THE EFFICIENT FRONTIER AND INTERNATIONAL PORTFOLIO DIVERSIFICATION IN TAXABLE AND TAX-PRIVILEGED ACCOUNTS

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Abstract. In this paper, we consider efficient frontiers associated to two and three fund portfolios consisting of total domestic bond funds, total domestic equity funds, and total international equity funds. These frontiers are intended to help inform investment decisions regarding international exposure in taxable and tax-privileged accounts.

1. Introduction

One of the most difficult decisions facing modern U.S. investors is to what extent they should invest in foreign companies. There is considerable diversity of opinion on the topic. Many advocates of the efficient market hypothesis believe that one's overall equity exposure should correspond to global market capitalization, and accordingly at the present time one should have approximately 58% of equities lying in domestic funds and 42% held in international funds.[1] Other prominent investors, such as the late John Bogle [1], have advocated that a U.S. investor's equity portfolio be composed entirely of U.S. companies. Issues in play include favorable demographic trends in certain emerging market economies such as India, political risks involving nationalization of industries, majority state-ownership of certain foreign companies, the notion that China might overtake the U.S. as the world's leading economy, and that shareholders of U.S. companies enjoy many protections unavailable to shareholders of companies incorporated overseas. Such geopolitical risks are difficult to analyze and we do not do so here.

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[1]These percentages are based on the allocation of domestic and international assets held in the Vanguard Total World Stock Index Fund (VTWAX).

Instead, we consider issues associated to *taxation* within both taxable and tax-privileged accounts.

U.S. owners of foreign stocks and international funds are undoubtedly aware that dividends associated to companies in foreign countries are taxed by the countries in which the companies are incorporated. So as to not encounter double-taxation on these dividends, the U.S. investor is often allowed to claim a Foreign Tax Credit on their personal income tax. (See *IRS Form 1116 and Instructions* for more detailed information in this regard.) However, if the investments are held within a tax-privileged account such as a SEP or Roth or Traditional IRA, such foreign taxes are never reimbursed. This leads the investor to recognize that, for all practical purposes, within tax-privileged accounts foreign investments have additional expenses not associated to domestic ones, and possible equity reallocations may be in order.

Let us for the remainder of the paper consider a U.S. investor who only holds three types of investments: a broad-based domestic equity fund such as the Vanguard Total Stock Market Index Fund (VTSAX), a total international stock index fund such as the Vanguard Total International Stock Index Fund (VTIAX), and a total domestic bond fund such as the Vanguard Total Bond Market Index Fund (VBTLX). Within a tax-privileged account, neither the bond market fund nor the total domestic stock market fund will be taxed. However, Vanguard reported that in 2018 an investor in VTIAX would have paid a foreign tax of 8% on dividends. The yield of VTIAX being approximately 3.2%, we find a VTIAX holding held in a tax-privileged account has an annual loss of approximately 0.25% due to foreign taxation that is not present in a VTIAX holding in a taxable account.

We recognize that the expenses associated to management fees are the same for funds held in either taxable or tax-privileged accounts. Accordingly, we choose here to disregard such expenses, instead focusing on how the expenses associated to foreign and domestic taxes paid might suggest differing allocations of the above three funds depending on whether they are held in a taxable or tax-privileged accounts.

To make fair comparisons between an asset allocation's performance based on whether it is situated in a tax-privileged or taxable account, it is important to pair the foreign tax contribution of VTIAX in a tax-privileged account to effective additions to the expense ratios due to domestic taxes on *all of* VTSAX, VTIAX, and VTBLX held in a taxable account. Here we consider the taxes associated to an investor with an effective qualified dividend income tax rate of 23.8%. As of the time of this writing, Schwab reports one year tax cost ratios of 0.42% for VTSAX, 0.78% for VTIAX, and 1.04% for VBTLX. Taking into account the foreign tax credit for VTIAX indicated above, we treat the *effective* tax cost ratio for VTIAX as $0.78\% - 0.25\% = 0.53\%$. To be concrete, for the remainder of the paper we will disregard managment fees on all funds considered, assume the effective annual returns for VTIAX, VTSAX, and VTBLX in tax-privileged accounts are those provided by Vanguard, and within taxable accounts adjust the annual returns for VTIAX, VTSAX, and VTBLX as provided by Vanguard downward respectively by 0.53%, 0.42%, and 1.04%. The time frame of data considered in this paper ranges from 2009–2019, and the reader should recognize that these adjustments of data may not be exact but nonetheless are relatively accurate approximations.

The goal of this paper is to use the above data to construct the *efficient frontier* of two and three fund portfolios consisting of the above funds in both taxable and tax-privileged acccounts. Many believe that portfolio allocations lying on the efficient frontier are optimal in terms of risk versus reward, and we hope that a close analysis of these frontiers will prove to be beneficial for investors and their financial advisors.

In the second section of this paper we review the underlying concepts of the efficient frontier of a two-fund portfolio and illustrate it by providing efficient frontiers of a portfolio consisting of a total domestic equity index fund and a total domestic bond fund (a typical two-fund portfolio) as well as a portfolio consisting of a total domestic equity index fund and a total international equity index fund, each portfolio being treated in both taxable and tax-privileged accounts from the U.S. investor standpoint. For the underlying data we consider the annual returns of the Vanguard funds VTIAX, VTSAX, and VTBLX indicated above over the years 2009–2019, with associated tax cost bases as provided by Schwab.

