The optimal investment-consumption-insurance strategy with Inflation Risk and Stochastic Income in an Itô Lêvy Setting

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Abstract

In this study, we focus on finding the possible optimal portfolio where there is consumption, stochastic income, purchasing of life insurance and investment in risky and risk-less investments are made. The stock follows a jump diffusion process and the bond is linked to inflation making the two risky and we have the investment position taken also on the money market account. The wealth process is determined by the different generations of the life of an investor, that is before the investor dies and after the investor dies. We also look into the optimal portfolio the beneficiaries benefit after the investor dies. We applied the concept of change of probability measures considering the Girsanov's and the Radon-Nikodym theorems, we found the generator of the Backward Stochastic differential equations defined and employed the Hamilton-Jacobi-Bellman dynamic programming (HJB) in finding the stochastic optimal controls of interest.

Keywords: Optimal portfolio, jump diffusion process, Girsanov's theorem, Radon-Nikodym theorem, Hamilton-Jacobi-Bellman dynamic programming (HJB), stochastic optimal controls.

1. Introduction

Over the past years life insurance have been studied extensively incorporating the different ways of making it sound to the field of applied mathematics and statistics in the financial context life insurance has also been analyzed, (Hakansson, 1969) and (Lewis, 1989) developed insights into the life insurance demand.They demonstrated how uncertain the nature of mortality is, the utility and strength of bequest desire, risk aversion, increasing household wealth and intertemporal rates of substitution impact the decision of life insurance and the type of life insurance to purchase. In 2015 Kronborg and Steffensen (Kronborg and Steffensen, 2015) studied optimal consumption, investment and life insurance with surrender option guarantee where an investor's lifetime was considered in investing with deterministic labour income and possibly willing to continue to invest in a Black-scholes market and buy life insurance or annuities,they also considered that the savings were not supposed to be negative which is a non-negative reserve constraint in the context of life insurance and one constraint taken into account was that the savings did earn a minimum return as an interest rate guarantee. Other references Guambe and Kufakunesu introduced optimal investment consumption and life insurance with capital constraints. (Huang et al., 2008) believes that there's a blinding light between the way financial advisers sell life insurance and how they sell or promote investment products that they're presented as if they were a "separation theorem" that justified their relative invariance. Their aim was to analyze the decision of how much life insurance a family unit should have to protect against the loss of its breadwinner and how the family can distribute their financial resources between risky assets and risk free assets.

Different scholars have in depth researched on the different approaches of studying optimal investment consumption. They further extended the study by including conditions and constraints considering the economic downturns eg (inflation), how the assets are to be allocated and how consumption is to occur to obtain maximal investment benefits. In 2019 (Li and Xia, 2019) studied the optimal investment strategy under the disordered return and random inflation where in it the optimal portfolio problem was vested in such a way that the inflation risk was taken into account, the risky asset price is changed by the impacts of major events which leads to the asset returns disorder at a random time and when investors have partial information in the financial market, (Maslov and Zhang, 1998) researched on optimal investment strategy for a risky assets where investment in a portfolio of assets subject to multiplicative Brownian motion, the strategy providing the maximal typical long term growth rate of the investors capital, determination of the optimal fraction of capital an investor is supposed to keep in risky assets and the different assets weights in an optimal portfolio where volatility and average return of an asset are considered to be indicators that determines its optimal weight.

(Li and Xia, 2019) studied the optimal investment strategy under the disordered return and random inflation on how to adjust the investment strategy in order to cope with the inflation risk as its important to the investor. The Martingale approach have been previously used by some scholars to study the portfolio with inflation risk. In 2011 (Siu, 2011) gave the optimal allocation strategy of the long term strategic assets with the inflation risk. (Brennan and Xia, 2002) studied the dynamic optimal portfolio with inflation, (Wachter, 2002) studied Portfolio and Consumption Decisions under Mean-Reverting Returns: An Exact Solution for Complete Markets. In 2015 the impacts of inflation and the jumps on optimal asset allocations of an investor under asset prices with a jump environment were investigated by (Fei et al., 2015). Two optimal consumption and portfolio problems with the financial markets of Markovian switching and inflation research was carried out by (Fei, 2013). In 2021 (Doctor, 2021) studied application of Generalized Geometric Itô-Lévy Process to Investment-Consumption-Insurance Optimization Problem under Inflation Risk using the martingale approach to maximize the utility of an agent investing in risky and risk-less assets.

In this study the objective is to use the stochastic programming approach to determine the optimal investment, consumption and insurance strategy considering the inflation risk and the stock's diffusion jump in a finite time interval [0, T]. The stochastic programming approach was also used by (Shapiro, 2008) on optimization under uncertainty where in it computational complexity and risk averse approaches to two and multistage stochastic programming problems were discussed in depth but on the recent work emphasis was on the study of diffusion jumps and BSDE equation in maximizing the portfolio using the Hamilton-Jacobi-Bellman dynamic programming (HJB) and the BSDE approach. Recommendations on other alternative approaches refer to (Doctor, 2021) and (Kronborg and Steffensen, 2015) on the martingale approach.
The article is structured as follows. In Section 2, the model framework is presented in detail with the characteristics of the positions of investment. Section 3 has the solutions to the different generations of the portfolio problem in obtaining the optimal strategies. Section 4 concludes followed by the references.

2. Model Description and Framework

We suppose that our model is built in a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ which follows the characteristics of the Brownian motion $B_t = \{B_t^0, B_t^1, B_t^2, 0 \leq t \leq T\}$ in a finite time horizon, i.e $0 \leq t \leq T$. to be considered. The Brownian motions $\langle B_t^0, B_t^1, B_t^2 \rangle$ are all correlated in a correlation matrix such that $dB_t^0 \cdot dB_t^1$, $dB_t^0 \cdot dB_t^2$,$dB_t^1 \cdot dB_t^2 = \rho_{i,j} dt$ for $i, j \in \{0, 1, 2\}$, are correlated with the correlation coefficient $-1 \leq \rho \leq 1$. For $\rho = 1$ then the Brownian motions are strongly correlated and for $\rho = -1$ then they are negatively correlated.

We take into account an important economic factor inflation, since the value for money in a long term investment has to be considered as inflation can lead to reduction in the value of investment returns of an investor. This was also discussed by (Zhang and Zheng, 2019) that inflation could lead to diminution of financial wealth of the insurer when studying optimal investment Reinsurance policy with stochastic interest and inflation rates. We can measure inflation by the inflation rate, by considering the consumer price index (CPI) which includes the consumer price and retail price indices for any change in the index.

The investor inherits in three assets namely, a money market account, an inflation-linked index bond and stock. The wealth of an insurance policy holder is indicated by a portfolio distributing the wealth accordingly with π_1 being the fraction of the wealth at time t held by the stock, π_2 being held in the inflation-linked index bond which shall be defined in depth along the descriptions and $(1-\pi_1-\pi_2)$ denotes the fraction held in the money market account such that all the fractions held all add up to 1. We denote the market value of the money market account R_t at a given time t as

$$dR_t = rR_t dt, \tag{1}$$

where $r > 0$ denote the interest rate to be compounded over t.

The dynamics of the stock follows an Itô Lêvy process;

$$dS_t = S_{t-}[\alpha dt + \sigma dB_t^0 + \int_{\mathbb{R}} \gamma(t,\eta)\tilde{J}(d\eta, dt)] \tag{2}$$

where α, σ and $r > 0$ are the stock's drift and volatility rates respectively, $\gamma > -1$ and B_t^0 is the Brownian motion driving the stock. We assume the inequality $\alpha - r > 0$ is satisfied for economic equilibrium purposes.

