X.$

(iii) (Symmetry)
$d(x,y) = d(y,x) \quad \forall x,y \in X.$

(iv) (Triangle inequality)
$d(x,y) \leq d(x,z) + d(z,y) \quad \forall x,y,z \in X.$

The distance between x and y is $d(x,y)$. The pair (X,d) consisting of the set X and the metric d is a **metric space**.
Examples 1

(i) $X = R, \quad d(x,y) = |x-y|.$

(ii) $X = R^2 = R \times R$,
$x = (x_1, x_2), \quad y = (y_1, y_2)$
$d_1(x,y) = |x_1 - y_1| + |x_2 - y_2|.$

(iii) $X \longrightarrow R^2$
$x = (x_1, x_2), \quad y = (y_1, y_2)$
$d_2(x,y) = (|x_1 - y_1|^2 + |x_2 - y_2|^2)^{\frac{1}{2}}$

Example 2
Consider R, define $d(x,y) = |x-y|$. This is called the Euclidean metric or standard metric or the usual metric. To show that the real line is a metric space, we have to proof these properties.

Proof. **(i)** $|x| \leq 0$ and $|x| = 0 \iff x = 0.$

(ii) $|-x| = |x|.$

(iii) $|x+y| \leq |x| + |y|.$
By (i) $d(x,y) = |x-y| \iff x - y = 0, \iff x = y.$
By (ii) $d(x,y) = |x-y|$
$= |-(y-x)|$
$= |y-x|$
$= d(y,x).$
By (iii) $d(x,y) = |x-y|$
$= |(x-z) + (z-y)|$
$\leq |x-z| + |z-y|$
$= d(x,z) + d(z,y)$
[6]

□

Lemma 3. ***Hölder's Inequality***
Let $x, y \in R^n$, then
$|\sum_{i=1}^n x_i y_i| \leq (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}} (\sum_{i=1}^n |y_i|^q)^{\frac{1}{q}}$
where $\frac{1}{p} + \frac{1}{q} = 1$, and thus (p,q) is called the conjugate pair. If $p = q = 2$, the Hölder's Inequality becomes Cauchy-Schwartz Inequality.

Lemma 4. *On R^n the metric*
$d_2(x,y) = (\sum_{i=1}^n |x_i - y_i|^2)^{\frac{1}{2}}$ satisfies the triangle inequality.

Proof. Let $x, y, z \in R^n$. Then we deduce from **Lemma 3** ,
$d(x,y)^2 = \sum_{i=1}^n |x_i - y_i|^2 = \sum_{i=1}^n |(x_i - z_i) - (y_i - z_i)|^2$
$= \sum_{i=1}^n |(x_i - z_i)|^2 - 2\sum_{i=1}^n (x_i - z_i)(y_i - z_i) + \sum_{i=1}^n |(y_i - z_i)|^2$
$\leq d(x,z)^2 + 2d(x,z)d(y,z) + d(y,z)^2$
$= (d(x,z) + d(y,z))^2$
Hence, $d(x,y) \leq d(x,z) + d(y,z)$ and the assertion is proved.
$\implies d(x,y) \leq d(x,z) + d(z,y).$ □

[7]

V. Isometry

Given a metric space, an **isometry** is a transformation which maps elements to the same or another metric space such that the distance between the image elements in the new metric space is equal to the distance between the elements in the original metric space.
In a two-dimensional Euclidean space, two geometric figures are congruent if they are related by an isometry.
Also, **isometry** is a transformation in which the original figure and its image are
congruent.
Moreover, isometry is invariant with respect to distance. That is, in an isometry, the distance between any two points in the original figure is the same as the distance between their corresponding image in the transformed

figure(image).

I. Formal Definitions

Let X and Y be a metric space with metrics d_x and d_y. A map $f : X \longrightarrow Y$ is called an **isometry or distance preserving** if for any $a, b \in X$
It implies that,
$d_y(f(a), f(b)) = d_x(a, b)$.
An isometry is automatically **injective**.
Clearly, every isometry between metric space is a topological embedding(i.e. homeomorphism).
A global isomtry, isometric isomorphism or congruence mapping is a **bijective isometry**. Two metric spaces X and Y are called isometric if there is a bijective isometry from X to Y. The set of bijective isometries from a metric space to itself forms a **group** with respect to function composition called the **isometry group**.[8]

II. Linear Isometries

Given two normed linear or vector spaces V and W, a **linear isometry** is a linear map $f : V \longrightarrow W$ that preserves the norms. Any isometry of normed linear or vector spaces over R is affine.