In the third section we review the underlying concepts of the efficient frontier of a three-fund portfolio and subsequently provide an analysis of three-fund portfolios consisting of the above funds in both taxable and nontaxable accounts.

Based on results appearing in the subsequent sections, we believe that, *based on considerations of the efficient frontier alone*, asset allocation within two and three fund portfolios should not depend on whether the portfolios lie in a taxable or tax-privileged accounts.

Many readers undoubtedly find the mathematics underlying the construction of efficient frontier curves to be mysterious. In this paper we explicitly derive such curves, featuring calculations that are particularly informative. It is our intention that our construction of the efficient frontier curve of a two-fund portfolio be accessible to the reader with a firm grasp of elementary algebra, and our construction of the efficient frontier curve of a three-fund portfolio be accessible to our readers who have taken a course in multivariable calculus (or are at least familiar with Lagrange multipliers.) Statistical concepts such as the mean, variance, and covariance are used, and the reader unfamiliar with these is encouraged to consult a standard text such as the one of Feller [2].

We remark that in this paper we allow for "unrestricted allocations", in particular, allowing the shorting of one fund to enable additional purchase of another. The results are disquieting (frequently calling for a significant shorting of the international equity fund) but undoubtably informative. Although at first glance exotic, such shorting strategies have been considered previously, e.g. by Markowitz *et al* in [3].

2. The efficient frontier of two fund portfolios

We begin by describing how one may calculate the efficient frontier of two-fund portfolios. The concept of the efficient frontier was discovered by Markowitz (see, e.g. [6] and other relevant works [4, 5, 7, 8].) Our treatment is very similar to his, and in the case of two-fund portfolios only requires as a mathematical prerequisite a firm grasp of high-school level algebra.

The efficient frontier of a two-fund portfolio is typically the top-half of a *parabola* on standard coordinate axes, where the "x-axis" corresponds to the variance of a portfolio and the "y-axis" corresponds to the return. In particular, a point on the efficent frontier typically corresponds to the greatest possible return for a portfolio given a fixed variance; or equivalently the least variance for a given fixed return. (There are admittedly some "degenerate" scenarios for which the efficient frontier might be viewed as either a horizontal or vertical line or a single point.) The efficient frontier is important in financial analysis as it reveals that certain portfolio reallocations enable higher return while simultaneously reducing variance (frequently viewed as a proxy for risk.)

Suppose a portfolio consists of holdings in assets A and B. (The reader could mentally substitute an equity index for A and a bond fund for B at this point with no loss of generality.) The portfolio weight allocated to A is denoted by ω_A and the weight allocated to B is denoted ω_B. Assuming that the entire portfolio is invested in positions A and B, we have

$$\omega_A + \omega_B = 1 \ .$$

We denote the average return of position A by $\bar{r}_A$ and the average return of position B by $\bar{r}_B$. Denoting the variance of the returns of A and B respectively by $Var(A)$ and $Var(B)$, the covariance of A and B by $Cov(A, B)$, the average return of the overall portfolio by $Ret(P)$, and the variance of the overall portfolio by $Var(P)$, we have

$$Ret(P) = \omega_A \bar{r}_A + \omega_B \bar{r}_B$$

and

$$Var(P) = \omega_A^2 Var(A) + 2\omega_A \omega_B Cov(A, B) + \omega_B^2 Var(A) \ .$$

Let us unpack the above equations a little bit. Suppose portfolio A has a sequence of, say, annual returns

$$r_{A,1}, \ldots, r_{A,N}$$

and portfolio B has a sequence of returns

$$r_{B,1}, \ldots, r_{B,N} \ .$$

The *mean* (or average) return of A is

$$\bar{r}_A = \frac{1}{N}\left(r_{A,1} + \cdots + r_{A,N}\right) \ ,$$

and the mean return of B is

$$\bar{r}_B = \frac{1}{N}\left(r_{B,1} + \cdots + r_{B,N}\right) \ .$$

The *variance* of the returns of A is given by

$$Var(A) = \frac{1}{N}\left((r_{A,1} - \bar{r}_A)^2 + \cdots (r_{A,N} - \bar{r}_A)^2\right) ,$$

and the variance of the returns of B is given by

$$Var(B) = \frac{1}{N}\left((r_{B,1} - \bar{r}_B)^2 + \cdots (r_{B,N} - \bar{r}_B)^2\right) \ .$$

The *covariance* of the returns of A and B is given by

$$Cov(A,B) = \frac{1}{N}\left((r_{A,1} - \bar{r}_A)(r_{B,1} - \bar{r}_B) + \cdots + (r_{A,N} - \bar{r}_A)(r_{B,N} - \bar{r}_B)\right) \ .$$

If, at the end of every year, the portfolio is rebalanced to its original weighting, the sequence of annual returns for the overall portfolio will be

$$\omega_A r_{A,1} + \omega_B r_{B,1} \ , \ \ldots \ , \ \omega_A r_{A,N} + \omega_B r_{B,N}$$

with mean portfolio rate of return

$$\begin{aligned} Ret(P) &= \frac{1}{N}\left(\omega_A r_{A,1} + \omega_B r_{B,1} + \cdots + \omega_A r_{A,N} + \omega_B r_{B,N}\right) \\ &= \frac{1}{N}\left((\omega_A r_{A,1} + \cdots + \omega_A r_{A,1}) + (\omega_B r_{B,1} + \omega_B r_{B,N})\right) \\ &= \omega_A \bar{r}_A + \omega_B \bar{r}_B \end{aligned}$$

and with the returns having variance

$$\begin{aligned} Var(P) &= \frac{1}{N}\Big(((\omega_A r_{A,1} + \omega_B r_{B,1}) - (\omega_A \bar{r}_A + \omega_B \bar{r}_B))^2 \\ &\qquad + \cdots + \ ((\omega_A r_{A,N} + \omega_B r_{B,N}) - (\omega_A \bar{r}_A + \omega_B \bar{r}_B))^2\Big) \\ &= \omega_A^2 Var(A) + 2\omega_A\omega_B Cov(A,B) + \omega_B^2 Var(B) \ . \end{aligned}$$