We consider the consumer good[1] an investor purchases in their time of life in the economy over a time interval satisfying a stochastic process:

$$dQ(t) = Q(t)[Z(t)dt + \varrho(t)dB^1(t)], \;\; Q(0) = Q_0, \tag{3}$$

For all $t \in [0,T]$ with $B^1(t)$ being a Brownian motion driving the consumption good, $\varrho(t)$ is the volatility price index rate of the commodity good and $Z(t)$ is the stochastic drift rate representing the expected inflation rate over time governed by the time dependent Ornstein- Uhlenbeck (OU) process also applied by (Chaiyapo and Phewchean, 2017) and is given by:

$$dZ(t) = \varsigma(t)[\beta(t) - Z(t)]dt + \bar{\varrho}(t)dB^1(t), \tag{4}$$

where $\beta(t)$ is the long-run mean of the inflation rate, $\bar{\varrho}(t)$ is the volatility rate and $\varsigma(t)$ is the rate of mean reversion of the inflation rate. The quantity functions $\varrho(t), \beta(t), \varsigma(t)$and $\bar{\varrho}(t)$ are continuous and deterministic with $t \in [0,T]$. We have the index bond $M(t)$ which is linked to inflation and has the price level process

$$\frac{dM(t)}{M(t)} = k(t)dt + \frac{dQ(t)}{Q(t)} = (k(t) + Z(t))dt + \varrho(t)dB^1(t), \tag{5}$$

where $k(t)$ is the interest rate at time t.

[1]Items that individuals and households buy, they include (packaged goods, clothing, beverages, automobiles, and electronics) for their own use and enjoyment.

The insured have an incoming source of funds which comes in at different time intervals therefore we consider it to be random over a given period of time. The income rate at time t is given by $\psi(Y_t, t)$ where Y_t is referred to as the state variable which is an Itô process such that

$$dY_t = \alpha_1(Y_t, t)dt + \sigma_1(Y_t, t)dX_t, \tag{6}$$

and follows the properties of Itô processes. With the assumption that $\alpha_1(Y_t, t)$ and $\sigma_1(Y_t, t)$ satisfies Lipschitz and growth conditions in Y_t and are continuous so we obtain a unique solution.

The policy holder purchases an insurance policy that pays premiums $P_t > 0$ over a unit time interval, supposedly on monthly basis which is an $\mathcal{F}_t$ adapted and measurable process with

$$\int_0^T P(s)ds < \infty. \tag{7}$$

We consider the lifetime of the policy holder at $t > 0$ defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. The insurance is determined by the retirement and the time of death of the policy holder τ which is finite. The death benefit is paid to the beneficiaries at the time of death of the policy holder. The continuation of the policy can be renegotiated by the family members when the holder dies based on the predetermined terms and conditions of the insurance policy. We consider the probability mass function $f(t_l)$ to be the lifetime of the holder, then the cumulative mass function of t is given by

$$F(t_l) = P(t < t_l) = \int_0^{t_l} f(s)ds,$$

then the probability that $t > t_l$ is the survival function given by

$$G(t_l) = P(t > t_l) = 1 - F(t_l).$$

The instantaneous force of mortality $\mu(t_l)$ for a policy holder to be alive at time t_l is given by

$$\mu(t_l) = \lim_{\Delta t_l \to 0} \frac{P(t_l \leq t < t_l + \Delta t_l | t \geq t_l)}{\Delta t_l}$$

$$= \lim_{\Delta t_l \to 0} \frac{P(t_l \leq t < t_l + \Delta t_l)}{\Delta t_l P(t \geq t_l)}$$

$$\frac{f(t_l)}{1 - F(t_l)} = -\frac{d}{dt_l}(ln(1 - F(t_l))),$$

which is the hazard rate function. The conditional probability of survival of the holder is defined as

$$\bar{F}(t_l) = 1 - F(t_l) = P(t > t_l|\mathcal{F}_t) = exp(-\int_0^{t_l} \mu(s)ds). \quad (8)$$

and the conditional survival probability density of the death of the holder is given by

$$f(t_l) = \mu(t_l)exp(-\int_0^{t_l} \mu(s)ds).$$

With the conditions above, the wealth process is given below where;

S_t is the price of the stock, R_t being the price of the bond, W_t is the wealth of the investor at time $t \in [0, T]$ and π_1, π_2, $(1 - \pi_1 - \pi_2)$ are as defined previously.

The total wealth process at time $t \in [0, T]$, where T is the time of retirement is given by:

1. **Pre-death case**
   For $t < \tau \wedge T$ that is $t \in [0, \tau \wedge T]$ we have,

$$\begin{aligned} dW_t &= \big[W_t\big(\pi_1(\alpha - r) + \pi_2(k(t) + Z(t) - r) + r + \mu(t)\big) \\ &+ \psi(Y_t, t) - \mu(t)P(t) - C(t)\big]dt + W_t\big[\pi_1\sigma dB_t^0 - \pi_2\varrho(t)dB_t^1\big] \\ &+ \pi_1 W_t \int_{\mathbb{R}} \gamma(t, \eta)\tilde{J}(d\eta, dt), \end{aligned} \quad (9)$$

2. **Post-death case**
   We have,

$$\begin{aligned} dW_t &= \big(W_t[\pi_1(\alpha - r) + \pi_2(k(t) + Z(t) - r) + r] - C(t) \\ &+ \psi(Y_t, t)\big)dt + W_t[\sigma\pi_1 dB_t^0 - \pi_2\varrho(t)dB_t^1] \\ &+ \pi_1 W_t \int_{\mathbb{R}} \gamma(t, \eta)\tilde{J}(dt, d\eta) \end{aligned} \quad (10)$$

where $\tau \wedge T = \min\{\tau, T\}$, and $\mu(t)(P(t) - W_t)dt$ corresponds to the risk premium rate to pay for the life insurance at time t, $\psi(Y_t, t)$ is the stochastic income rate and the consumption rate $C(t)$ which are non negative and continuous.

3. Portfolio Selection

3.1 Pre-death case

We employ the Stochastic programming approach in finding the suitable investment strategy $(\pi_1, \pi_2, P(t), C(t))$ that maximizes the terminal wealth of an investor, the consumption and premiums to be paid during the minimum time $[\tau, T]$. Let the hurdle rate be a positive $\mathcal{F}_t$ adapted process defined by $\Phi(t)$ and the utility function of the terminal wealth, consumption and premiums be given as U, we then select a strategy that maximize the performance function given by:

$$
\begin{aligned}
J(0, W(0), \pi_1, \pi_2, C, P) \quad &:= \sup_{(\pi_1,\pi_2,C,P)\in\chi} \mathbb{E}\Bigg[\int_0^{\tau\wedge T} e^{-\int_0^s \Phi(u)du} \\
&\times \quad (U_1(C(s)) + U_2(P(s)))ds \\
&+ \quad e^{-\int_0^T \Phi(u)du}(U_3(W(T))_{\{\tau<T\}}\Bigg],
\end{aligned} \tag{11}
$$

where $U_i, i = 1, 2, 3$ is the utility function for the consumption, premiums and the terminal wealth of an investor and χ is a set of admissible strategies that is

$$
\chi := \{\pi_1, \pi_2, C, P := (\pi_1(t), \pi_2(t), C(t), P(t))_{t\in[0,T]}\}. \tag{12}
$$

At the terminal time t the set (π_1, π_2, C, P) is $\mathcal{F}_t$ adapted with P as defined in (7) and

$$
\int_0^T [C(t) + \pi_1^2(t) + \pi_2^2(t)]dt < \infty.
$$

Using the conditional probability of survival in (8) we redefine the performance function as:

$$
\begin{aligned}
J(0, W(0), \pi_1, \pi_2, C, P) \quad &:= \sup_{(\pi_1,\pi_2,C,P)\in\chi} \mathbb{E}\Bigg[\int_0^{T} e^{-\int_0^s \Phi(u)du} \\
&\times \quad \bar{F}(s)(U_1(C(s)) + U_2(P(s)))ds \\
&+ \quad e^{-\int_0^T \Phi(u)du}\bar{F}(T)(U_3(W(T))\Bigg],
\end{aligned} \tag{13}
$$

therefore we obtain

$$
\begin{aligned}
J(0, W(0), \pi_1, \pi_2, C, P) := \sup_{(\pi_1,\pi_2,C,P)\in\chi} \mathbb{E}\Bigg[\int_0^{T} e^{-\int_0^s (\Phi(u)+\mu(u))du}(U_1(C(s)) \\
+U_2(P(s)))ds + e^{-\int_0^T (\Phi(u)+\mu(u))du}(U_3(W(T))\Bigg].
\end{aligned} \tag{14}
$$

3.2 Conditions on the random parameters considered

We impose conditions since we are dealing with random parameters and diffusion jumps which will guarantee the unique and existent solutions, they are as follows:

- $r \neq \alpha$
- The interest rate r, the force of mortality $\mu(t)$ and the discount rate $\Phi(t)$ are bounded away from zero, thus there exist $\epsilon > 0$ such that for $\wp := r, \mu, \Phi$, we have $|\wp| \geq \epsilon$.
- The exponential integrability conditions are satisfied by the random parameters, that is $\mathbb{E}\Big[e^{(\Theta \int_0^T |\wp(t)|dt)}\Big] < \infty$, for large Θ as defined by (Guambe and Kufakunesu, 2015).

