A New Class of Generalized Inverse Weibull Distribution with Applications

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Abstract

The gamma-inverse Weibull (GIW) distribution which includes inverse Weibull, inverse exponential, gamma-inverse exponential, gamma inverse Rayleigh, inverse Rayleigh, gamma Fréchet and Fréchet distributions as special cases is proposed and studied. This new distribution might be useful for failure time data analysis. Some mathematical properties of the new distribution including moments, distribution of the order statistics, Shannon and Rényi entropies are presented. Maximum likelihood estimation technique is used to estimate the model parameters and applications to real data sets to illustrate its usefulness are presented.

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1 Introduction

The inverse Weibull distribution has been used to model degradation of mechanical components such as pistons, crankshafts of diesel engines, as well as breakdown of insulating fluid to mention just a few areas. The usefulness and applications of inverse Weibull (IW) distribution in various areas including reliability, and branching processes can be seen in Keller and Kamath (1982)and in references therein. The authors used the distribution to describe the degradation phenomena of mechanical components such as pistons, crank shaft of

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diesel engines. The model also provides a reasonably good fit to data on times to breakdown of an insulating fluid, subject to constant tension, Nelson (1982). Additional results on the inverse Weibull distribution including work on reliability and tolerance limits, Bayes 2-sample prediction, and maximum likelihood and least squares estimation are given by Calabria and Pulcini (1989, 1994, 1990). In this note we generalize the inverse Weibull distribution via the use of the gamma distribution function. There are several generalizations of distributions including those of Eugene et al. (2002) dealing with the beta-normal distribution, as well results on the moments of the beta-normal distribution given by Gupta and Nadarajah (2004). Famoye et al. (2005) discussed and presented results on the beta-Weibull distribution. Kong and Sepanski (2007) developed the beta-gamma distribution. Results on the length-biased inverse Weibull can be seen in Kersey and Oluyede (2012). Additional results on the generalizations of the inverse Weibull and related distributions with application are given by Oluyede and Yang (2014).

In this note we present and analyze the gamma-inverse Weibull (GIW) distribution. First we discuss some properties of the inverse Weibull distribution. The inverse Weibull (IW) cumulative distribution function (cdf) is given by

$$F(x;\alpha,\beta) = exp\Big[-(\alpha(x-x_0))^{-\beta}\Big], \qquad x \geq 0,\ \alpha > 0,\ \beta > 0, \tag{1}$$

where α, x_0 and β are the scale, location and shape parameters, respectively. Often the parameter x_0 is called the minimum life or guarantee time. When $\alpha = 1$ and $x = x_0 + \alpha$, then $F(\alpha + x_0; 1, \beta) = F(\alpha + x_0; 1) = e^{-1} = 0.3679$. This value is in fact the characteristic life of the distribution. In what follows, we assume that $x_0 = 0$, and the inverse Weibull cdf becomes

$$F(x;\alpha,\beta) = exp[-(\alpha x)^{-\beta}], \qquad x \geq 0,\ \alpha > 0,\ \beta > 0. \tag{2}$$

The corresponding inverse Weibull probability density function (pdf) is given by

$$f(x;\alpha,\beta) = \beta\alpha^{-\beta}x^{-\beta-1}exp(-(\alpha x)^{-\beta}), \qquad x \geq 0,\ \alpha > 0,\ \beta > 0. \tag{3}$$

Note that when $\alpha = 1$, we have the Fréchet distribution function. Also, for the inverse Weibull pdf, we have the following relationship:

$$xf(x;\alpha,\beta) = \beta F(x;\alpha,\beta)(-\ln(F(x;\alpha,\beta)), \qquad x \geq 0,\ \alpha > 0,\ \beta > 0. \tag{4}$$

Let X be a continuous random variable with cdf F, and pdf f, then the hazard function, reverse hazard function and mean residual life functions are given by $h_F(x) = f(x)/\overline{F}(x)$, $\tau_F(x) = f(x)/F(x)$, and $\delta_F(x) = \int_x^\infty \overline{F}(u)du/\overline{F}(x)$ respectively. The functions $h_F(x)$, $\delta_F(x)$, and $\overline{F}(x)$ are equivalent (Shaked and Shanthikumar (1994)).