KEYS IN ISOMETRIES

1. **Metric space**
   Metric space is a set for which distances between all members of the set are defined.

2. **Congruent**
   In geometry, two figures or objects are **congruent** if they have the same shape and size, or if one has the same shape and size as the mirror image of the other. More formally, two sets of points are called **congruent**, if and only if one can be transformed into the other by an isometry i.e. a combination of rigid motions, namely a **translation**, a **rotation**, and a **reflection**.

   (i) Two line segments are congruent if they have the same length.

   (ii) Two angles are congruent if they have the same measure.

   (iii) Two circles are congruent if they have the same diameter.

3. **Affine**
   It involves the transformations which preserve collinearity, especially in classical geometry , those of translation, rotation and reflection in an axis.

III. Translation Operator

T_δ such that $T_\delta f(v) = f(v + \delta)$.
If v is a fixed vector, then translation T_v will work as $T_v(p) = p + v$.
In Euclidean geometry, a translation is a function that moves every point a constant distance in a specified direction.
Translations,denoted by T_v, where v is a vector in R^2. This has the effect of shifting the plane in the direction of v. That is, for any point p in the plane.
$T_v(p) = p + v$, or in terms of (x, y) coordinates,

$$T_v(P) = \begin{bmatrix} P_x + V_x \\ P_y + V_y \end{bmatrix}$$

. [9]

1. **Example in Isometries**
   The figure below shows a translation, an isometry.
   An irregular polygon $ABCDEF$ is translated to $A^1B^1C^1D^1E^1F^1$.
   It is clearly seen that, the distance between A and B is the same as the distance between their image A^1 and B^1.

Let v= (4,-3) be the fixed vector.

(1) For $A = (-4, 3)$,

$$T_v(A) = \begin{bmatrix} A_x + V_x \\ A_y + V_y \end{bmatrix} = \begin{bmatrix} -4 + 4 \\ 3 + (-3) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} = A^1$$

.

(2) For $B = (-1, 3)$,

$$T_v(B) = \begin{bmatrix} B_x + V_x \\ B_y + V_y \end{bmatrix} = \begin{bmatrix} -1 + 4 \\ 3 + (-3) \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 0 \end{bmatrix} = B^1$$

.

(3) For $C = (-1, 1)$,

$$T_v(C) = \begin{bmatrix} C_x + V_x \\ C_y + V_y \end{bmatrix} = \begin{bmatrix} -1 + 4 \\ 1 + (-3) \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -2 \end{bmatrix} = C^1$$

.

(4) For $D = (-2, 1)$,

$$T_v(D) = \begin{bmatrix} D_x + V_x \\ D_y + V_y \end{bmatrix} = \begin{bmatrix} -2 + 4 \\ 1 + (-3) \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2 \end{bmatrix} = D^1$$

.

(5) For $E = (-2, -1)$,

$$T_v(E) = \begin{bmatrix} E_x + V_x \\ E_y + V_y \end{bmatrix} = \begin{bmatrix} -2 + 4 \\ -1 + (-3) \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -4 \end{bmatrix} = E^1$$

.

(6) For $F = (-4, -1)$,

$$T_v(F) = \begin{bmatrix} F_x + V_x \\ F_y + V_y \end{bmatrix} = \begin{bmatrix} -4 + 4 \\ -1 + (-3) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -4 \end{bmatrix} = F^1$$

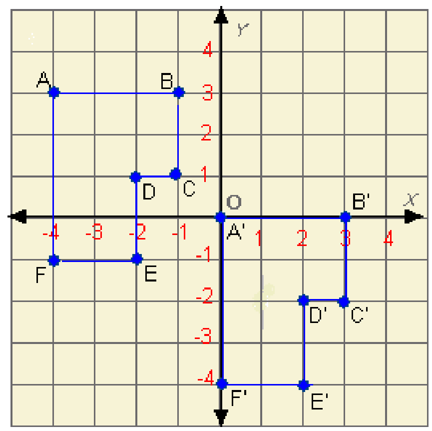

Figure 8: *Example of isometry*

VI. Groups

A group is a **set** combined with an **operation**.
Formal Definition of a Group.
A group is a set **G**, combined with an operation $*$, such that:

(i) The group contains an **identity**.

(ii) The group contains **inverse**.

(iii) The operation is **associative**.

(iv) The group is closed under the operation.

A group **G** consists of a set **G** together with a **binary operation,**$*$ for which the following properties are satisfied. **[?]**

• $(x * y) * z = x * (y * z)$ for all elements x, y and z of **G**. (**Associative Law**).