3.3 Value Function

The value function is the maximal payoff function, hence in the present problem our value function is an $\mathcal{F}_t$ measurable random process which cannot be obtained by partially differentiating that is always used but the Hamilton-Jacobi-Bellman (HJB) equation incorporation with the backward stochastic differential equation (BSDE) and is given by:

$$V(t,W) = \sup_{(\pi_1,\pi_2,C,P)\in\chi} J(t,w,\pi_1,\pi_2,C,P) = J(t,w,\pi_1^*,\pi_2^*,C^*,P^*), \tag{15}$$

with $(\pi_1^*,\pi_2^*,C^*,P^*) \in \chi$ being the main strategy of interest to be established.

3.4 The HJB equation and the BSDE with the diffusion jump

Incorporating the diffusion jump we consider the following conditions:

- The terminal condition $\xi \in \mathbb{L}^2(\Omega,\{\mathcal{F}_t\},\mathbb{P},\mathbb{R})$, where $H(T)=\xi$.
- A mapping f(generator) $f:\Omega\times[0,T]\times\mathbb{R}\times L_v^2(\mathbb{R})\mapsto\mathbb{R}$ is predictable.
- $\mathbb{E}[\int_0^T |f(t,0,0)|^2 dt] < \infty$; with $|f(t,y,v)-f(t,y,v')|^2 \le h(|y-y'|^2+\int_{\mathbb{R}} |v(\eta)-v'(\eta)|^2\nu(d\eta))$,

thus the BSDE with a diffusion jump is given by:

$$f(t,H(t),\Upsilon(t))dt = \int_{\mathbb{R}} \Upsilon(t,\eta)\tilde{J}(d\eta,dt) - dH(t). \tag{16}$$

With the conditions above we are assured that there is a unique solution (H,Υ) to (16). The parameter Υ is a control process that controls the process H so it satisfies the first condition above.

The following HJB equation is satisfied by the value function defined above. We let φ be defined as $\varphi:\mathbb{R}^n\mapsto\mathbb{R}$ such that $\varphi\in C_\circ^2(\mathbb{R}^n)$. We assume that partial derivatives φ_t, φ_w, φ_z,φ_y, φ_h, φ_{zz}, φ_{yy}, φ_{wy}, φ_{wz}, φ_{yz}, φ_{ww} w.r.t t, w, z, y, h exists, where $\varphi_t = \dfrac{\partial\varphi}{\partial t}$, $\varphi_z = \dfrac{\partial\varphi}{\partial z}$, $\varphi_{zz} = \dfrac{\partial^2\varphi}{\partial z^2}$, etc., therefore the generator is given by:

$$\begin{aligned}
\mathcal{L}^{\pi_1,\pi_2,c,p}[\varphi(t,w,z,y,h)] &= \varphi_t + \varphi_w\big(w\big[r+\mu(t)\big] + \psi(Y_t,t) - \mu(t)P(t) - C(t)\big) \\
&+ \varphi_y\alpha_1(y_t,t) + \frac{1}{2}\varphi_{yy}\sigma_1^2(Y_t,t) + \varphi_z\varsigma(t)[\beta(t)-Z(t)] \\
&+ \frac{1}{2}\varphi_{zz}\bar{\varrho}^2(t) - \varphi_{wy}\rho\varrho(t)\sigma_1(Y_t,t)w - \varphi_{wz}\bar{\varrho}(t)\varrho(t)w \\
&+ \varphi_{yz}\rho\sigma_1(Y_t,t)\bar{\varrho}(t) + \sup_{\pi_1\in\mathbb{R}}\{\frac{1}{2}\varphi_{ww}\sigma^2\pi_1^2w^2 + \varphi_w w\pi_1(\alpha-r) \\
&- \varphi_{wz}\sigma\varrho(t)\rho w^2\pi_1 + \varphi_{wy}\sigma(t)\sigma_1(Y_t,t)w\pi_1 \\
&+ \varphi_{wz}\rho\bar{\varrho}(t)\sigma(t)w\pi_1\} + \sup_{\pi_2\in\mathbb{R}}\{\frac{1}{2}\varphi_{ww}\varrho^2(t)\pi_2^2w^2 \\
&+ \varphi_w w\pi_2(k(t)+Z(t)-r)\} - \varphi_h f(t,H(t),\Upsilon(t)) \\
&+ \int_{\mathbb{R}}[\varphi(t,w+\pi_1 w\gamma(t,\eta),z,y,h) - \varphi(t,w,z,y,h) \\
&- \pi_1 w\gamma(t,\eta)\varphi_w]\nu(d\eta) + \int_{\mathbb{R}}[\varphi(t,w,z,y,h+\Upsilon(t,\eta)) \\
&- \varphi(t,w,z,y,h) - \Upsilon(t,\eta)\varphi_w]\nu(d\eta),
\end{aligned} \tag{17}$$

in consideration with the following equations;

$$dY_t = \alpha_1(Y_t,t)dt + \sigma_1(Y_t,t)dX_t \tag{18}$$

$$dZ(t) = \varsigma(t)[\beta(t) - Z(t)]dt + \bar{\varrho}(t)dB^1(t) \tag{19}$$

$$\begin{aligned} dW_t &= \big(W_t\big[\pi_1(\alpha - r) + \pi_2(k(t) + Z(t) - r) + r + \mu(t)\big] + \psi(Y_t,t) \\ &- \mu(t)P(t) - C(t)\big)dt + W_t\big[\pi_1\sigma dB_t^0 - \pi_2\varrho(t)dB_t^1\big] \\ &+ \pi_1 W_t \int_{\mathbb{R}} \gamma(t,\eta)\nu(d\eta,dt) \end{aligned} \tag{20}$$

We define a function G as follows:

$$G(t,w,\pi_1,\pi_2,c,p) = \mathcal{L}^{\pi_1,\pi_2,c,p}[\varphi(t,w,z,y,h)] + U(c) + U(p). \tag{21}$$

$U(x)$ is defined as the utility function, therefore in general differentiating equation (21) with respect to π_1, π_2, c, p and equating the derivatives to zero gives the following general optimal controls:

The proportion of wealth of an investor invested in stock π_1^* is given by;

$$\begin{aligned} \frac{\partial G(\cdot)}{\partial \pi_1} &= \varphi_{ww}\sigma^2\pi_1 w^2 + \varphi_w w(\alpha - r) - \varphi_{wz}\sigma\varrho(t)\rho w^2 \\ &+ \varphi_{wy}\sigma(t)\sigma_1(Y_t,t)w + \varphi_{wz}\rho\bar{\varrho}(t)\sigma(t)w \\ &+ \int_{\mathbb{R}}[\varphi_{\pi_1}(t,w+\pi_1 w\gamma(\eta,t),z,y,H(t)) - w\gamma(t,\eta)\varphi_w]\nu(d\eta) = 0 \end{aligned} \tag{22}$$

π_1^* is the solution of the above equation.