Zografos and Balakrishnan (2009), proposed the gamma-generated family. Based on a baseline continuous distribution $F(x)$ with survival function $\overline{F}(x)$ and pdf $f(x)$, they defined the gamma-generated cdf and pdf as

$$K(x) = \frac{1}{\Gamma(\delta)} \int_0^{-\log \overline{F}(x)} t^{\delta-1} e^{-t} dt, \quad x \in \mathbf{R}, \, \delta > 0, \tag{5}$$

and

$$k(x) = \frac{1}{\Gamma(\delta)} [-\log(\overline{F}(x)]^{\delta-1} f(x), \tag{6}$$

respectively, where $\Gamma(\delta)$ is the gamma function. Ristić and Balakrishnan (2011) proposed an alternative gamma-generator defined by the cdf and pdf given by

$$G(x) = 1 - \frac{1}{\Gamma(\delta)} \int_0^{-\log F(x)} t^{\delta-1} e^{-t} dt, \quad x \in \mathbf{R}, \, \delta > 0, \tag{7}$$

and

$$g(x) = \frac{1}{\Gamma(\delta)} [-\log(F(x))]^{\delta-1} f(x), \tag{8}$$

respectively. We will work with the family of distributions defined by Ristić and Balakrishnan (2011). The motivations for this new family of distributions were given in Ristić and Balakrishnan (2011), that is for $n \in \mathbf{N}$, the last equation above is the pdf of the n^{th} lower record value of a sequence of i.i.d. variables from a population with density $f(x)$. Ristić and Balakrishnan (2011) considered the exponentiated exponential (EE) distribution with cdf $F(x) = (1 - e^{-\lambda x})^{\alpha}$, where $\alpha > 0$ and $\lambda > 0$, (see Gupta and Kundu (1999) for details) in equation (7), obtained and studied the gamma-exponentiated exponential (GEE) model. In this note, we obtain a natural extension for the inverse Weibull distribution, which we called the gamma-inverse Weibull (GIW) distribution.

In section 2 we present the gamma-inverse Weibull (GIW) distribution and its sub models. This section also contains further analysis of the distribution function including results on the hazard and reverse hazard functions. The moments and moment generating function are given in section 3. Mean deviations, Lorenz and Bonferroni curves are given in section 4. Section 5 contains some additional useful results including entropies. In section 6, results on the estimation of the parameters of the GIW distribution are presented. Applications are given in section 7, followed by concluding remarks.

2 GIW Distribution and Sub-models

In this section, the GIW distribution and some of its sub-models are presented. The mode, quantile function, hazard and reverse hazard functions are also presented. Let $\lambda = \alpha^{-\beta}$ and consider the inverse Weibull (IW) distribution given by

$$F_{IW}(x;\lambda,\beta) = \exp[-\lambda x^{-\beta}], \quad x \geq 0,\ \lambda > 0,\ \beta > 0. \tag{9}$$

Inserting the IW distribution in equation (7) gives the GIW survival function

$$\overline{G}(x) = \frac{1}{\Gamma(\delta)} \int_0^{-\log(\exp[-\lambda x^{-\beta}])} t^{\delta-1} e^{-t} dt = \frac{\gamma(-\log(exp[-\lambda x^{-\beta}]),\delta)}{\Gamma(\delta)}, \tag{10}$$

for $x > 0$, $\lambda > 0$, $\beta > 0$, $\delta > 0$, and $\gamma(x,\delta) = \int_0^x x^{\delta-1} e^{-t} dt$ is the lower incomplete gamma function. The cdf of the GIW distribution is given by $G(x) = 1 - \overline{G}(x)$. The corresponding pdf is given by

$$g_{GIW}(x) = \frac{\beta x^{-1}}{\Gamma(\delta)} [\lambda x^{-\beta}]^{\delta} \exp[-\lambda x^{-\beta}], \tag{11}$$

for $x > 0$, $\lambda > 0$, $\beta > 0$, $\delta > 0$. If a random variable X has the density above, we write $X \sim GIW(\beta,\lambda,\delta)$. From here on we will set $g_{GIW}(x) = g(x)$.