• $\exists$ an element e of **G** (known as the identity element of **G**) such that $e * x = x = x * e$, for all element x of **G** (**Identity**).

• For each element x of **G**, $\exists$ an element x^{-1} of **G** (known as the inverse of x) such that

$x * x^{-1} = e = x^{-1} * x$ (where e is the identity element of **G**) (**Iverse**).

• The order $|\mathbf{G}|$ of a finite group G is the number of elements of G.

• A group G is **Abelian**(or commutative) if $x * y = y * x$ for all elements x and y of G.

I. Multiplication notation for Groups

The product $x * y$ of two elements x and y of a group G is denoted by xy. The inverse of an element x of G is then denoted by x^{-1}.
The identity element is usually denoted by e. Sometimes the identity element is denoted by 1.
Thus, when multiplication notation is adopted, the group axioms are written as follows:

• $(xy)z = x(yz)$ for all elements x, y and z of G(**Associative Law**).

• $\exists$ an element e of G (the identity element of G) such that $ex = x = xe$, for all elements x of G.

• For each element x of G $\exists$ an element x^{-1} of G (the inverse of x) such that

$xx^{-1} = e = x^{-1}x$ (e is the identity element of G).

The group G is said to be **Abelian**(or **commutative**) if $xy = yx \quad \forall x, y$ of G.

II. Addition Notation For Groups

The group operation is denoted by $+$, the identity element of the group is denoted by 0, the inverse of an element x of the group is denoted by $-x$.
Note: Additive notation is only used for **Abelian groups**.
Abelian Group
An Abelian group is a group satisfying the *commutative law.*

III. Axioms for Abelian Group in Additive notation

• $x + y = y + x \quad \forall x, y$ of G (**Commutative Law**).
• $(x + y) + z = x + (y + z) \quad \forall x, y$ of G (**Associative Law**).
• $\exists$ 0 of G (identity element or zero element of G) such that $0 + x = x = x + 0, \forall x \in G$.(**Identity Law**).
• For each element $x \in G, \exists - x \in G$ (inverse of x) such that $x + (-x) = 0 = (-x) + x$.(**Inverse Law**)

VII. Discontinuous Group

A discontinuous group is one for which there exists an ordinary point. Ordinary point is a point of a plane which is not a limit point.

I. Limit point

Let A be a subset of a topological space X. A point $x \in X$ is a limit point of A if every neighborhood of x intersects A in a point than that in x itself.

Theorem 5. *Let B be a group of linear fractional transformation all of whose elements(except the identity) possess isometric circle. If there exists an open set of points that is exterior to all isometric circle, then B is* ***discontinuous****.*

VIII. Overview of Riemann Geometry

Riemannian geometry is one way of looking at distances on manifolds. This seems an easy enough concept when you first think of it, but after further though we realize it is not so easy. Sure we know how to measure distances on a plane. The shortest distance between two points is a straight line. So just draw the line and measure the distance (first we set what unit measure is, for instance 1 meter, and then compare the distance we want to measure to our set standard unit distance, say the meter stick). But on a sphere how do we measure distance? Or on a torus (the surface of a doughnut)? A sphere we can think of as living in Euclidean 3-space and then just say that the distance between 2 points on the sphere is just the distance between those points (x_1, y_1) in 3-space, as described by the Pythagorean theorem, which says that for points (x_2, y_2) and the distance between them is $\sqrt{(x_1 - x_2) + (y_1 - y_2)}$ which is the fact that the length of the hypotenuse (longest side) of a right triangle is equal the square root of the sum of the squares of the other two sides.

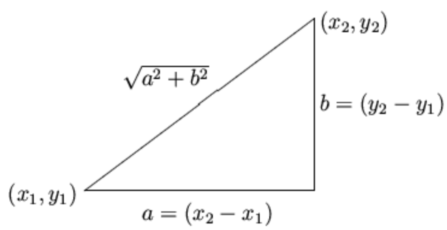

Figure 9: *Pythagoras theorem*

We are going to skirt these issues by taking another approach to distances. Suppose we could measure the length of curves. Then we could simply define the distance between two points to be the length of the shortest path, if one exists. Even if one does not exist, we could express the distance as the largest lower bound for lengths of paths between the two points (this is called the **infimum**). This isn't too important so let's assume that we can find a shortest path. So we could measure distances if we could measure the lengths of paths. This is where a Riemannian metric comes in.