To obtain the proportion invested in the inflation linked bond π_2^* we have;

$$\frac{\partial G(\cdot)}{\partial \pi_2} = \varphi_{ww}\varrho^2(t)\pi_2 w^2 + \varphi_w w(k(t) + Z(t) - r) = 0,$$

$$\pi_2^* = -\frac{\varphi_w}{\varphi_{ww}}\left[\frac{(k(t) + Z(t) - r)}{w\varrho^2(t)}\right]. \tag{23}$$

In general we have the optimal consumption process c^* and premiums p^* to be paid are given as;

$$\begin{aligned} \frac{\partial G(\cdot)}{\partial c} &= -w\varphi_w + U^{'}(c) = 0 \\ U^{'}(c) &= w\varphi_w. \end{aligned} \tag{24}$$

$$\begin{aligned} \frac{\partial G(\cdot)}{\partial p} &= -\mu(t)w\varphi_w + U^{'}(p) = 0 \\ U^{'}(p) &= \mu(t)w\varphi_w. \end{aligned} \tag{25}$$

We consider the CRRA utility defined as follows:

$$U(x) = \begin{cases} \dfrac{x^\lambda}{\lambda}, & \text{if } \lambda \in \mathbb{R}\setminus\{0\} \\ \ln x, & \text{if } \lambda = 0. \end{cases} \tag{26}$$

3.4.1 The Power Utility

Suppose we have the type of a constant relative risk aversion (CRRA) utility function given by the power utility;

$$U(x) = \frac{x^{\lambda}}{\lambda}, \;\; \lambda \in \mathbb{R} \setminus \{0\}. \tag{27}$$

We choose the value function as,

$$\varphi(w) = \frac{w^{\lambda}}{\lambda} e^{(1-\lambda)H(t)}, \;\; \lambda \in \mathbb{R} \setminus \{0\}. \tag{28}$$

Consider the following differentials;

$$\varphi_{wz}(w) = 0 \tag{29}$$

$$\varphi_{wy}(w) = 0 \tag{30}$$

$$\varphi_{w}(w) = w^{\lambda-1} e^{-(\lambda-1)H(t)} \tag{31}$$

$$\varphi_{ww}(w) = \frac{w^{\lambda-2}}{\lambda-1} e^{-(\lambda-1)H(t)} \tag{32}$$

$$\varphi_{\pi_1}(w) = (w + \pi_1 w\gamma(\eta,t))^{\lambda-1} w\gamma(\eta,t) e^{-(\lambda-1)H(t)} \tag{33}$$

Substituting (29), (30), (31), (32), and π_1^* into (22), (23), (24) and (25) then we have the following optimal strategies;

The optimal proportion of wealth invested in stock $\pi_1^*(t)$ is the solution of the following equation:

$$\frac{w^{\lambda}}{\lambda-1} e^{-(\lambda-1)H(t)} \sigma^2 \pi_1 + \int_{\mathbb{R}} [(w + \pi_1 w\gamma(\eta,t))^{\lambda-1} w\gamma(\eta,t) e^{-(\lambda-1)H(t)} \\ -w^{\lambda}\gamma(\eta,t) e^{-(\lambda-1)H(t)}]\nu(d\eta) = -w^{\lambda} e^{-(\lambda-1)H(t)}(\alpha - r) \tag{34}$$

For the proportion of wealth invested in the inflation linked bond $\pi_2^*(t)$ we have;

$$\pi_2^* \;\; = \;\; \frac{(\lambda-1)}{\varrho(t)} \left[\frac{(k(t) + Z(t) - r)}{\varrho(t)} \right] \tag{35}$$

We have the optimal consumption process $c^*(t)$ given as;

$$c^*(t) \;\; = \;\; w^{\frac{\lambda}{(\lambda-1)}} e^{-H(t)}. \tag{36}$$

The investor pays premiums $p^*(t)$ as given by;

$$p^*(t) \;\; = \;\; [\mu(t) w^{\lambda}]^{\frac{1}{(\lambda-1)}} e^{-H(t)}. \tag{37}$$

With the optimal strategies in section (3.4.1), substituting π_1^*, (35), (36) and (37) into equation (21) we obtain the following;

$$\begin{aligned} 0 &= \frac{(1-\lambda)}{\lambda}e^{(1-\lambda)H(t)}\Bigg\{w^{\lambda}\Bigg[\lambda\Bigg(\frac{(r+\mu)}{(1-\lambda)}+\frac{1}{w(1-\lambda)}\psi(Y_t,t) \\ &- \frac{e^{-H(t)}}{(1-\lambda)}(\mu^{\lambda}w)^{\frac{1}{\lambda-1}}-\frac{e^{-H(t)}}{(1-\lambda)}w^{\frac{1}{\lambda-1}}-\frac{1}{2}\sigma^2(\pi_1^*)^2 \\ &+ \pi_1^*\frac{(\alpha-r)}{(1-\lambda)}+\frac{1}{(\varrho(t))^2}((\lambda-1)(k(t)+Z(t)-r))^2\Big(\frac{1}{2}+\frac{1}{(\lambda-1)^2}\Big)\Bigg) \\ &- f(t,H(t),\Upsilon(t))+\frac{1}{(1-\lambda)}\Big\{\int_{\mathbb{R}}[(1+\pi_1^*\gamma(t,\eta))^{\lambda}-1 \\ &- \lambda\pi_1^*\gamma(t,\eta)]\nu(d\eta)+\int_{\mathbb{R}}[e^{(\lambda-1)\gamma(t,\eta)}-1-\frac{\gamma(t,\eta)}{w}]\nu(d\eta)\Big\}\Bigg] \\ &+ w^{\frac{\lambda^2}{\lambda-1}}\frac{e^{-\lambda H(t)}}{(1-\lambda)}\Big(1+\mu^{\frac{\lambda}{\lambda-1}}\Big)\Bigg\} \end{aligned} \tag{38}$$

Note that $\frac{1-\lambda}{\lambda}e^{(1-\lambda)H(t)} \neq 0$ for $\lambda \notin \{0,1\}$ we have ;

$$\begin{aligned} 0 &= w^{\lambda}\Bigg[\lambda\Bigg(\frac{(r+\mu)}{(1-\lambda)}+\frac{1}{w(1-\lambda)}\psi(Y_t,t)-\frac{e^{-H(t)}}{(1-\lambda)}(\mu^{\lambda}w)^{\frac{1}{\lambda-1}} \\ &- \frac{e^{-H(t)}}{(1-\lambda)}w^{\frac{1}{\lambda-1}}-\frac{1}{2}\sigma^2(\pi_1^*)^2+\pi_1^*\frac{(\alpha-r)}{(1-\lambda)}+\frac{(\lambda-1)^2}{(\varrho(t))^2}(k(t)+Z(t)-r)^2 \\ &\times \Big(\frac{1}{2}+\frac{1}{(\lambda-1)^2}\Big)\Bigg)-f(t,H(t),\Upsilon(t))+\frac{1}{(1-\lambda)}\Big\{\int_{\mathbb{R}}[(1+\pi_1^*\gamma(t,\eta))^{\lambda} \\ &- 1-\lambda\pi_1^*\gamma(t,\eta)]\nu(d\eta)+\int_{\mathbb{R}}[e^{(\lambda-1)\gamma(t,\eta)}-1-\frac{\gamma(t,\eta)}{w}]\nu(d\eta)\Big\}\Bigg] \\ &+ w^{\frac{\lambda^2}{\lambda-1}}\frac{e^{-\lambda H(t)}}{(1-\lambda)}\Big(1+\mu^{\frac{\lambda}{\lambda-1}}\Big), \end{aligned} \tag{39}$$

then we have;

$$\begin{aligned} f(t,H(t),\Upsilon(t)) &= \Bigg[\lambda\Bigg(\frac{(r+\mu)}{(1-\lambda)}+\frac{1}{w(1-\lambda)}\psi(Y_t,t)-\frac{e^{-H(t)}}{(1-\lambda)}(\mu^{\lambda}w)^{\frac{1}{\lambda-1}} \\ &- \frac{e^{-H(t)}}{(1-\lambda)}w^{\frac{1}{\lambda-1}}-\frac{1}{2}\sigma^2(\pi_1^*)^2+\pi_1^*\frac{(\alpha-r)}{(1-\lambda)}+\frac{(\lambda-1)^2}{(\varrho(t))^2} \\ &\times (k(t)+Z(t)-r)^2 \\ &\times \Big(\frac{1}{2}+\frac{1}{(\lambda-1)^2}\Big)\Bigg)+\frac{1}{(1-\lambda)}\Big\{\int_{\mathbb{R}}[(1+\pi_1^*\gamma(t,\eta))^{\lambda} \\ &- 1-\lambda\pi_1^*\gamma(t,\eta)]\nu(d\eta)+\int_{\mathbb{R}}[e^{(\lambda-1)\gamma(t,\eta)}-1-\frac{\gamma(t,\eta)}{w}]\nu(d\eta)\Big\}\Bigg] \\ &- w^{\frac{\lambda}{\lambda-1}}\frac{e^{-\lambda H(t)}}{\lambda(1-\lambda)}\Big(1+\mu^{\frac{\lambda}{\lambda-1}}\Big), \end{aligned} \tag{40}$$

for $\lambda \in (0,1)$.