2.1 Shapes and Stochastic Orders

In this section, we present the mode and discuss the shape, as well as stochastic orders of the GIW distribution. To obtain the mode, we solve the equation $\frac{d\ln(g(x))}{dx} = 0$, for x. Note that

$$\ln(g(x)) = \ln(\beta) + \delta\ln(\lambda) - (\beta\delta + 1)\ln(x) - \lambda x^{-\beta} - \ln(\Gamma(\delta)), \tag{12}$$

so that the mode occurs at

$$x_0 = \left(\frac{\lambda\delta}{1 + \beta\lambda}\right)^{\frac{1}{\beta}}. \tag{13}$$

Note that $\lim_{x\to 0} g(x) = 0$, and $\lim_{x\to\infty} g(x) = 0$.

Let X_i be distributed according to $GIW(\lambda,\beta,\delta)$, with cdf and pdf G_i and g_i, respectively, $i = 1,2$. We say X_2 is stochastically greater than X_1 in likelihood ratio if $g_2(x)/g_1(x)$ is an increasing function of x. It is well known that likelihood ratio order implies failure rate order which in turn implies stochastic order, see Shaked and Shanthikumar (1994) for additional details.

- If $\beta_1 = \beta_2$ and $\delta_1 = \delta_2$, then X_2 is stochastically greater than X_1 with respect to likelihood ratio order if and only if $\lambda_2 > \lambda_1$.
- If $\beta_1 = \beta_2$ and $\lambda_1 = \lambda_2$ then X_2 is stochastically larger than X_1 with respect to likelihood ratio order if and only if $\delta_1 > \delta_2$.

Note that

$$\frac{g_2(x)}{g_1(x)} = \frac{\beta_2 x^{-1}(\lambda_2 x^{-\beta_2})^{\delta_2} \exp(-\lambda_1 x^{-\beta_2})(\Gamma(\delta_2))^{-1}}{\beta_1 x^{-1}(\lambda_1 x^{-\beta_1})^{\delta_1} \exp(-\lambda_1 x^{-\beta_1})(\Gamma(\delta_1))^{-1}}. \tag{14}$$

If $\beta_1 = \beta_2$, and $\delta_1 = \delta_2$, then

$$K(x) = \frac{\lambda_2}{\lambda_1} \exp(x^{-\beta}(\lambda_1 - \lambda_2)), \tag{15}$$

is such that $K'(x) = \frac{\lambda_2}{\lambda_1} \exp(-x^{-\beta}(\lambda_2 - \lambda_1))\beta x^{-\beta-1}(\lambda_2 - \lambda_1) > 0$, if and only if $\lambda_2 - \lambda_1 > 0$. Similarly, if $\beta_1 = \beta_2$ and $\lambda_1 = \lambda_2$, then X_2 is stochastically larger than $X_1$with respect to likelihood ratio order if and only if $\delta_1 > \delta_2$.

2.2 Quantile Function

The quantile function is the solution of the equation

$$\frac{\gamma(-\log(exp[-\lambda x^{-\beta}]), \delta)}{\Gamma(\delta)} = 1 - u, \tag{16}$$

that is, $\lambda x^{-\beta} = \gamma^{-1}((1-u)\Gamma(\delta), \delta)$ and

$$x = Q(u) = G^{-1}(u) = \left[\frac{\gamma^{-1}((1-u)\Gamma(\delta), \delta)}{\lambda}\right]^{-1/\beta}, \tag{17}$$

where u is uniformly distributed on the interval $(0, 1)$.

The graphs of the cdf and pdf of the GIW for selected values of the model parameters are given below.