I. Riemannian metric

Riemannian metric is a function defined at every point that takes two vectors and gives a number. Technically, a Riemannian metric must be a symmetric bilinear form.
We express the Riemannian metric as **g** and we express vectors at a point **p** in our manifold as V_p, W_p, U_p. then we can measure quantities like $g(V_p, W_p)$. The fact that is symmetric means:
$g(V_p, W_p) = g(W_p, V_p)$
and bilinearity (when coupled with symmetry) is the following condition:
$g(V_p + W_p, U_p) = (V_p, U_p) + (W_p, U_p)$.
The important thing is that the Riemannian metric gives us a way to measure lengths of vectors at each point in the manifold, and also gives us a way of measuring lengths on the manifold.[10]

II. Manifold

A manifold is a topological space that is locally Euclidean.

IX. Topological Space

Let X be a set A topology. T on X is a collection of subsets of X each called an open set such that

(a) ϕ and X are open sets.

(b) The intersection of finitely many open set is an open set.

(c) The union of any arbitrary collection of open set is an open set.

I. Example of topological space

Let X be a nonempty set.Define $T = \{\phi, X\}$ .Check whether T is a topology.
sol.
Let $T = \{\phi, X\}$, $\implies$ $\phi \cap X = \phi$, $\phi \cup X = \{\phi, X\}$
$\therefore$ T is a topology.

II. Locally Euclidean

A topological space X is called locally Euclidean if there is a non-negative integer n such that every point in X has a neighborhood which is **homeomorphic** to the Euclidean space R^n.

III. Properties of locally Euclidean

The property of being locally Euclidean is preserved by local homeomorphisms. That is , if X is locally Euclidean of dimension n and $f : Y \longrightarrow X$ is a local homeomorphism, then Y is a locally Euclidean of dimension n.

Theorem 6. *A manifold is locally connected, locally compact and the union of countably many compact subsets. A manifold is normal and metrizable.*

Note: A manifold can be either compact or non-compact, which we refer to as closed or open manifold respectively.

IV. Connectedness

A topological space or manifold X is connected if it cannot be broken down into distinct places that form the union.

Definition 7. Let X be topological manifold.

(i) We call X connected if there does not exists a pair of disjoint non-empty open sets whose union is X.

(ii) We call X disconnected if X is not connected .

(iii) If X is disconnected then a pair of disjoint open sets whose union is X is called a **separation of** X.

V. Compactness

A topological space or manifold X is compact if every open cover of X ha a finite subcover.

Definition 8. Let A be a subset of a topological space X, and let O be a collection of subset of X.

(i) The collection O is said to cover A or to be a cover of A if A is contained in the union of the sets in O.

(ii) If O covers A , and each set in O is open , then we call O an open cover of A.

(iii) If O covers A and $0'$ is a subcollection of O that also cover A, then O' is called a subcover of O.

X. Topological Manifolds

A topological space M is called topological manifold or m-dimensional manifolds if the following conditions hold.

(i) M is a Hausdorff space.

(ii) For any $P \in M, \exists$ a neighborhood U of P which is homeomorphic to an open subsets $V \subset R^m$ and

(iii) M has a countable basis of open sets.

I. Hausdorff Space

A topological space M is called **Hausdorff** if for any pair $p, q \in M, \exists$ open neighborhood $U \ni p$, and $U^1 \ni q$ such that $U \cap U^1 = \phi$.

II. Homeomorphism

Let N and M be manifolds and let $f : N \longrightarrow M$ be continuous mapping. A mapping f is called a **homeomorphism** between N and M if f is continuous and has a continuous inverse $f^{-1} : M \longrightarrow N$. In this case,the manifold N and M are said to be **homeomorphic**.
[11]

Using coordinate chart to show homeomorphism.

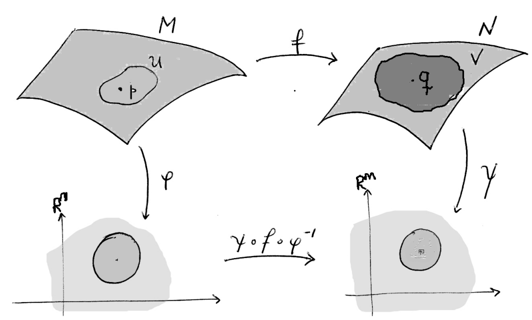

Figure 10: *coordinate chart*

Using charts (U, φ) and (V, ψ) for N and M respectively,coordinate expression for f i.e. $\tilde{f} = \psi \circ f \circ \varphi - 1$.

III. Basis

For a topological space M with topology τ , a collection $\beta \subset \tau$ is a **basis** if and only if each $U \in \tau$ can be written as union of sets in β. A bsis is called a **countable basis** if β is a countable set. [?]