We observe that equation (40) is the generator of the equation (16), we consider the conditions stated by (Guambe and Kufakunesu, 2015), conditions "(C1 to C3)". In finding the unique solution (H, Υ) we let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F}_t)$ such that we have a function $L(\omega)$ that can define a new measure through the relation $d\mathbb{Q}(\omega) = L(\omega)d\mathbb{P}$. The probability measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent written $\mathbb{P} \sim \mathbb{Q}$ whenever $\forall\ \omega \in \Omega\ \mathbb{P}(\omega) > 0$ iff $\mathbb{Q}(\omega) > 0$. $\mathbb{Q}$ is considered a probability measure on $(\Omega, \mathcal{F}_t)$ if the probability measure $\mathbb{P}$ is defined and if the function L have the following properties;
$\mathcal{K}_1 : L(t) \geq 0$ a.s w.r.t $\mathbb{P}$.
$\mathcal{K}_2 : \displaystyle\int_\Omega L(t)d\mathbb{P}(t) = 1.$

Given the equivalent probability measures above on the $\sigma-$algebra $\mathcal{F}$ we obtain

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = L, \tag{41}$$

L is referred to as the Radon-Nikodym derivative and is given by;

$$\frac{dL(t)}{L(t)} = \int_{\mathbb{R}} \left[\frac{w(e^{(1-\lambda)\Upsilon(s,\eta)} - 1)}{(1-\lambda)\Upsilon(s,\eta)} - 1 \right] \tilde{J}(ds, d\eta) \tag{42}$$

The Girsanov's theorem explains the variability of the stochastic process under the new equivalent measure to the original probability measure, in this case we consider the change from probability measure $\mathbb{P}$ to $\mathbb{Q}$. We let

$$\nu(t,\eta) = \frac{w(e^{(1-\lambda)\Upsilon(t,\eta)} - 1)}{(1-\lambda)\Upsilon(t,\eta)} - 1. \tag{43}$$

Using the Girsanov's theorem we define the $(\mathbb{Q}, \mathcal{F}_t)$ compensated jump measure $\tilde{J}^{\mathbb{Q}}(dt, d\eta)$ as;

$$\tilde{J}^{\mathbb{Q}}(dt, d\eta) = J(dt, d\eta) - (\nu(t,\eta) + 1)\nu(d\eta)dt,$$

Therefore in the probability measure $\mathbb{Q}$ we have equation (16) with the generator (40) is given as;

$$\begin{aligned} dH(t) &= -\Bigg\{ \frac{\lambda(r+\mu)}{(1-\lambda)} + \frac{\lambda\psi(Y_t,t)}{w(1-\lambda)} - \frac{\lambda}{2}\sigma^2(\pi_1^*)^2 + \pi_1^* \frac{\lambda(\alpha - r)}{(1-\lambda)} \\ &+ (k(t) + Z(t) - r)^2 \frac{\lambda(\lambda-1)^2}{(\varrho(t))^2} \Big(\frac{1}{2} + \frac{1}{(\lambda-1)^2} \Big) \\ &+ \frac{\lambda}{(1-\lambda)} \Bigg\{ \int_{\mathbb{R}} [(1 + \pi_1^*\gamma(t,\eta))^\lambda - 1 - \lambda\pi_1^*\gamma(t,\eta)]\nu(d\eta) \Bigg\} \\ &- \Bigg(\frac{\lambda(\mu^\lambda w)^{\overline{\lambda - 1}^{\,1}}}{(1-\lambda)} + \frac{\lambda w^{\overline{\lambda-1}^{\,1}}}{(1-\lambda)} \Bigg) e^{-H(t)} \\ &+ w^{\overline{\lambda-1}^{\,\lambda}} \frac{\Big(1 + \mu^{\overline{\lambda-1}^{\,\lambda}} \Big) e^{-\lambda H(t)}}{\lambda(1-\lambda)} \Bigg\} dt + \int_{\mathbb{R}} \Upsilon(t,\eta)\tilde{J}^{\mathbb{Q}}(d\eta, dt) \end{aligned} \tag{44}$$

$H(T) = 0.$

The solution of the above equation (44) is represented below as:

$$
\begin{aligned}
\int_t^T dH(s) &= -\int_t^T \Bigg\{ \frac{\lambda(r+\mu)}{(1-\lambda)} + \frac{\lambda\psi(Y_s,s)}{w(1-\lambda)} - \frac{\lambda}{2}\sigma^2(\pi_1^*)^2 \\
&+ \pi_1^* \frac{\lambda(\alpha-r)}{(1-\lambda)} + \frac{(k(s)+Z(s)-r)^2\lambda(\lambda-1)^2}{(\varrho(s))^2}\Big(\frac{1}{2}+\frac{1}{(\lambda-1)^2}\Big) \\
&+ \frac{\lambda}{(1-\lambda)}\Big\{\int_{\mathbb{R}}[(1+\pi_1^*\gamma(s,\eta))^\lambda - 1 - \lambda\pi_1^*\gamma(s,\eta)]\nu(d\eta)\Big\}\Bigg\}ds \\
&+ \int_t^T \int_{\mathbb{R}} \Upsilon(s,\eta)\tilde{J}^{\mathbb{Q}}(d\eta, ds)
\end{aligned} \tag{45}
$$

With $H(T) = 0$ the solution reduces to;

$$
\begin{aligned}
H(t) &= -\int_t^T \Bigg\{ \frac{\lambda(r+\mu)}{(1-\lambda)} + \frac{\lambda\psi(Y_s,s)}{w(1-\lambda)} - \frac{\lambda}{2}\sigma^2(\pi_1^*)^2 \\
&+ \pi_1^* \frac{\lambda(\alpha-r)}{(1-\lambda)} + \frac{(k(s)+Z(s)-r)^2\lambda(\lambda-1)^2}{(\varrho(s))^2}\Big(\frac{1}{2}+\frac{1}{(\lambda-1)^2}\Big) \\
&+ \frac{\lambda}{(1-\lambda)}\Big\{\int_{\mathbb{R}}[(1+\pi_1^*\gamma(s,\eta))^\lambda - 1 - \lambda\pi_1^*\gamma(s,\eta)]\nu(d\eta)\Big\}\Bigg\}ds \\
&+ \int_t^T \int_{\mathbb{R}} \Upsilon(s,\eta)\tilde{J}^{\mathbb{Q}}(d\eta, ds)
\end{aligned} \tag{46}
$$

defined by (Łukasz Delong, 2013) in consideration to the special condition that $w^{\frac{\lambda}{\lambda-1}} \dfrac{\Big(1+\mu^{\frac{\lambda}{\lambda-1}}\Big)e^{-\lambda H(t)}}{\lambda(1-\lambda)} = 0$. Note that since, $\dfrac{w^{\frac{\lambda}{\lambda-1}} e^{-\lambda H(t)}}{\lambda(1-\lambda)} \neq 0$ implies that $1+\mu^{\frac{\lambda}{\lambda-1}} = 0$. Thus

$$
\mu(t) = (-1)^{\frac{\lambda-1}{\lambda}} = \mu^\star, \;\; \mu^\star \text{ a constant} \tag{47}
$$

for some λ. The above equation (47) is the hazard rate function of a policy holder to be alive at a certain time t. Therefore the policy holder's conditional probability of survival as;

$$
\bar{F}(t_l) = e^{-\mu^\star t_l} \tag{48}
$$

the conditional survival probability density of the death of the policy holder given by;

$$
f(t_l) = \mu^\star e^{-\mu^\star t_l} \tag{49}
$$

Theorem 3.1. *In an Itô Lêvy setting the optimal investment-consumption-insurance strategy with the conditions imposed on random parameters considered in the model with the power utility* $U(w) = \dfrac{w^\lambda}{\lambda};\ \lambda \in \mathbb{R}\setminus\{0\}$ *and value function* $\varphi(t,w,z,y,h) = \dfrac{w^\lambda}{\lambda}e^{(1-\lambda)H(t)}$ *are:*