Since $\omega_B = 1 - \omega_A$, we can express $Var(P)$ in terms of ω_A by

$$Var(P) = \omega_A^2[Var(A) - 2Cov(A,B) + Var(B)] + 2\omega_A[Cov(A,B) - Var(B)] + Var(B) \ . \tag{2.1}$$

Suppose that $\bar{r}_A \neq \bar{r}_B$. Then

$$\omega_A = \frac{Ret(P) - \bar{r}_B}{\bar{r}_A - \bar{r}_B} \ , \tag{2.2}$$

and we can express $Var(P)$ in terms of $Ret(P)$ by

$$\begin{aligned} Var(P) &= \left(\frac{Ret(P) - \bar{r}_B}{\bar{r}_A - \bar{r}_B}\right)^2 [Var(A) - 2Cov(A,B) + Var(B)] \\ &\quad + 2\left(\frac{Ret(P) - \bar{r}_B}{\bar{r}_A - \bar{r}_B}\right)[Cov(A,B) - Var(B)] + Var(B) \ . \end{aligned}$$

An important observation regarding the quantity

$$Var(A) - 2Cov(A,B) + Var(B)$$

is in order here. Using the elementary fact that $0 \leq (x-y)^2 = x^2 - 2xy + y^2$, implying $xy \leq \frac{1}{2}(x^2 + y^2)$ with equality only occuring when $x = y$, we recognize that

$$Cov(A,B) \leq \frac{1}{2}\left(Var(A) + Var(B)\right)$$

with equality only occuring when $Var(A) = Var(B) = Cov(A,B)$. To see this, observe that

$$\begin{aligned}
Cov(A,B) &= \frac{1}{N}\left((r_{A,1}-\bar{r}_A)(r_{B,1}-\bar{r}_B)+\cdots+(r_{A,N}-\bar{r}_A)(r_{B,N}-\bar{r}_B)\right) \\
&\le \frac{1}{N}\left(\frac{1}{2}((r_{A,1}-\bar{r}_A)^2+(r_{B,1}-\bar{r}_B)^2)+\cdots+\frac{1}{2}((r_{A,N}-\bar{r}_A)^2+(r_{B,N}-\bar{r}_B)^2)\right) \\
&= \frac{1}{2}\left(\frac{1}{N}((r_{A,1}-\bar{r}_A)^2+\cdots+(r_{A,N}-\bar{r}_A)^2)\right) \\
&\quad + \frac{1}{2}\left(\frac{1}{N}(r_{B,1}-\bar{r}_B)^2+\cdots+(r_{B,N}-\bar{r}_B)^2)\right) \\
&= \frac{1}{2}\left(Var(A)+Var(B)\right) \ .
\end{aligned}$$

where equality is possible only if $\bar{r}_{A,i} - \bar{r}_A$ equals $\bar{r}_{B,i} - \bar{r}_B$ for every i, which in turn implies $Var(A) = Var(B) = Cov(A,B)$.

Expanding the expression for $Var(P)$ above, we have

$$Var(P) = C_1(Ret(P))^2 + C_2 Ret(P) + C_3$$

where

$$\begin{aligned}
C_1 &= \left(\frac{1}{\bar{r}_A-\bar{r}_B}\right)^2 \left[Var(A)-2Cov(A,B)+Var(B)\right] , \\
C_2 &= \frac{-2\bar{r}_B}{(\bar{r}_A-\bar{r}_B)^2}\left[Var(A)-2Cov(A,B)+Var(B)\right] \ + \ 2\frac{Cov(A,B)-Var(B)}{\bar{r}_A-\bar{r}_B} , \\
C_3 &= \left(\frac{\bar{r}_B}{\bar{r}_A-\bar{r}_B}\right)^2 \left[Var(A)-2Cov(A,B)+Var(B)\right] \\
&\quad - \ \frac{2\bar{r}_B}{\bar{r}_A-\bar{r}_B}\left[Cov(A,B)-Var(B)\right] \ + \ Var(B) \ .
\end{aligned}$$

Note $C_1 \geq 0$ and $C_2 = 0$ if $C_1 = 0$. Accordingly, the set of points in the plane

$$\{(Var(P), Ret(P)) : \omega_A + \omega_B = 1\}$$

forms a *parabola* opening to the right if

$$Var(A) - 2Cov(A,B) + Var(B) > 0$$

and otherwise is identically $Var(B)$ (which in this case also equals $Var(A)$.)