IV. Locally Homeomorphic

If $p \in M$ has an open neighborhood $U \ni p$ homeomorphic to the open disc D^m in R^m , we say that M is locally homeomorphic to R^m.

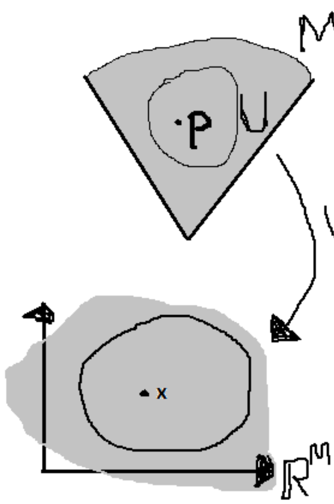

Figure 11: *(Coordinate charts*(U, φ)*)*

Figure 3.3 displays coordinate charts (U, φ), where U is coordinate neighborhood or charts and φ is (coordinate) homeomorphism.

Transition maps(φ_{ij})

Transition between different choices of coordinates are called **transition maps**,
φ_{ij}= $\varphi_j o \varphi_i^{-1}$ which are again homeomorphism. See **Figure 3.4**.

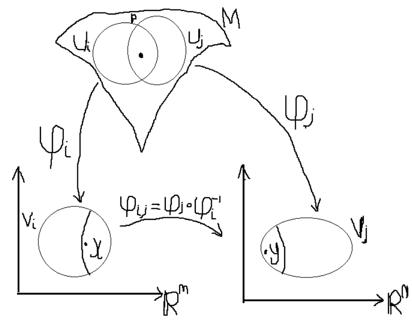

Figure 12: *Transition maps*(φ_{ij})

.
From **Figure 3.4**, we usually write,
$x = \varphi(p)$
$\varphi\colon U \longrightarrow V \subset R^m$ as coordinate for U.
$p = \varphi^{-1}(x)$,
$\varphi^{-1}\colon V \longrightarrow U \subset M$ as parametrization of U.
$\implies \varphi(p) \longrightarrow x$——— coordinatization .
$\varphi^{-1}(x) \longrightarrow p$——parametrization.

V. Topological Manifold with Boundary

A topological space M is called an **m-dimensional or topological manifold with boundary,** $\partial m \subset M$,if the following conditions hold:

- i M is a Hausdorff space.
- ii For any point $p \in M$, $\exists$ a neighborhood U of p which is homeomorphic to an open subset $V \subset H$ and
- iii M has a countable basis of open sets.

  Manifolds with boundary are also either compact or non-compact. In both cases the boundary can be compact.

Open subsets of a topological manifold are called **open submanifold** and can be given a manifold structure again.

VI. Topological Embedding

A topological embedding is a continuous injective mapping $f\colon X \longrightarrow Y$, which is a home-

omorphism onto its image $f(x) \subset Y$ with respect to the subspace topology.
Let $f\colon N \longrightarrow M$ be an embedding, then its image $f(N) \subset M$ is called a **submanifold** of M.

XI. Differential Manifold

Given a topological manifold M which is such that
$\varphi_{ij} = \varphi_j o \varphi_i^{-1}\colon \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$ are diffeomorphism is called a **differentiable or smooth manifold**.
Diffeomorphism between open subsets of R^m are C^∞-maps, whose inverses are also C^∞-maps.

• **Atlas** is the union of all coordinate charts.

I. Diffeomorphism

A function $f\colon X \longrightarrow Y$ is said to be **diffeomorphic** if f and f^{-1} or both functions are differentiable.
Two atlas $(C^\infty), A, A^1$ are said to be equivalent if their C^∞ coordinate maps are **compatible**.
In that case , $A \cup A^1$ is also an **atlas**.
• **Maximal Atlas**
Maximal atlas is the union of all equivalent atlases and is denoted as A_D.

Definition 9. Let M be a topological manifold, and let D be a differentiable structure on M with maximal atlas A_D. Then the pair (M, A_D) is called a **differentiable manifold**.
Also a differentiable manifold is a manifold which has a differentiable structure.

II. Theorem(Unique differentiable structure)

Given a set M , a collection $\{U_\alpha\}$ of subsets, and injective maps $\varphi_\alpha\colon U_\alpha \longrightarrow R^m$ such that the following are satisfied.