$\pi_1^*(t)$ *is the solution of the following equation:*

$$\left(\frac{\sigma^2\pi_1}{\lambda-1}+\int_{\mathbb{R}}[(1+\pi_1\gamma(\eta,t))^{\lambda-1}\gamma(\eta,t)-\gamma(\eta,t)]\nu(d\eta)+(\alpha-r)\right)=0; \textit{ for} \tag{50}$$

$w^{\lambda}e^{-(\lambda-1)H(t)}\neq 0$

$\pi_2^*=\frac{(\lambda-1)}{\varrho(t)}\left[\frac{(k(t)+Z(t)-r)}{\varrho(t)}\right]$, $c^*(t)=w^{\overline{(\lambda-1)}}e^{-H(t)}$ *and*

$p^*(t)=[\mu(t)w^{\lambda}]^{\frac{1}{(\lambda-1)}}e^{-H(t)}$.

3.4.2 The Logarithmic Utility

We consider the exponential utility function defined by equation (26) where $\lambda=0$ therefore;

$$U(x)=\ln x,\ \ 0<x<\infty$$

We define the value function $\varphi(w)$ as;

$$\varphi(w)=e^{H(t)}\ln w. \tag{51}$$

Consider the following differentials

$$\varphi_{wz}(w)=0 \tag{52}$$

$$\varphi_{wy}(w)=0 \tag{53}$$

$$\varphi_{w}(w)=\frac{e^{H(t)}}{w} \tag{54}$$

$$\varphi_{ww}(w)=-\frac{e^{H(t)}}{w^2} \tag{55}$$

$$\varphi_{\pi_1}(w)=\frac{e^{H(t)}\gamma(\eta,t)}{(1+\pi_1\gamma(\eta,t))}. \tag{56}$$

With the above differentials we obtain the controls as follows;

π_1^* is the proportion of wealth invested in stock and lies in $(0,1)$, is the solution of the following equation;

$$e^{H(t)}\big[\sigma^2\pi_1-\int_{\mathbb{R}}\Big[\gamma\Big(\frac{-\pi_1\gamma}{(1+\pi_1\gamma)}\Big)\Big]\nu(d\eta)\big]=e^{H(t)}(\alpha-r), \tag{57}$$

since $e^{H(t)}\neq 0$, we have;

$$\big[\sigma^2\pi_1-\int_{\mathbb{R}}\Big[\gamma\Big(\frac{-\pi_1\gamma}{(1+\pi_1\gamma)}\Big)\Big]\nu(d\eta)\big]=(\alpha-r). \tag{58}$$

We have the optimal proportion $\pi_2^*(t)$ given by;

$$\pi_2^*(t)\ \ =\ \ \left[\frac{(k(t)+Z(t)-r)}{\varrho^2(t)}\right] \tag{59}$$

We have the optimal consumption process $c^*(t)$ and the premium $p^*(t)$ given by;

$$c^* = e^{-H(t)} \tag{60}$$

For p^* we have

$$p^* = \frac{e^{-H(t)}}{\mu(t)} \tag{61}$$

Taking the controls in example 2, substituting π_1^*, (59), (60), (61) into (21) we obtain the following;

$$\begin{aligned} 0 &= e^{H(t)} \ln w \Bigg\{ \frac{1}{\ln w}\Bigg((r+\mu) + \frac{\Psi(Y_t,t)}{w} - \frac{2e^{-H(t)}}{w} - \frac{\sigma^2(\pi_1^*)^2}{2} \\ &+ \pi_1^*(\alpha - r) + \frac{(k(t)+Z(t)-r)^2}{2\varrho^2} - f(t,H(t),\Upsilon(t)) \\ &+ \int_{\mathbb{R}} [\ln(1+\pi_1^* w\gamma(t,\eta)) - \pi_1^*\gamma(t,\eta)]\nu(d\eta) \\ &+ \int_{\mathbb{R}} [\ln w(e^{\gamma(t,\eta)}-1) - \frac{\gamma(t,\eta)}{w}]\nu(d\eta) - e^{-H(t)}\big(2H(t) - \ln\mu\big)\Bigg)\Bigg\} \end{aligned} \tag{62}$$

Taking into consideration that $e^{H(t)}\ln w \neq 0$ therefore we have ;

$$\begin{aligned} f(t,H(t),\Upsilon(t)) &= \frac{1}{\ln w}\Bigg((r+\mu) + \frac{\Psi(Y_t,t)}{w} - \frac{2e^{-H(t)}}{w} - \frac{\sigma^2(\pi_1^*)^2}{2} \\ &+ \pi_1^*(\alpha - r)\Bigg) + \frac{(k(t)+Z(t)-r)^2}{2\varrho^2 \ln w} \\ &+ \frac{1}{\ln w}\Bigg(\int_{\mathbb{R}} [\ln(1+\pi_1^* w\gamma(t,\eta)) - \pi_1^*\gamma(t,\eta)]\nu(d\eta) \\ &+ \int_{\mathbb{R}} [\ln w(e^{\gamma(t,\eta)}-1) - \frac{\gamma(t,\eta)}{w}]\nu(d\eta)\Bigg) \\ &- \frac{e^{-H(t)}}{\ln w}\big(2H(t) - \ln\mu\big). \end{aligned} \tag{63}$$

We have the generator above, equation (63) for the BDSE with diffusion jump in equation (16). We use the definition of equivalent measures defined in section (3.4.1) in finding the unique solution (H,γ). We define the Radon-Nikodym derivative given by;

$$\frac{dL(t)}{L(t)} = \int_{\mathbb{R}} \left[\frac{w \ln w(e^{\gamma}-1)}{\gamma(t,\eta)} - 1 \right] \tilde{J}(ds,d\eta). \tag{64}$$

In consideration to the Girsanov's theorem we consider the change of probability measure $\mathbb{P}$ to $\mathbb{Q}$.We suppose that

$$\nu(\eta,t) := \frac{w \ln w(e^{\gamma}-1)}{\gamma(t,\eta)} - 1 \tag{65}$$

then we have the compensated random measure defined under the probability measure $\mathbb{Q}$, we then have the BSDE in equation (16) given by;

$$\begin{aligned} dH(t) &= \Bigg\{\frac{1}{\ln w}\Bigg((r+\mu)+\frac{\Psi(Y_t,t)}{w}-\frac{\sigma^2(\pi_1^*)^2}{2}+\pi_1^*(\alpha-r) \\ &+ \frac{(k(t)+Z(t)-r)^2}{2\varrho^2}+\int_{\mathbb{R}}[\ln(1+\pi_1^* w\gamma(t,\eta))-\pi_1^*\gamma(t,\eta)]\nu(d\eta)\Bigg) \\ &+ e^{-H(t)}\Bigg[\frac{-2}{w\ln w}-\frac{\big(2H(t)+\ln\mu\big)}{\ln w}\Bigg]\Bigg\}dt+\int_{\mathbb{R}}\Upsilon(t,\eta)\tilde{J}^{\mathbb{Q}}(d\eta,dt) \end{aligned} \tag{66}$$

The solution of the above equation (66) is given by:

$$\begin{aligned} H(t) &= \mathbb{E}\Bigg\{\int_t^T \ln\Bigg[1+\int_t^s\Bigg(\frac{-2}{w\ln w}\Bigg)d\tilde{u}\Bigg]\times\frac{1}{\ln w}\Bigg((r+\mu) \\ &+ \frac{\Psi(Y_t,t)}{w}-\frac{\sigma^2(\pi_1^*)^2}{2}+\pi_1^*(\alpha-r)+\frac{(k(t)+Z(t)-r)^2}{2\varrho^2} \\ &+ \int_{\mathbb{R}}[\ln(1+\pi_1^* w\gamma(t,\eta))-\pi_1^*\gamma(t,\eta)]\nu(d\eta)\Bigg)\Bigg\} \end{aligned} \tag{67}$$

whenever

- $\mu(t)=e^{-2H(t)}$.
- $1+\pi_1^* w\gamma>0$; for $\pi_1^*>\dfrac{-1}{w\gamma}$ and $w\gamma\neq 0$.