Let us consider the case that $C_1 > 0$. Then the efficient frontier is the upper half of the above parabola, having as a boundary the vertex, or far left point, of the parabola above. This can be found only using high school algebra as follows. Note that

$$\begin{aligned} Var(P) &= C_1(Ret(P))^2 + C_2 Ret(P) + C_3 \\ &= C_1\left(Ret(P))^2 + \frac{C_2}{C_1}Ret(P) + \frac{C_3}{C_1}\right) \\ &= C_1\left(\left(Ret(P) + \frac{C_2}{2C_1}\right)^2 - \left(\frac{C_2}{2C_1}\right)^2 + \frac{C_3}{C_1}\right), \end{aligned}$$

implying that the minimum value of $Var(P)$ occurs when $Ret(P) = -\frac{C_2}{2C_1}$, and hence is

$$C_3 - \frac{C_2^2}{4C_1}.$$

We can actually find the asset allocation associated to this minimum variance. Using (2.1), one can substitute $C_3 - \frac{C_2^2}{4C_1}$ for $Var(P)$ and solve for ω_A using the quadratic formula, ultimately yielding

$$\omega_A = \frac{Var(B) - Cov(A,B)}{Var(A) - 2Cov(A,B) + Var(B)},$$

and, plugging this value for ω_A into 2.1, yielding a minimum variance

$$\frac{Var(A)Var(B) - (Cov(A,B))^2}{Var(A) - 2Cov(A,B) + Var(B)}$$

with associated return

$$\frac{\bar{r}_A(Var(B) - Cov(A,B)) + \bar{r}_B(Var(A) - Cov(A,B))}{Var(A) - 2Cov(A,B) + Var(B)}.$$

We deal with the case that $C_1 = 0$, $\bar{r}_A \neq \bar{r}_B$ as follows. If $C_1 = 0$ the portfolio would always have variance $Var(A) = Var(B)$ regardless of the asset allocation. Any desired *return* could be obtained via (2.3), yielding an efficient frontier curve consisting of a vertical line crossing the x-axis at $Var(A)$. (In financial practice this should never happen; it would be the equivalent of a scenario where one could short a guaranteed investment paying one percentage in order to purchase another guaranteed investment paying a higher percentage, in arbitrarily large amounts.)

We now consider the special case that $\bar{r}_A = \bar{r}_B$. In this case the return is automatically $\bar{r}_A$, and the efficient frontier is a point in the

Cartesian plane whose y coordinate is $\bar{r}_A$ and whose x coordinate is the minimum possible variance of the portfolio.

If $Var(A) + Var(B) = 2Cov(A, B)$, then by following the preceding argument we find that the efficient frontier is the point $(Var(A), \bar{r}_A)$. If $Var(A) + Var(B) \neq 2Cov(A, B)$, then, using (2.1), we have

$$\frac{Var(P)}{Var(A) - 2Cov(A, B) + Var(B)}$$
$$= \omega_A^2 + \frac{2[Cov(A, B) - Var(B)]}{Var(A) - 2Cov(A, B) + Var(B)}\omega_A + \frac{Var(B)}{Var(A) - 2Cov(A, B) + Var(B)}$$
$$= \left(\omega_A + \frac{Cov(A, B) - Var(B)}{Var(A) - 2Cov(A, B) + Var(B)}\right)^2$$
$$- \left(\frac{Cov(A, B) - Var(B)}{Var(A) - 2Cov(A, B) + Var(B)}\right)^2 + \frac{Var(B)}{Var(A) - 2Cov(A, B) + Var(B)} .$$

Setting $\omega_A = -\dfrac{Cov(A, B) - Var(B)}{Var(A) - 2Cov(A, B) + Var(B)}$ yields a minimum portfolio variance of

$$\frac{Var(A)Var(B) - (Cov(A, B))^2}{Var(A) - 2Cov(A, B) + Var(B)} .$$

Note in this scenario, the efficient frontier might be viewed as a horizontal ray in the xy-plane terminating at the point whose x coordinate is the above minimum variance and y coordinate is $\bar{r}_A$.

We make an aside that might be of interest to our more theoretically inclined readers. The variance, being a sum of nonnegative numbers, is of course greater than or equal to zero. We have already observed that $Var(A) - 2Cov(A, B) + Var(B) \geq 0$. Hence the numerator above for the minimum variance must be nonnegative, yielding the well-known *covariance inequality*

$$(Cov(A, B))^2 \leq Var(A)Var(B) .$$

Let us summarize the above discussion of the efficient frontier of two-fund portfolios.

The Efficient Frontier of Two Fund Portfolios. *Suppose a portfolio is distributed between assets A and B, with the corresponding weights of allocation given by ω_A and ω_B, where $\omega_A + \omega_B = 1$. Suppose the mean returns of assets A and B are denoted by $\bar{r}_A$ and $\bar{r}_B$, the variances of the returns of assets of A and B are denoted by $Var(A)$ and $Var(B)$,* *and the covariance of the returns of the assets A and B is given by $Cov(A,B)$.*

If $\bar{r}_A \neq \bar{r}_B$ and $Var(A) - 2Cov(A,B) + Var(B) \neq 0$, then the efficient frontier is the upper half of the parabola

$$x = \left(\frac{y - \bar{r}_B}{\bar{r}_A - \bar{r}_B}\right)^2 [Var(A) - 2Cov(A,B) + Var(B)] + 2\left(\frac{y - \bar{r}_B}{\bar{r}_A - \bar{r}_B}\right)[Cov(A,B) - Var(B)] + Var(B).$$