Properties

i . $\varphi_\alpha(\varphi_\alpha \subseteq R^m)$ is open for all α.

ii . $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in R^m.

iii . For $U_\alpha \cap U_\beta \neq \phi$, the transition maps, $\varphi_\alpha o \varphi_\beta^{-1}\colon \varphi_\beta(U_\alpha \cap U_\beta) \longrightarrow \varphi_\alpha(U_\alpha \cap U_\beta)$ are diffeomorphism for every α, β.

iv . Countably many set U_α cover M.

v . For any pair $p \neq q \in M$, either $p, q \in U_\alpha$ or $\exists$ disjoint sets U_α, U_β such that $p \in U_\alpha$ and $q \in U_\beta$[This property is Hausdorff].

Then M has a unique differentiable manifold structure and (U_α, U_β) are smooth charts. A function $f\colon N \longrightarrow M$ is said to be **smooth** if for every $p \in N$, $\exists$ charts (U, φ) of p and (V, ψ) of $f(p)$ with $f(U) \subset V$ such that $\tilde{f} = \psi o f o \varphi^{-1}\colon \varphi(U) \longrightarrow \psi(V)$ is a C^∞-mapping (R^n to R^m).

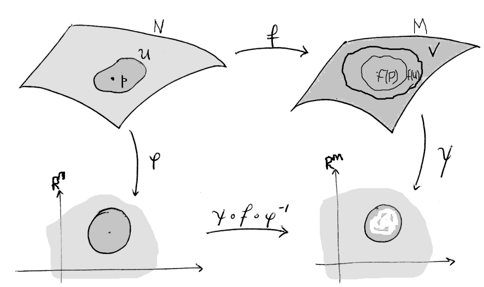

Figure 13: *Coordinate representation for f,with $f(U) \subset V$*

.

III. Differentiable Homeomorphism

A smooth map $f\colon N \longrightarrow M$ is said to be **diffeomorphic** if f^{-1} is also smooth.
The associated differentiable structures on N are also said to be **diffeomorphic**.

IV. Local Diffeomorphic

A function f is said to be a **local diffeomorphism** if for every $p \in N, \exists$ a neighborhood U, with $f(U)$ open in M such that $f\colon U \longrightarrow f(V)$ is a diffeomorphism.

Example 1
Let $N = R$ with atlas $(R, \varphi(p) = p)$ and let $M = R$ with atlas $(R, \psi(q) = q^3)$, then

$f\colon N \longrightarrow M, p \longrightarrow f(p) = p^{\frac{1}{3}}$.

Solution

Take $U = V = R$, then

$f\colon R \longrightarrow R$

$\tilde{f} = \psi o f o \varphi^{-1} = \psi o f(p) = \psi(p^{\frac{1}{3}}) = (p^{\frac{1}{3}})^3 = p$

$\tilde{f}^{-1} = (\psi o f o \varphi^{-1})^{-1}$

$= \psi^{-1} o f o (\varphi^{-1})^{-1}$

$= \varphi o f o \psi^{-1}$

$= \varphi o f^{-1}(q)$

$= \varphi(q^3)$

$= (q^3)^{\frac{1}{3}}$

$= q$

If f is smooth and its inverses i.e. f^{-1} is also smooth, it implies that f is a **diffeomorphism**. It also implies that N and M are diffeomorphic to each other and their differentiable structures are diffeomorphic and equivalent.

V. Rank of Mapping

Rank of f,denoted by $rank(f)$ is given by $rank(J(f))$.

For $f\colon N \longrightarrow M$,

$J(f) =$

$(x_1, ..., x_n) \longrightarrow (f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_m))$.

$\implies J(f)$ is of order $m \times n$.

$$J(f) = \begin{bmatrix} \frac{\delta f_1}{\delta x_1} & \cdots & \frac{\delta f_1}{\delta x_n} \\ \vdots & \ddots & \vdots \\ \frac{\delta f_m}{\delta x_1} & \cdots & \frac{\delta f_m}{\delta x_n} \end{bmatrix}$$

A largest square matrix which is non-zero is called **rank**.

VI. Definition(Immersion)

A mapping $f\colon N \longrightarrow M$ is called an **immersion** if $rank(f) = n$ (domain).

VII. Definition(Submersion)

A mapping $f\colon N \longrightarrow M$ is called a **submersion** if $rank(f) = m$(co-domain).

VIII. Definition(Smooth Embedding)

An immersion that is injective or $1-1$ and is a homeomorphism onto (surjective mapping) its image $f(N) \subset M$, with respect to the subspace topology is called
a **smooth embedding**.

A smooth embedding is an (injective) immersion that is a topological embedding.

A mapping $f\colon N \longrightarrow M$ is **injective** if for all $p, p^1 \in N$ for which $f(p) = f(p^1)$ it holds that $p = p^1$.