Theorem 3.2. *The optimal investment-consumption-insurance strategy in an Itô Lêvy setting with the considered conditions on random parameters in the model where we have the logarithmic utility $U(w)=\ln w;\ \ for\ \ \lambda=0$ and the value function given by $\varphi(t,w,z,y,h)=\ln we^{H(t)}$, we have the strategies given by:*

$$p^*=\frac{e^{-H(t)}}{\mu(t)}\ ,\ c^*=e^{-H(t)}\ ,\ \pi_2^*=\Bigg[\frac{(k(t)+Z(t)-r)}{\varrho^2(t)}\Bigg] \tag{68}$$

and, $\pi_1^*(t)$ is the solution of the following equation:

$$\sigma^2\pi_1-\int_{\mathbb{R}}\Big[\gamma(\eta,t)\Big(\frac{w}{(w+\pi_1 w\gamma(\eta,t))}-1\Big)\Big]\nu(d\eta)-(\alpha-r)=0. \tag{69}$$

3.5 Post-death case

In these section we present the case when the investor dies at time τ after the retirement time T, that is $\tau>T$. Note that here the investor does not pay premiums thus $p=0$. We have the wealth process as given by equation (10) and the beneficiaries obtain a share as given by the following;

$$W(\tau)+\frac{p(\tau)}{\tilde{\pi}(\tau)}, \tag{70}$$

where $\tilde{\pi}(\tau)$ is the premium insurance ratio, $\frac{p(\tau)}{\tilde{\pi}(\tau)}$ is the insurance benefit and $W(\tau)$ is the wealth of the investor in their time of life to their time of death τ. We have the investment-consumption strategy for the beneficiaries given by

$$\chi := (\bar{\pi}_1, \bar{\pi}_2, c_a) \tag{71}$$

where $\bar{\pi}_1, \bar{\pi}_2$ are the proportions of wealth to be obtained by the beneficiaries from stock and the inflation linked bond respectively and c_a is the consumption rate after the investor dies. It is as well defined by;

$$\int_0^T c_a(s)ds < \infty \tag{72}$$

The performance function is given below as:

$$\begin{aligned} J(0, W(0), \bar{\pi}_1, \bar{\pi}_2, c_a) \quad := \quad & \sup_{(\bar{\pi}_1, \bar{\pi}_2, c_a) \in \chi} \mathbb{E}_{t,w} \Bigg[\int_t^T e^{-\int_0^{\tau \wedge T} \Phi(u)du} \\ \times \quad & (U_1(c_a(s)))ds + e^{-\int_t^T \Phi(u)du} (U_2(W(T))_{\{\tau > T\}} \Bigg], \end{aligned} \tag{73}$$

where U_1 and U_2 are the utility functions for the consumption of the beneficiaries and the terminal wealth of an investor respectively. The set $(\bar{\pi}_1, \bar{\pi}_2, c_a)$ is $\mathcal{F}_t$ adapted. The performance function is then redefined using the probability of survival as:

$$\begin{aligned} J(0, W(0), \bar{\pi}_1, \bar{\pi}_2, c_a) \quad := \quad & \sup_{(\bar{\pi}_1, \bar{\pi}_2, c_a) \in \chi} \mathbb{E}_{t,w} \Bigg[\int_t^T e^{-\int_0^{\tau \wedge T} (\Phi(u) + \mu(u))du} \\ \times \quad & (U_1(c_a(s)))ds \\ + \quad & e^{-\int_t^T (\Phi(u) + \mu(u))du} (U_2(W(T)) \Bigg]. \end{aligned} \tag{74}$$

We consider the conditions imposed on the parameters in section (3.2). The defined function below is given as:

$$\bar{G} = \sup_{\bar{\pi}_1^*, \bar{\pi}_2^*, c_a^*} \{ \mathcal{L}^{\bar{\pi}_1, \bar{\pi}_2, c_a} [\varphi(t, w, z, y, h)] + U(c_a) \} \tag{75}$$

We assume that the partial derivatives $\varphi_t, \varphi_w, \varphi_z, \varphi_y, \varphi_h, \varphi_{zz}, \varphi_{yy}, \varphi_{wy}, \varphi_{wz}, \varphi_{yz}, \varphi_{ww}$ w.r.t t, w, z, y, h exists, where $\varphi_t = \frac{\partial \varphi}{\partial t}$, $\varphi_z = \frac{\partial \varphi}{\partial z}$, $\varphi_{zz} = \frac{\partial^2 \varphi}{\partial z^2}$, etc. exists, then we have the generator of the wealth process is defined below as:

$$
\begin{aligned}
\mathcal{L}^{\bar{\pi}_1,\bar{\pi}_2,c_a}[\varphi(t,w,z,y,h)] &= \varphi_t + \varphi_w\big(w\big[r+\mu\big] + \psi - c_a\big) + \varphi_y\alpha_1 \\
&+ \frac{1}{2}\varphi_{yy}\sigma_1^2 + \varphi_z\varsigma[\beta - Z] + \frac{1}{2}\varphi_{zz}\bar{\varrho}^2 \\
&- \varphi_{wy}\rho\varrho(t)\sigma_1 w - \varphi_{wz}\bar{\varrho}\varrho w + \varphi_{yz}\rho\sigma_1\bar{\varrho} \\
&+ \sup_{\bar{\pi}_1\in\mathbb{R}}\{\frac{1}{2}\varphi_{ww}\sigma^2\bar{\pi}_1^2 w^2 + \varphi_w w\bar{\pi}_1(\alpha - r) \\
&- \varphi_{wz}\sigma\varrho\rho w^2\bar{\pi}_1 + \varphi_{wy}\sigma\sigma_1 w\bar{\pi}_1 \\
&+ \varphi_{wz}\rho\bar{\varrho}\sigma w\bar{\pi}_1\} + \sup_{\bar{\pi}_2\in\mathbb{R}}\{\frac{1}{2}\varphi_{ww}\varrho^2\bar{\pi}_2^2 w^2 \\
&+ \varphi_w w\bar{\pi}_2(k + Z - r)\} - \varphi_h f(t,H,\Upsilon) \\
&+ \int_{\mathbb{R}}[\varphi(t, w + \bar{\pi}_1 w\gamma(t,\eta), z, y, h) - \varphi(t,w,z,y,h) \\
&- \bar{\pi}_1 w\gamma(t,\eta)\varphi_w]\nu(d\eta) + \int_{\mathbb{R}}[\varphi(t,w,z,y,h + \Upsilon(t,\eta)) \\
&- \varphi(t,w,z,y,h) - \Upsilon(t,\eta)\varphi_w]\nu(d\eta),
\end{aligned} \tag{76}
$$

considering the equations (18), (19) and (10). We obtain the optimal controls $\bar{\pi}_1^*, \bar{\pi}_2^*, c_a^*$ by differentiating (75) with respect to $\bar{\pi}_1, \bar{\pi}_2$ and c_a and equating them to zero.