Here the minimum variance is

$$\frac{Var(A)Var(B) - (Cov(A,B))^2}{Var(A) - 2Cov(A,B) + Var(B)}$$

associated to the return

$$\frac{\bar{r}_A(Var(B) - Cov(A,B)) + \bar{r}_B(Var(A) - Cov(A,B))}{Var(A) - 2Cov(A,B) + Var(B)}$$

and asset allocation

$$\omega_A = \frac{Var(B) - Cov(A,B)}{Var(A) - 2Cov(A,B) + Var(B)}\ ,\ \omega_B = 1 - \omega_A\ . \tag{2.3}$$

If $\bar{r}_A = \bar{r}_B$ and $Var(A) + Var(B) \neq 2Cov(A,B)$, then $Ret(P) = \bar{r}_A = \bar{r}_B$ and the efficient frontier is a horizontal ray in the Cartesian plane. The asset allocation minimizing the variance is given by

$$\omega_A = \frac{Var(B) - Cov(A,B)}{Var(A) - 2Cov(A,B) + Var(B)}\ ,\ \omega_B = 1 - \omega_A,$$

with the minimum variance being

$$\frac{Var(A)Var(B) - (Cov(A,B))^2}{Var(A) - 2Cov(A,B) + Var(B)}$$

with the same asset allocation as in (2.3)*.*

If $\bar{r}_A \neq \bar{r}_B$ and $Var(A) + Var(B) = 2Cov(A,B)$, then the efficient frontier corresponds to a vertical line in the Cartesian plane. Here, $Var(P) = Var(A) = Var(B)$ regardless of asset allocation, and the return is given by $Ret(P) = \omega_A\bar{r}_A + \omega_B\bar{r}_B$.

If $\bar{r}_A = \bar{r}_B$ and $Var(A) + Var(B) = 2Cov(A,B)$, then the efficient frontier is the single point $(Var(A), \bar{r}_A)$ in the Cartesian plane. For *any choice of* ω_A *and* ω_B *satisfying* $\omega_A + \omega_B = 1$*, the portfolio variance will be* $Var(A) = Var(B)$ *with return* $\bar{r}_A = \bar{r}_B$*.*

We now provide two examples of efficient frontiers associated to portfolios containing two assets. Figure 1 provides the efficient frontiers associated to the Vanguard Total Stock Market Index Fund (VTSAX) and the Vanguard Total Bond Market Index Fund (VBTLX) in both taxable and tax-privileged accounts, the taxable account indicated in green and the tax-privileged account in blue.

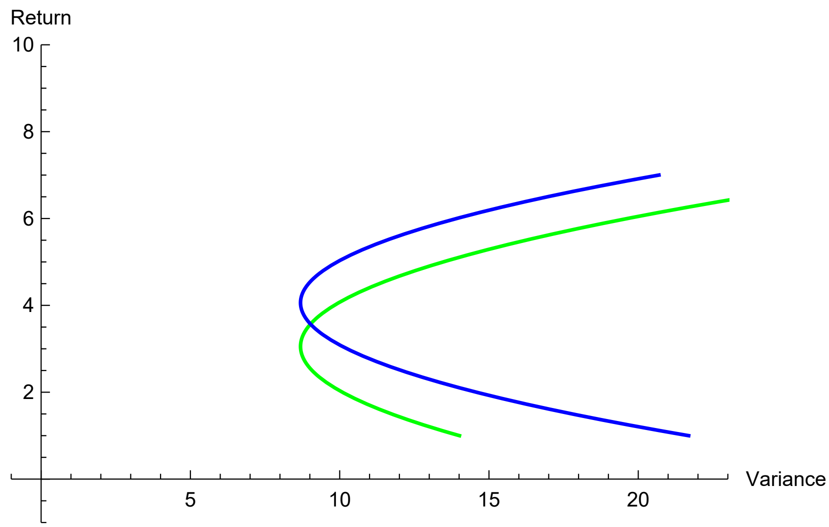

FIGURE 1. Efficient Frontiers of VTSAX, VTBLX (Taxable and Tax-Privileged)

Figure 2 provides the efficient frontiers associated to the Vanguard Total Stock Market Index Fund (VTSAX) and the Vanguard Total International Stock Index Fund (VTIAX) in both taxable and tax-privileged accounts, the taxable account indicated in green and the tax-privileged account in blue.

The formula provided indicates the asset allocation that would minimize the variance in these portfolios. Using the market data at the time of this writing, for the two-fund portfolio consisting of VTSAX and VTBLX in either a taxable or tax-privileged account, the portfolio variance is minimized with the allocation VTSAX (6.8%), VTBLX (93.2%).

For a two-fund portfolio consisting of VTSAX and VTIAX in either a taxable or tax-privileged account, the portfolio variance is minimized with the allocation VTSAX (133.1%), VTIAX (-33.1%). This indicates that the variance would be minimized by *shorting* a position in VTIAX