A mapping $f\colon N \longrightarrow M$ is **surjective** if $f(N) = M$ or equivalently for every $q \in M, \exists p \in N$ such that $q = f(p)$.

Theorem 10. *Let $f\colon N \longrightarrow M$ be smooth with constant rank, $rk(f) = k$, then for each $p \in N$, and $f(p) \in M, \exists$ coordinate (U, φ) for p and (V, ψ) for $f(p)$, with $f(u) \subset V$, such that $(\psi o f o \varphi^{-1})(x_1, ..., x_k, x_{k+1}, ..., x_n) = (x_1, ..., x_k, 0, ..., 0)$.*

Example 1

Let $N = R, M = R^2$ and $f\colon N \longrightarrow M$ defined by $f(t) = (t^2, t^3)$.

Solution

$$Jf|_t = 2 \times 1 = \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix}$$

$t \neq 0$, none of the elements will varnish, therefore the largest square matrix is 1.

$\implies t \neq 0, rank(Jf) = 1$

$\therefore f$ is immersion and injective.

But for $t = 0, f$ is not immersion.

Example 2

Let $N = R^2$ and $M = R$ with projection mapping $f\colon N \longrightarrow M$ that is $(x, y) \longrightarrow f(x, y) = x$.

Solution

$$Jf|_{(x,y)} = 1 \times 2 = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$Rank(f) = 1$.

$\therefore$ submersion.

This is because the rank of f is the same as the dimension of the co-domain i.e. $m = 1$.

Example 3

Let $N = M = R^2$ and consider the mapping $f(x, y) = (x^2, y)^t$

Solution

$$Jf|_{(x,y)} = 2 \times 2 = \begin{pmatrix} 2x & 0 \\ 0 & 1 \end{pmatrix}$$

.

$Rank(Jf) = 2$ for $x \neq 0$.
$Rank(Jf) = 1$ for $x = 0$.[19]

XII. Riemannian Manifold

To measure distances and angles on manifolds, the manifold must be Riemannian. A Riemannian manifold is a differentiable manifold in which each tangent space is equipped with an inner product $< .,. >$ in a manner which varies smoothly from point to point. All differentiable manifold (of constant dimension) can be given the structure of a Riemannian manifold. The Euclidean space itself carries a natural structure of Riemannian manifold(the tangent spaces are naturally identified with the Euclidean space itself and carry the standard scalar product of the space).

I. Tangent Space

Let x be a point in an n-dimensional compact manifold M, and attach at x a copy of R^n tangential to M. The resulting structure is called the **tangent space** of M of x and is denoted T_xM.
If γ is a smooth curve passing through x,then the derivative of γ at x is a vector in T_xM.
A manifold possessing a metric tensor.
For a complete Riemannian manifold, the metric $d(x,y)$ is defined as the length of the shortest curve**(geodesic)** between x and y.

II. Riemannian space

Riemannian space (M, g) is a real smooth manifold M equipped with an inner product g_x on the tangent space T_xM at each point that varies smoothly from point to point in the sense that if X and Y are vector field on M , then

$x \longrightarrow g_x(X(x), Y(x))$ is a smooth function.
The family g_x of inner products is called a **Riemannian metric(tensor)**

III. Riemannian manifolds as metric spaces

Riemannian manifold is defined as a smooth manifold with a smooth section of the positive-definite quadratic forms on the tangent bundle. Then one has to work to show that it can be turned to a **metric space**.
If $\gamma\colon [a,b] \longrightarrow M$ is a continuously differentiable curve in the Riemannian manifold M, then we define its length $L(\gamma)$ as:
$L(\gamma) = \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))dt} = \int_a^b \|\gamma'(t)\|dt$.
With this definition of length, every connected Riemannian manifold M becomes a metric space in a natural fashion and the distance $d(x,y)$ between the points x and y of M is define as:
$d(x,y) = inf\{L(\gamma)\colon \gamma$ is a continuously differentiable curve joining x and $y\}$.
Even though Riemannian manifolds are usually "curved", there is still a notion of "straight line " on them, that is **geodesics**.
Assuming the manifold is compact, any two points x and y can be connected with a geodesic whose length is $d(x,y)$.

Properties

In Riemannian manifolds, the notions of **geodesic completeness, topological completeness and metric completeness** are the same.

- **Diameter**

The diameter of a Riemannian manifold M is defined as
$diam(M)\colon \sup_{x,y \in M} d(x,y) \in R \geq 0 \cup \{+\infty\}$.
Riemannian manifold M is **compact** if and only if it is **complete** and has **finite diameter**. M is geodesically complete if and only if it is **complete** as a **metric space**.