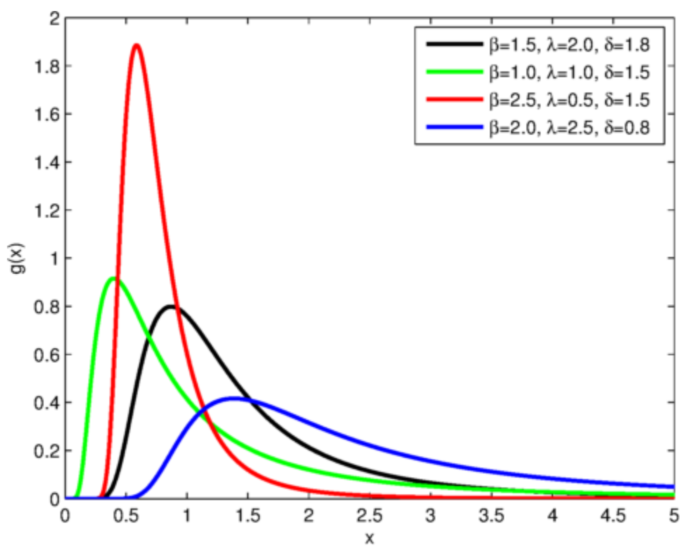

Figure 2: Graphs of GIW Density

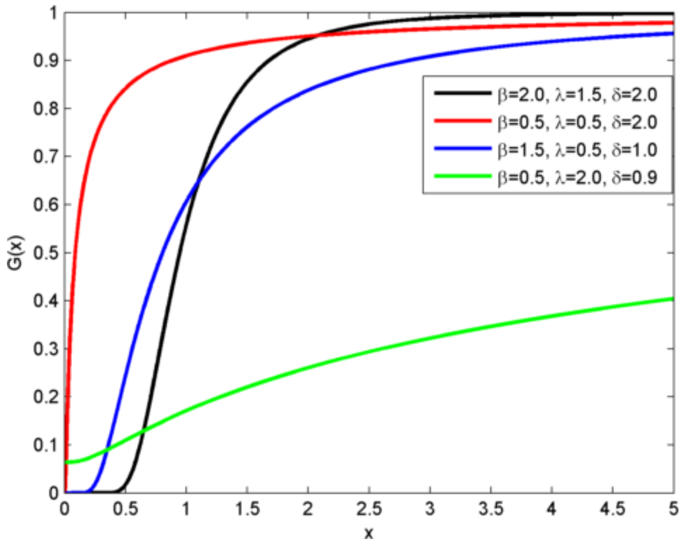

Figure 1: Graphs of GIW cdf

2.3 GIW Sub-models

Some of the sub-models of the GIW distribution are listed below:

- When $\delta = 1$, we have the inverse Weibull (IW) distribution.
- When $\lambda = 1$, we have the gamma-Fréchet (GF) distribution.
- When $\delta = \lambda = 1$, we have the Fréchet (F) distribution.
- When $\beta = 2$, we have gamma-inverse Rayleigh (GIR) distribution.
- When $\beta = 2$, $\delta = 1$, we have inverse Rayleigh (IR) distribution.
- When $\beta = 1$, we have the gamma-inverse exponential (GIE) distribution.
- When $\delta = \beta = 1$, we get the inverse exponential (IE) distribution.

2.4 Hazard and Reverse Hazard Functions

In this section, the hazard and reverse hazard functions of the GIW distribution are presented. The hazard and reverse hazard functions are

$$h_G(x) = \frac{\beta x^{-1}[\lambda x^{-\beta}]^{\delta}\exp[-\lambda x^{-\beta}]}{\gamma(-\log(exp[-\lambda x^{-\beta}]), \delta)}, \tag{18}$$

and

$$\tau_G(x) = \frac{\beta x^{-1}[\lambda x^{-\beta}]^{\delta}\exp[-\lambda x^{-\beta}]}{1-\gamma(-\log(exp[-\lambda x^{-\beta}]), \delta)} \tag{19}$$

for $x \geq 0$, $\lambda > 0$, $\beta > 0$, $\delta > 0$, respectively. We apply Glaser's (1980) Lemma to the GIW distribution. Note that

$$\eta(x) = \frac{-g'(x)}{g(x)} = (\beta\delta + 1)x^{-1} - \lambda\beta x^{-\beta-1},$$

and $\eta'(x) = 0$ implies $x_0 = \left(\frac{\lambda\beta(\beta+1)}{\beta\delta+1}\right)^{1/\beta}$. Consequently, there exists x_0 such that $\eta'(x) > 0$ for $0 < x < x_0$ and $\eta'(x) < 0$ for $x > x_0$, so that $h_G(x)$ is upside down bathtub (UBT) shape.