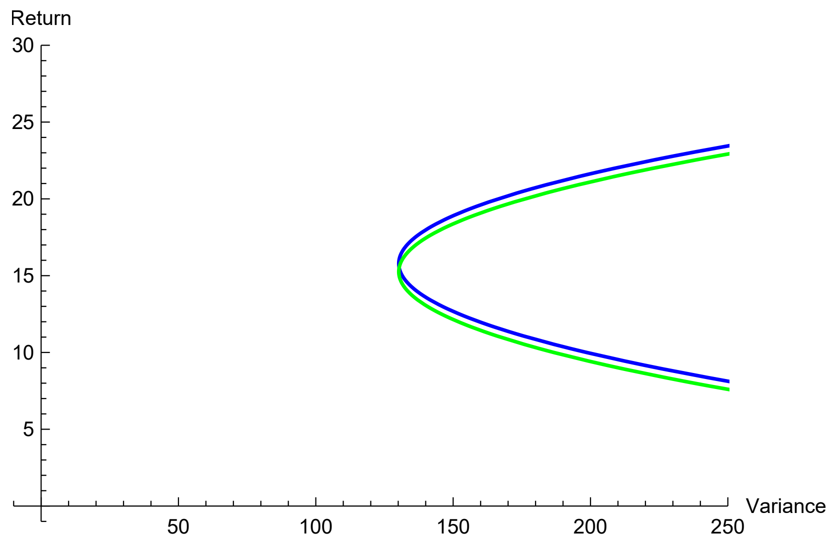

FIGURE 2. Efficient Frontiers of VTSAX, VTIAX (Taxable and Tax-Privileged)

and using the associated funds to have a position in VTSAX exceeding that of the overall portfolio value.

In each figure the two efficient frontier curves are vertical translates of each other. The reason why this occurs is that, in the formula for the efficient frontier curve given previously, once $Var(A)$, $Var(B)$, and $Cov(A, B)$ are provided, x is a function of $\left(\frac{y-\bar{r}_B}{\bar{r}_A-\bar{r}_B}\right)$, and although $\bar{r}_A$ and $\bar{r}_B$ are affected by taxes, $Var(A)$, $Var(B)$, and $Cov(A, B)$ are not.

It is also worthwhile to know that the asset allocation typically minimizing the variance, namely

$$\omega_A = \frac{Var(B) - Cov(A, B)}{Var(A) - 2Cov(A, B) + VarB}$$

with $\omega_B = 1 - \omega_A$, does *not* depend on either $\bar{r}_A$ or $\bar{r}_B$, and hence is unaffected by the taxes on the respective assets. This fact, together with the apparant close similarity of the efficient frontier curves exhibited above, leads us to believe that considerations of the efficient frontier in and of itself should not lead to adjustments of a portfolio based on whether it lies in a taxable or tax-privileged account. (Of course, the choice of whether to *place* assets in a taxable or tax-privileged account is an entirely diffent matter.)

3. The efficient frontier of three fund portfolios

We now describe how one may calculate the efficient frontier of *three*-fund portfolios. The mathematics is more involved here, taking advantage of fundamental concepts in multivariable calculus. To simplify the calculations, we will assume the quantities considered lie in real-world scenarios where no pair of assets is perfectly correlated, all assets have returns with nonzero variance, and no pair of mean returns is identical. In exotic scenarios where either of these holds, one can proceed in a case-by-case basis as we did previously with the two-fund portfolios.

The portfolio we consider consists of holdings in assets A, B, and C, with a respective allocation of weights ω_A, ω_B, and ω_C, where

$$\omega_A + \omega_B + \omega_C = 1 \ . \tag{3.1}$$

The return of the portfolio is given by

$$Ret(P) = \omega_A \bar{r}_A + \omega_B \bar{r}_B + \omega_C \bar{r}_C$$

and the variance of the portfolio is in this case

$$\begin{aligned} Var(P) = {} & \omega_A^2 Var(A) + \omega_B^2 Var(B) + \omega_C^2 Var(C) \\ & + 2[\omega_A\omega_B Cov(A,B) + \omega_A\omega_C Cov(A,C) + \omega_B\omega_C Cov(B,C)] \ . \end{aligned}$$

Using (3.1), we express $Ret(P)$ and $Var(P)$ in terms of just ω_A and ω_B, yielding

$$Ret(P) = \omega_A(\bar{r}_A - \bar{r}_C) + \omega_B(\bar{r}_B - \bar{r}_C) + \bar{r}_C$$

and

$$\begin{aligned} Var(P) &= \omega_A^2 Var(A) + \omega_B^2 Var(B) + (1-\omega_A-\omega_B)^2 Var(C) \\ &\quad + 2[\omega_A\omega_B Cov(A,B) + \omega_A(1-\omega_A-\omega_B) Cov(A,C) \\ &\quad + \omega_B(1-\omega_A-\omega_B) Cov(B,C)] \\ &= \omega_A^2[Var(A) + Var(C) - 2Cov(A,C)] \\ &\quad + \omega_B^2[Var(B) + Var(C) - 2Cov(B,C)] \\ &\quad + 2\omega_A\omega_B[Var(C) + Cov(A,B) - Cov(A,C) - Cov(B,C)] \\ &\quad + 2\omega_A[-Var(C) + Cov(A,C)] \\ &\quad + 2\omega_B[-Var(C) + Cov(B,C)] + Var(C) \ . \end{aligned}$$

Note we can express $Ret(P)$ and $Var(P)$ as

$$Ret(P) = c_1\omega_A + c_2\omega_B + \bar{r}_C \ , \tag{3.2}$$

$$Var(P) = c_3\omega_A^2 + c_4\omega_A\omega_B + c_5\omega_B^2 + c_6\omega_B + c_7\omega_A + c_8 \ , \tag{3.3}$$

where

$$\begin{aligned}
c_1 &= \bar{r}_A - \bar{r}_C \\
c_2 &= \bar{r}_B - \bar{r}_C \\
c_3 &= Var(A) + Var(C) - 2Cov(A, C) \\
c_4 &= 2[Var(C) + Cov(A, B) - Cov(A, C) - Cov(B, C)] \\
c_5 &= Var(B) + Var(C) - 2Cov(B, C) \\
c_6 &= 2[-Var(C) + Cov(B, C)] \\
c_7 &= 2[-Var(C) + Cov(A, C)] \\
c_8 &= Var(C) \ .
\end{aligned}$$