The proportion π_1^* of wealth held from the invested stock is given by;

$$
\begin{aligned}
\frac{\partial G(\cdot)}{\partial \bar{\pi}_1} &= \varphi_{ww}\sigma^2\bar{\pi}_1 w^2 + \varphi_w w(\alpha - r) - \varphi_{wz}\sigma\varrho\rho w^2 \\
&+ \varphi_{wy}\sigma\sigma_1 w + \varphi_{wz}\rho\bar{\varrho}\sigma w \\
&+ \int_{\mathbb{R}}[\varphi_{\bar{\pi}_1}(t, w + \bar{\pi}_1 w\gamma(\eta,t), z, y, H(t)) - w\gamma(t,\eta)\varphi_w]\nu(d\eta) = 0
\end{aligned} \tag{77}
$$

$\bar{\pi}_1^*$ is the solution of the above equation.

We get the proportion of wealth held in the inflation linked bond $\bar{\pi}_2^*$ as;

$$
\frac{\partial G(\cdot)}{\partial \bar{\pi}_2} = \varphi_{ww}\varrho^2\bar{\pi}_2 w^2 + \varphi_w w(k + Z - r) = 0,
$$

$$
\bar{\pi}_2^* = -\frac{\varphi_w}{\varphi_{ww}}\left[\frac{(k + Z - r)}{w\varrho^2}\right]. \tag{78}
$$

We have the optimal consumption process c_a^* to be consumed as;

$$
\begin{aligned}
\frac{\partial G(\cdot)}{\partial c_a} &= -w\varphi_w + U^{'}(c_a) = 0 \\
U^{'}(c_a) &= w\varphi_w,
\end{aligned} \tag{79}
$$

given that $U^{'}(c_a)$ is the differential of the utility function $w.r.t\ c_a$.

The following theorems states the solutions above given a specific utility function $U(w)$ and a specific value function $\varphi(t,w,z,y,h)$. The steps are omitted as they are obtained exactly like those in the pre-death case.

Theorem 3.3. *Given a power utility function* $U(w) = \dfrac{w^\lambda}{\lambda}$*; for* $\lambda \in \mathbb{R}\backslash\{0\}$ *and the value function* $\varphi(t,w,z,y,h) = \dfrac{w^\lambda}{\lambda} e^{-(\lambda-1)H(t)}$*, the optimal investment-consumption strategy of the beneficiaries in an Itô Lêvy setting with the conditions given on the parameters are given by;*

$\bar{\pi}_1^*(t)$ *is the solution of the following equation:*

$$\left(\frac{\sigma^2 \bar{\pi}_1}{\lambda-1} + \int_{\mathbb{R}} [(1+\bar{\pi}_1 \gamma(\eta,t))^{\lambda-1}\gamma(\eta,t) - \gamma(\eta,t)]\nu(d\eta) + (\alpha - r)\right) = 0; \text{ for} \tag{80}$$

$w^\lambda e^{-(\lambda-1)H(t)} \neq 0$

$$\bar{\pi}_2^* = \frac{(\lambda-1)}{\varrho(t)}\left[\frac{(k(t)+Z(t)-r)}{\varrho(t)}\right], \ c_a^*(t) = w^{\overline{(\lambda-1)}^{\lambda}} e^{-H(t)}.$$

Theorem 3.4. *The optimal investment-consumption strategies of the beneficiaries in an Itô Lêvy setting with the considered conditions on random parameters with the logarithmic utility* $U(w) = \ln w$; for $\lambda = 0$ *and the value function* $\varphi(t,w,z,y,h) = \ln we^{H(t)}$*, we have the strategies given by:*

$$c_a^* = e^{-H(t)}, \ \bar{\pi}_2^* = \left[\frac{(k(t)+Z(t)-r)}{\varrho^2(t)}\right] \tag{81}$$

and $\bar{\pi}_1^*(t)$ *is the solution of the following equation:*

$$\sigma^2\bar{\pi}_1 - \int_{\mathbb{R}} \left[\gamma(\eta,t)\Big(\frac{1}{(1+\bar{\pi}_1\gamma(\eta,t))} - 1\Big)\right]\nu(d\eta) - (\alpha - r) = 0. \tag{82}$$

The theorems above are obtained similarly as the theorems 3.1 and 3.2 respectively. Substituting $\bar{\pi}_1^*$ and the strategies above into (75) we obtain the generators of the BSDE equation (16) with the considered utility functions and value functions.

$$\begin{aligned} f(t,H(t),\Upsilon(t)) &= \Big\{\lambda\Big[\frac{1}{1-\lambda}\big((r+\mu) + \frac{\psi(y,t)}{w} + \pi_1^*(\alpha - r)\big) - \frac{\sigma^2(\pi_1^*)^2}{2} \\ &- \frac{3}{2}\Big(\frac{k+z-r}{\varrho}\Big)^2\Big]\Big\} - \frac{1}{1-\lambda}\Big(\lambda e^{-H(t)} - w^{\overline{\lambda-1}^{1}} e^{-H(2\lambda-1)}\Big) \\ &+ \frac{1}{(1-\lambda)}\Big\{\int_{\mathbb{R}}[(1+\pi_1^*\gamma(t,\eta))^\lambda - 1 - \lambda\pi_1^*\gamma(t,\eta)]\nu(d\eta) \\ &+ \int_{\mathbb{R}}[e^{(\lambda-1)\gamma(t,\eta)} - 1 - \frac{\gamma(t,\eta)}{w}]\nu(d\eta)\Big\}\Big], \end{aligned} \tag{83}$$

and,

$$\begin{aligned} f(t,H(t),\Upsilon(t)) &= \frac{1}{\ln w}\Big((r+\mu) + \frac{\Psi(Y_t,t)}{w} - \frac{2e^{-H(t)}}{w} - \frac{\sigma^2(\pi_1^*)^2}{2} \\ &+ \pi_1^*(\alpha-r)\Big) + \frac{(k(t)+Z(t)-r)^2}{2\varrho^2\ln w} \\ &+ \frac{1}{\ln w}\Big(\int_{\mathbb{R}}[\ln(1+\pi_1^* w\gamma(t,\eta)) - \pi_1^*\gamma(t,\eta)]\nu(d\eta) \\ &+ \int_{\mathbb{R}}[\ln w(e^{\gamma(t,\eta)} - 1) - \frac{\gamma(t,\eta)}{w}]\nu(d\eta)\Big) \\ &- \frac{1}{\ln w}\big(H(t) + \frac{1}{w}\big)e^{-H(t)}. \end{aligned} \tag{84}$$

respectively.

4. Conclusion

The paper entirely focused on the study of optimal investment-consumption-insurance strategy in an Itô Lêvy setting with diffusion jumps. The wealth process is determined at different generations of the life of an investor, where investment position is taken on both risky and risk-less assets. The assets comprises of stock with a diffusion jump, bond, and a money market account. Inflation as the most important economic factor is taken into account as it affects the wealth of an investor at any given time, thus the inflation is linked to the bond which makes the bond risky. The investor's incoming source of funds is taken to be a stochastic process that also contributes to his/her's wealth. The investor's consumption is finite and does not exceed the limit of what they have. We managed to employ both the BSDE with jumps and the HJB equation in finding the optimal investment strategies π_1^*, π_2^*, c^*, and p^* with the different conditions imposed on the parameters throughout the paper. When the investor dies the proportions invested are then the proportions that beneficiaries can decide whether to sell or buy looking at the market the investment has been made on, and the insurance pays out the benefit they have agreed on with the holder in their time of life. The beneficiaries can either continue with the investment by increasing or decreasing the fractions looking at the level of risk in each one of them and continue to consume the proportion that will allow the investment to go on. With the use and understanding of the theorems employed, that is the Radon-Nikodym theorem and the Girsanov's theorem, the consideration of the utility functions and value functions chosen, the power and logarithmic functions gave us the optimal strategies given by theorems (3.1) and (3.2).
The study can further be improved by using other approaches or techniques in finding the optimal strategies. The study on diffusion jumps could help in giving a function on the solution of π_1^* to improve the solutions obtained. Different assumptions can be made on the values of the parameters and the utility functions chosen and compare the solutions obtained. One can consider the different times of life of an investor and choose a more specific insurance an investor buys.

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