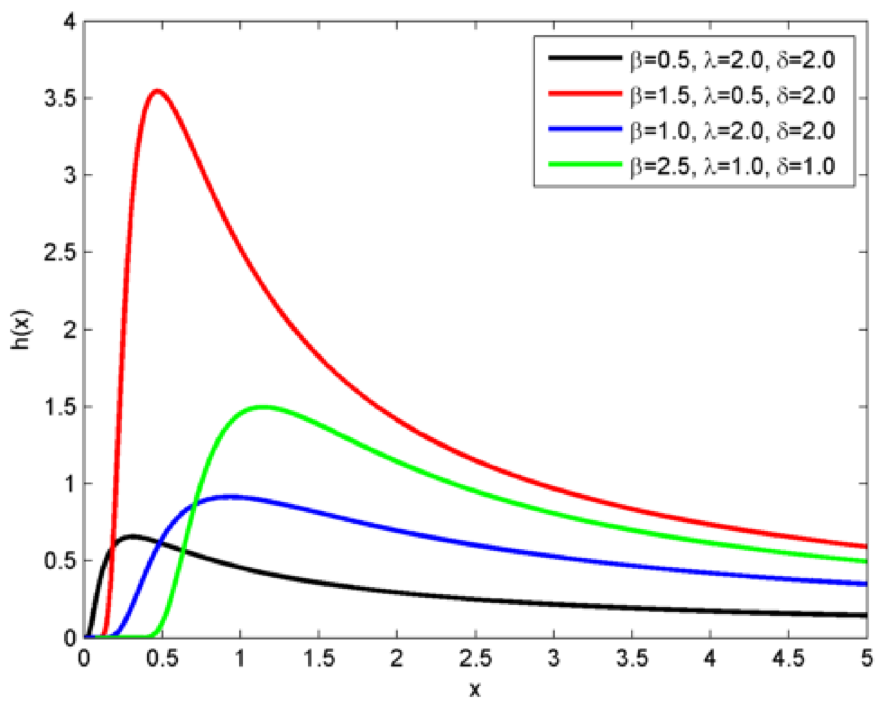

Figure 3: Graph of GIW Hazard Rate Function

The graphs of the hazard function for four combinations of the values of the model parameters are also presented. The plots of the hazard rate function show various shapes including, uni-modal and upside down bathtub shapes with four combinations of the values of the parameters. This attractive flexibility makes the GIW hazard rate function useful and suitable for non-monotone empirical hazard behaviors which are more likely to be encountered or observed in real life situations.

3 Moments and Moment Generating Function

In this section we obtain moments for the GIW distribution. The r^{th} raw moment is obtained as follows:

$$\begin{aligned} E(X^r) &= \int_0^\infty x^r g(x;\beta,\lambda,\delta)dx \\ &= \frac{1}{\Gamma(\delta)}\int_0^\infty x^r \beta x^{-1}[\lambda x^{-\beta}]^\delta \exp[-\lambda x^{-\beta}]dx \\ &= \frac{\lambda^{\frac{r}{\beta}}\Gamma(\delta-\frac{r}{\beta})}{\Gamma(\delta)}, \quad \beta \geq r. \end{aligned} \tag{20}$$

where we use $u=\lambda x^{-\beta}$ in the integral. Let $C_r=\Gamma(\delta-\frac{r}{\beta})$ and $C_0=\Gamma(\delta)$. Using equation (20), the mean, variance, coefficient of variation (CV), coefficient of Skewness (CS) and coefficient of Kurtosis (CK) are readily obtained. The mean and variance are given by

$$\mu = E(X) = \frac{\lambda^{\frac{1}{\beta}}\Gamma(\delta-\frac{1}{\beta})}{\Gamma(\delta)} = \frac{\lambda^{\frac{1}{\beta}}C_1}{C_0}, \tag{21}$$

and

$$\sigma^2 = E(X^2)-[E(X)]^2 = \frac{\lambda^{\frac{2}{\beta}}[C_0C_2-C_1^2]}{C_0^2}, \tag{22}$$

respectively. The coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) are given by

$$CV = \frac{\sigma}{\mu} = \sqrt{\frac{C_0C_2}{C_1^2}-1}, \tag{23}$$

$$CS = \frac{E[(X-\mu)^3]}{\sigma^3} = \frac{C_0^2C_3-3C_0C_1C_2+2C_1^3}{[C_0C_2-C_1^2]^{3/2}} \tag{24}$$

and

$$CK = \frac{E[(X-\mu)^4]}{\sigma^4} = \frac{C_0^3C_4-4C_0^2C_1C_3+6C_0C_1^2C_2-3C_1^4}{(C_0C_2-C_1^2)^2}, \tag{25}$$

respectively. Recall the Taylor's series expansion of the function e^{tx}, that is $e^{tx}=\sum_{j=0}^\infty \frac{(tx)^j}{j!}$, so the moment-generating function (MGF) of the GIW distribution is given by