IV. Riemannian Metric(tensor)

A Riemannian metric on a manifold M is a smoothly varying positive definite inner product on the tangent spaces T_x.
A Riemannian metric g on a smooth manifold is a smoothly chosen inner product
$g_x\colon T_xM \times T_xM \longrightarrow R$ on each of the tangent spaces T_xM of M. In other words, for each $x \in M, g = g_x$ satisfies:

(1) $g(u,v) = g(v,u)\ \forall u,v \in T_xM$.

(2) $g(u,u) \geq 0 \forall u \in T_xM$.

(3) $g(u,u) = 0$ if and only if $u = 0$.

Furthermore, g is smooth in the sense that for any smooth vector fields X and Y , the function $x \longrightarrow g_x(X_x, Y_x)$ is smooth.
Locally a metric can be described in terms of its coefficients in a local chart, defined by $g_{ij} = g(\partial_i, \partial_j)$.
The smoothness of g is equivalent to the smoothness of all the coefficient functions g_{ij} in some chart.
In a metric tensor, the tensor product is $T_x^* \otimes T_x^*$.

Definition 11. A Riemannian metric on a manifold M is a section g of $T^* \otimes T^*$ which at each point is symmetric and positive definite.
In a local coordinate system we can write,
$g = \sum_{ij} g_{ij}(x) dx_i \otimes dx_j$ as a metric tensor where $g_{ij}(x) = g_{ji}(x)$ and is a smooth function, with g_{ij} positive definite and g_{ij} are the component of the metric tensor at each point x.
Frequently, the tensor product symbol is omitted and we simply writes
$g = \sum_{i,j} g_{ij}(x) dx_i dx_j$. Then g is a **Riemannian metric**

Example
Given a smooth map
$F\colon M \longrightarrow N$ and a metric g on N ,we can pull g to a section F^*g of $T^*M \otimes T^*M$: $(F^*g)_x(X,Y) = g_F(x)(DF_x(X), DF_x(Y))$.
If DF_x is invertible , this will again be positive definite if F is a **diffeomorphism**.

Definition(Isometry)
Isometry in manifolds is injective and isometry between metric space is a topological embedding(i.e. homeomorphism).
A diffeomorphism $F\colon M \longrightarrow N$ between two Riemannian manifolds is an **isometry** if $F^*g_N = g_N$.[12]

Example
Let $M = \{(x,y) \in R^2 \colon y > 0\}$ and
$g = \frac{dx^2+dy^2}{y^2}$.
If $z = x + iy$ and $F(z) = \frac{az+b}{cz+d}$
with a, b, c, d are complex numbers and $ad - bc > 0$, then
$F^*dz = (ad - bc)\frac{dz}{(cz+d)^2}$ and
$F^*y = yoF = \frac{1}{i}(\frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d})$
$F^*y = \frac{ad-bc}{|cz+d|^2}y$
Then,

$$F^*g = (ad-bc)^2 \frac{dx^2+dy^2}{|(cz+d)^2|^2} \frac{|cz+d|^4}{(ad-bc)^2y^2}$$
$$= \frac{dx^2+dy^2}{y^2}$$
$$= g$$

So these Mӧbius transformations are isometries of a Riemannian metric on the upper half-plane.[13]

XIII. Conclusion

In this discussion,we find out that Non-Euclidean which is the same as hyperbolic geometry was develop from Euclidean Geometry. However, 2-dimensional hyperbolic geometry is also known as Non-Euclidean geometry. The 2-dimensional hyperbolic which is also known as Non-Euclidean geometry was obtain due to the violation of the fifth postulate of the Euclidean Geometry.
In describing the 2-dimensional hyperbolic geometry in a Reimannian geometry, was found out that it is associated with a manifold which is one way of measuring distance. The manifold is in two forms, Riemannian manifolds and differential manifolds.
However, these two manifolds are the same in terms of their geodesics. The Riemannian manifolds were also describe as metric spaces. Which by this we worked to show that it can be turned to a metric space.
Also, even though Riemannian manifolds are usually "curved", there is still a notion of straight line on them that is **geodesics**.
Therefore, geodesics was describe as shortest line between two points in a curve or flat surfaces. With this, we say that both the Riemannian geometry and hyperbolic geometry violate the fifth postulate of the Euclidean geometry.
In a nutshell, diffeomorphism between two Riemannian manifolds is an **isometry**.

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