We now assume that the return $Ret(P)$, viewed as a function of ω_A and ω_B, is fixed and find the minimal possible value of $Var(P)$, also viewed as a function of ω_A and ω_B. Via the method of Lagrange multipliers, the minimum will occur for values ω_A and ω_B for which there exists a constant λ satisfying the simultaneous equations

$$\begin{aligned}
\frac{\partial}{\partial \omega_A} Ret(P)(\omega_A, \omega_B) &= \lambda \frac{\partial}{\partial \omega_A} Var(P)(\omega_A, \omega_B) \\
\frac{\partial}{\partial \omega_B} Ret(P)(\omega_A, \omega_B) &= \lambda \frac{\partial}{\partial \omega_B} Var(P)(\omega_A, \omega_B) \\
Ret(P)(\omega_A, \omega_B) &= Ret(P) \ .
\end{aligned}$$

Taking the partial derivatives, this reduces to the simulaneous equations

$$\begin{aligned}
c_1 &= \lambda[2c_3\omega_A + c_4\omega_B + c_7] \\
c_2 &= \lambda[c_4\omega_A + 2c_5\omega_B + c_6] \\
Ret(P)(\omega_A, \omega_B) &= Ret(P) \ .
\end{aligned}$$

Hence

$$\lambda = \frac{c_1}{2c_3\omega_A + c_4\omega_B + c_7} = \frac{c_2}{c_4\omega_A + 2c_5\omega_B + c_6} \ ,$$

implying

$$\omega_A(c_1c_4 - 2c_2c_3) + \omega_B(2c_1c_5 - c_2c_4) + c_1c_6 - c_2c_7 = 0 \ .$$

As

$$\omega_A c_1 + \omega_B c_2 + \bar{r}_C - Ret(P) = 0 \ ,$$

we can solve for ω_A and ω_B, yielding

$$\omega_A = \frac{(Ret(P) - \bar{r}_C)(2c_1c_5 - c_2c_4) + c_1c_2c_6 - c_2^2c_7}{2c_1^2c_5 + 2c_2^2c_3 - 2c_1c_2c_4} \ ,$$

$$\omega_B = \frac{(Ret(P) - \bar{r}_C)(2c_2c_3 - c_1c_4) + c_1c_2c_7 - c_1^2c_6}{2c_2^2c_3 + 2c_1^2c_5 - 2c_1c_2c_4} \ .$$

Substituting these expressions for ω_A and ω_B into (3.3) for the variance, we yield the minimum variance associated to a desired return $Ret(P)$. It is important to recognize that in the two-fund portfolio scenario, the variance of the portfolio in general was a function of the return; here we are obtaining the *minimum possible variance associated to a given return.* Note that as ω_A and ω_B are linear in $Ret(P)$ and (3.3) provides an expression for the minimal variance associated to a given return as a quadratic function of that return, we again find that the efficient frontier constitutes the upper half of a parabola in the Cartesian plane; the x-axis being associated to the minimal variance associated to a given return, and the y-axis being associated to that given return.

In principle, having expressed the minimal variance of the return as a quadratic in that return, we could find the minimal variance over all asset allocations from that quadratic expression. This is rather unwieldy, however, and we proceed differently. We take (3.3) and minimize $Var(P)$ over all possible ω_A and ω_B, recognizing that optimizing ω_A and ω_B will have to provide *some* portfolio return, whose associated minimum variance would of course have to be the minimal variance over all asset allocations. The beauty of this approach is that it provides us not only the minimal possible variance, but also the asset allocations ω_A, ω_B, and ω_C associated to it as well as the associated return.

From (3.3), we recognize that

$$\frac{\partial Var(P)}{\partial \omega_A} = 2c_3\omega_A + c_4\omega_B + c_7$$

$$\frac{\partial Var(P)}{\partial \omega_B} = c_4\omega_A + 2c_5\omega_B + c_6 \ .$$

Setting both of these to zero and solving for ω_A and ω_B, we find the minimum variance occurs for

$$\omega_A = \frac{2c_5c_7 - c_4c_6}{c_4^2 - 4c_3c_5}$$

$$\omega_B = \frac{2c_3c_6 - c_4c_7}{c_4^2 - 4c_3c_5} \ .$$

Using (3.3), the minimum variance is then

$$\frac{c_3c_6^2 - c_4c_6c_7 + c_5c_7^2 + c_4^2c_8 - 4c_3c_5c_8}{c_4^2 - 4c_3c_5}$$

with associated return

$$\frac{2c_2c_3c_6 - c_1c_4c_6 - c_2c_4c_7 + 2c_1c_5c_7 + c_4^2\bar{r}_C - 4c_1c_5\bar{r}_C}{c_4^2 - 4c_3c_5} .$$

We summarize our discussion of the efficent frontier of a portfolio containing three assets with the following.