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \frac{1}{\Gamma(\delta)} \sum_{j=0}^{\infty} \frac{t^j}{j!} \int_0^{\infty} \beta x^{j-1} [\lambda x^{-\beta}]^{\delta} \exp[-\lambda x^{-\beta}] dx \\ &= \frac{1}{\Gamma(\delta)} \sum_{j=0}^{\infty} \frac{(t\lambda^{\frac{1}{\beta}})^j}{j!} \Gamma(\delta - \frac{j}{\beta}), \quad \beta > j. \end{aligned}$$

4 Mean Deviations, Lorenz and Bonferroni Curves

In this section, we present the mean deviation about the mean, the mean deviation about the median, Lorenz and Bonferroni curves. Bonferroni and Lorenz curves are income inequality measures that are also useful and applicable to other areas including reliability, demography, medicine and insurance. The mean deviation about the mean and mean deviation about the median are defined by

$$D(\mu) = \int_0^{\infty} |x - \mu| g(x) dx, \quad D(M) = \int_0^{\infty} |x - M| g(x) dx, \tag{26}$$

respectively, where $\mu = E(X)$ and $M = Median(X) = G^{-1}(1/2)$ is the median of G. These measures $D(\mu)$ and $D(M)$ can be calculated using the relationships:

$$D(\mu) = 2\mu G(\mu) - 2\mu + 2\int_{\mu}^{\infty} x g(x) dx = 2\mu G(\mu) - 2\int_0^{\mu} x g(x) dx, \tag{27}$$

and

$$D(M) = -\mu + 2\int_M^{\infty} x g(x) dx = \mu - 2\int_0^M x g(x) dx. \tag{28}$$

Lorenz and Bonferroni curves are given by

$$L(G(x)) = \frac{\int_0^x t g(t) dt}{E(X)}, \quad \text{and} \quad B(G(x)) = \frac{L(G(x))}{G(x)}, \tag{29}$$

or

$$L(p) = \frac{1}{\mu} \int_0^q t g(t) dt, \quad \text{and} \quad B(p) = \frac{1}{p\mu} \int_0^q t g(t) dt, \tag{30}$$

respectively, where $q = G^{-1}(p)$. Let $T(x) = \int_0^x t g(t) dt$, and set $u = \lambda t^{-\beta}$, then

$$T(x) = \frac{\lambda^{1/\beta}}{\Gamma(\delta)} \int_{\lambda x^{-\beta}}^{\infty} u^{\delta - \frac{1}{\beta} - 1} e^{-u} du = \frac{\lambda^{1/\beta} \Gamma(\delta - 1/\beta, \lambda x^{-\beta})}{\Gamma(\delta)}. \tag{31}$$

Consequently, the mean deviation about the mean is $D(\mu) = 2\mu G(\mu) - 2T(\mu)$, where μ is obtained from equation (18), and the mean deviation about the median is $D(M) = \mu - 2T(M)$, where $M = G^{-1}(1/2)$. Lorenz and Bonferroni curves are given by

$$L(G(x)) = \frac{\lambda^{1/\beta}\Gamma(\delta - 1/\beta, \lambda x^{-\beta})}{\mu\Gamma(\delta)}, \quad \text{and} \quad B(G(x)) = \frac{\lambda^{1/\beta}\Gamma(\delta - 1/\beta, \lambda x^{-\beta})}{G(x)\Gamma(\delta)}, \tag{32}$$

respectively, for $\beta > 1$.

5 Some Measures of Uncertainty

In this section, we present Shannon entropy and Rényi entropy for the GIW distribution. The concept of entropy plays a vital role in information theory. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty.

5.1 Shannon Entropy

Shannon entropy, $H[g(x)]$ is defined to be $H\left[g(X;\beta,\lambda,\delta)\right] = E_G\left[-\log(g(X;\beta,\lambda,\delta)\right]$. That is,

$$\begin{aligned} H\left[g(X;\beta,\lambda,\delta)\right] &= -\log\left[\frac{\beta}{\Gamma(\delta)}\right] + (1+\beta\delta)\int_0^\infty \log(x)g(x)dx \\ &- \delta\log(\lambda)\int_0^\infty g(x)dx + \int_0^\infty \lambda x^{-\beta}g(x)dx \\ &= -\log\left[\frac{\beta}{\Gamma(\delta)}\right] + \delta + \frac{\log(\lambda)}{\beta} - \frac{(\beta\delta+1)\Gamma'(\delta)}{\beta\Gamma(\delta)}. \end{aligned}$$

