A Note on Game Theoretic Approach to Detect Arbitrage Strategy: Application in the Foreign Exchange Market

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Abstract. A game theoretic approach to detect arbitrage strategy in a foreign exchange market is proposed. 5 propositions are given and then the optimal arbitrage path is derived.

Keywords: Arbitrage strategy; Exchange rate matrix, Foreign exchange market; Game theory

1 Introduction. Consider a foreign exchange market with n currencies. Let a_{ij} is the exchange rate of i-th currency with respect to j-th currency. Denote the exchange rate matrix, that is $A = (a_{ij})$. Under no arbitrage assumption, then the direct, triangular, and any other types of arbitrage don't exist. Ma (2008) detected arbitrage opportunities in a foreign exchange market. He used maximum eigenvalue of the exchange rates matrix A. Ma (2008) argued that Λ is an arbitrage indicator. When there is an arbitrage, then $a_{ij} = \frac{w_i}{w_j}\varepsilon_{ij}$ for some i, j, where w_i is the intrinsic value of i-th currency. Hao (2009) detected the applied the method of Ma's foreign exchange market considering the bid-ask transactions. The following propositions are useful.

Proposition 1. If direct arbitrage doesn't exist, then $a_{ij} = \frac{1}{a_{ji}}$, $b_{ij} + b_{ji} = 0$, where $b_{ij} = \log(a_{ij})$. Considering b_{ij} as a game, then it is a sum zero game.

Proposition 2. When there is no triangular arbitrage , then $a_{ik}a_{kj} = a_{ij}$ then

$$A_{ij}^2 = \sum_{k=1}^{n} a_{ik}a_{kj} = na_{ij} = nA_{ij}.$$

Therefore, $A^3 = n^2 A, \ldots A^k = n^{k-1}A$.

Remark 1. Under the no direct and triangular arbitrage opportunities, then

$$\sum_{k=1}^{n}(a_{ik}a_{kj} - a_{ij}) = 0 \text{ and } \sum_{g=1}^{n}\sum_{k=1}^{n}(a_{ig}a_{gk}a_{kj} - a_{ij}) = 0.$$

Remark 2. The arbitrage path is obtained for some m such that $A^m \neq n^{m-1}A$. Hence, some relations like them exist in Remark 1 exist and show the arbitrage path (strategy).

Proposition 3. Under the no arbitrage assumption, $A^k = n^{k-1}A$ then for some $h \in (-\delta, \delta)$, the moment generating function of matrix A

$$M_A(h) = M(h) = e^{hA} = \sum_{j=0}^{\infty}\frac{(hA)^j}{j!} = I + \frac{e^{nh} - 1}{n}A$$

where I is n*n identity matrix and

$$\frac{\partial^k M(h)}{\partial h^k}|_{h=0} = A^k, \qquad k = 1,2, \ldots$$

Proposition 4. If there is triangular arbitrage, an optimal weighted arbitrage path with weights $\pi_k, k = 1,2, \ldots$ is obtained by

$$max \sum_{k=1}^{n}\pi_k(a_{ik}a_{kj} - a_{ij}),$$

such that $\sum_{k=1}^{n} \pi_k = 1$, where $\pi_k = 1$ for $k = argmax_{k^*}(a_{ik^*}a_{k^*j} - a_{ij})$ and zero otherwise.

2 Game theoretic approach. It is well-known that there is an equivalence between linear programming (LP) problems and zero-sum games in the sense that any two-person zero-sum game can convert to an LP problem and vice-versa (see, Raghavan,1994). A natural question arises is which row of A should be selected? Here, based on game theory solution is proposed. That is, a randomized strategy $\{q_j\}_{j=1}^{n}$ (probability measure) is designed such that for each row the expectation of investor's profit is maximized. The necessary condition to this end is

$$\sum_{j=1}^{n} q_j a_{1j} = \cdots = \sum_{j=1}^{n} q_j a_{nj},$$

where a matrix equivalent form is $Aq = a^* 1$, for some arbitrary a^* where q is the vector of probabilities. When there is an arbitrage opportunity, then the largest eigenvalue is n and others are negative. If all of them are non-zero, then the inverse of matrix A exists and

$$q = a^{**} A^{-1} 1.$$

Proposition 5. If there is triangular arbitrage, an optimal weighted arbitrage path with weights $\pi_k, k = 1,2, \dots$ is obtained by

$$max_{\pi_k} \sum_{k=1}^{n} \pi_k (\varepsilon_{ik}\varepsilon_{kj} - \varepsilon_{ij})$$

Such that $\sum_{k=1}^{n} \pi_k = 1$, where $\pi_k = 1$ for $k = argmax_{k^*}(\varepsilon_{ik^*}\varepsilon_{k^*j} - \varepsilon_{ij})$ and zero otherwise.

Remark 3. The algorithm for finding the arbitrage path is given as follows

1. Find i_1 as described ib above.
2. Find $k_1^* = argmax_{k^*}(\varepsilon_{ik^*}\varepsilon_{k^*j} - \varepsilon_{ij})$ and let $i_2 = k_1^*$.
3. Continue to find time points $i_3, \dots, i_m$.
4. Compute $a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_{(m-1)} i_m} a_{i_m i_1}$.
5. Set $i_1 = 2$ and repeat steps 2,3,4.
6. Find the maximum of $a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_{(m-1)} i_m} a_{i_m i_1}$. It is the optimal arbitrage path.

References

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