The Efficient Frontier of Three Fund Portfolios. *Suppose a portfolio is distributed between assets A, B, and C, with the corresponding weights of allocation being given by ω_A, ω_B, and ω_C where $\omega_A + \omega_B + \omega_C = 1$. Suppose the mean returns of assets A, B, and C are respectively denoted by $\bar{r}_A$, $\bar{r}_B$, and $\bar{r}_C$; the variances of the returns of the assets A, B, and C are denoted by $Var(A)$, $Var(B)$, and $Var(C)$; and the covariance of the returns between these assets are denoted by $Cov(A,B)$, $Cov(A,C)$, and $Cov(B,C)$. Then the efficient frontier is the upper half of the parabola*

$$x = c_3\omega_A^2 + c_4\omega_A\omega_B + c_5\omega_B^2 + c_6\omega_B + c_7\omega_A + c_8 , \tag{3.4}$$

where the constants $c_1, \ldots, c_8$ are as above and

$$\omega_A = \frac{(y - \bar{r}_C)(2c_1c_5 - c_2c_4) + c_1c_2c_6 - c_2^2c_7}{2c_1^2c_5 + 2c_2^2c_3 - 2c_1c_2c_4} ,$$

$$\omega_B = \frac{(y - \bar{r}_C)(2c_2c_3 - c_1c_4) + c_1c_2c_7 - c_1^2c_6}{2c_2^2c_3 + 2c_1^2c_5 - 2c_1c_2c_4} .$$

The asset allocation minimizing the variance is given by

$$\omega_A = \frac{2c_5c_7 - c_4c_6}{c_4^2 - 4c_3c_5} ,$$

$$\omega_B = \frac{2c_3c_6 - c_4c_7}{c_4^2 - 4c_3c_5}$$

$$\omega_C = \frac{c_4^2 + c_4(c_6 + c_7) - 2(c_3(2c_5 + c_6) + c_5c_7)}{c_4^2 - 4c_3c_5} ,$$

with minimum variance

$$\frac{c_3c_6^2 - c_4c_6c_7 + c_5c_7^2 + c_4^2c_8 - 4c_3c_5c_8}{c_4^2 - 4c_3c_5}$$

and associated return

$$\frac{2c_2c_3c_6 - c_1c_4c_6 - c_2c_4c_7 + 2c_1c_5c_7 + c_4^2\bar{r}_C - 4c_1c_5\bar{r}_C}{c_4^2 - 4c_3c_5} .$$

We illustrate this result by providing the efficient frontiers associated to two portfolios. In Figure 3, we have the efficient frontiers associated to VTSAX, VBTLX, and VTIAX in both taxable and tax-privileged accounts, the taxable account indicated in green and the tax-privileged account in blue. It is noteworthy that, as in the two-fund case treated earlier, the efficient frontier associated to the tax-privileged account is a vertical translate of the parabolic efficient frontier of the taxable account. Note this follows as the formulas for the variances for the returns of assets of a portfolio as well as the associated covariances between assets are invariant under uniform translations of the returns of the individual asset classes. In other words, replacing each $r_{A,i}$ by $r_{A,i} - \sigma_A$ and similarly for assets B and C does not alter either the variance of any asset class or any of the associated covariances.

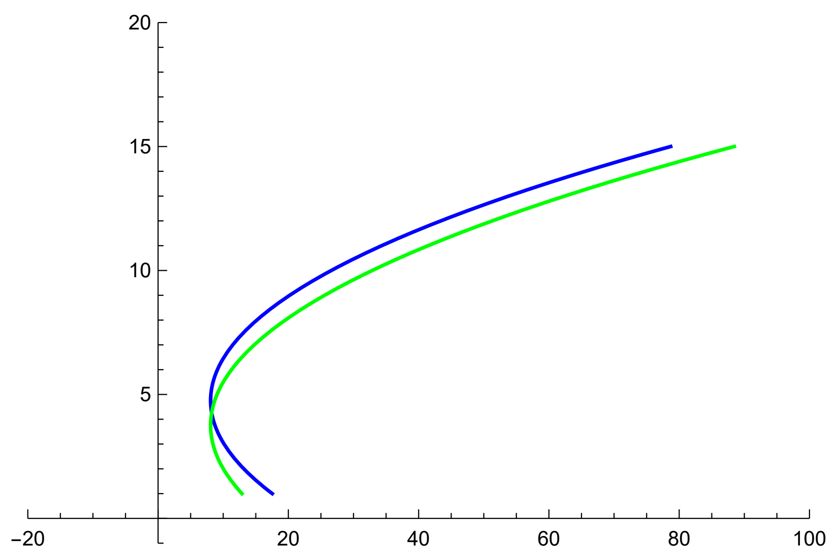

FIGURE 3. Efficient Frontiers of VTSAX, VTIAX, and VTBLX (Taxable and Tax-Privileged)

The efficient frontiers are remarkably similar, both having the same minimum variance, a variance that moreover can be obtained by the same asset allocation regardless of whether or not the portfolio lies in a taxable account. Using the above result we readily compute, using the market data at the time of writing of this paper, the asset allocation minimizing the variance: VTSAX (17.1%), VTBLX (92.0%), and VTIAX (−9.1%).

Observe that the values for ω_A, ω_B, and ω_C provided that minimize the variance depend on the coefficients $c_3, \ldots, c_7$ but *not* on c_1 and

c_2, and hence do not depend on any of $\bar{r}_A$, $\bar{r}_B$, and $\bar{r}_C$. Hence, as in the two-fund case, the portfolio allocation that minimizes the variance will be the same regardless of whether the portfolio lies in a taxable or tax-privileged account. This leads us to conclude that, based on considerations involving the efficient frontier alone, investors should not allocate assets in a portfolio differently based on whether the portfolio lies in a taxable or tax-privileged account.

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