5.2 Rényi Entropy

Rényi entropy is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v}\log\left(\int_0^\infty [g(x;\beta,\lambda,\delta)]^v dx\right), \quad v \neq 1, v > 0. \tag{33}$$

Rényi entropy tends to Shannon entropy as $v \to 1$. Note that Rényi entropy is given by

$$\begin{aligned} I_R(v) &= \left(\frac{1}{1-v}\right)\log\left[\frac{\beta^{v-1}}{[v^\delta\Gamma(\delta)]^v}\int_0^\infty \beta x^{-v}[\lambda v x^{-\beta}]^{\delta v}\right. \\ &\times \left. exp[-\lambda v x^{-\beta}\right]dx \\ &= \left(\frac{1}{1-v}\right)\log\left[\frac{\beta^{v-1}(\lambda v)^{\frac{1-v}{\beta}}}{[v^\delta\Gamma(\delta)]^v}\Gamma\left(\delta v+\frac{v-1}{\beta}\right)\right], \quad v\neq 1, v>0. \end{aligned}$$

6 Maximum Likelihood Estimation

Let $x_1, x_2, ..., x_n$ be a random sample of size n from the GIW(λ, β, δ) distribution. The log-likelihood function, which we will denote by L is given by

$$\begin{aligned} L = \log[l(\beta,\lambda,\delta)] &= n\log(\beta) - n\log(\Gamma(\delta)) + n\delta\log(\lambda) \\ &- (\beta\delta+1)\sum_{i=1}^{n} log(x_i) - \lambda\sum_{i=1}^{n} x_i^{-\beta}. \end{aligned} \tag{34}$$

The elements of the score vector are given by

$$\frac{\partial L}{\partial\beta} = \frac{n}{\beta} - \delta\sum_{i=1}^{n}\log(x_i) + \lambda\sum_{i=1}^{n}\frac{\log(x_i)}{x_i^\beta}, \tag{35}$$

$$\frac{\partial L}{\partial\lambda} = \frac{n\delta}{\lambda} - \sum_{i=1}^{n} x_i^{-\beta}, \tag{36}$$

and

$$\frac{\partial L}{\partial\delta} = \frac{-n\Gamma'(\delta)}{\Gamma(\delta)} + n\log(\lambda) - \beta\sum_{i=1}^{n}\log(x_i). \tag{37}$$

The maximum likelihood estimates, $\hat{\Theta}$ of $\Theta = (\beta, \lambda, \delta)$ are obtained by solving the nonlinear equations $\frac{\partial L}{\partial \beta} = 0$, $\frac{\partial L}{\partial \lambda} = 0$, and $\frac{\partial L}{\partial \delta} = 0$. These equations are not in closed form and the values of the parameters β, λ, δ must be found by using iterative methods.

The mixed second partial derivatives of the GIW distribution are given by

$$\frac{\partial^2 L}{\partial\beta^2} = \frac{-n}{\beta^2} - \lambda\sum_{i=1}^{n}\frac{[\log(x_i)]^2}{x_i^\beta}, \tag{38}$$

$$\frac{\partial^2 L}{\partial\beta\partial\lambda} = \sum_{i=1}^{n}\frac{\log(x_i)}{x_i^\beta}, \quad \frac{\partial^2 L}{\partial\beta\partial\delta} = -\sum_{i=1}^{n}\log(x_i), \tag{39}$$

$$\frac{\partial^2 L}{\partial \lambda^2} = \frac{-n\delta}{\lambda^2}, \quad \frac{\partial^2 L}{\partial \lambda \partial \delta} = \frac{n}{\lambda}, \tag{40}$$

and

$$\frac{\partial^2 L}{\partial \delta^2} = n\left[\frac{(\Gamma'(\delta))^2 - \Gamma(\delta)\Gamma''(\delta)}{(\Gamma(\delta))^2}\right]. \tag{41}$$

The elements of the information matrix are given by the negative expected values of the second mixed partial derivatives. These are given below:

$$\begin{aligned} I_{11} &= -E\left[\frac{\partial^2 L}{\partial \beta^2}\right] \\ &= \frac{n}{\beta^2}\left(1 + \delta[\log(\lambda)]^2 + \frac{\Gamma''(\delta+1) - 2\log(\lambda)\Gamma'(\delta+1)}{\Gamma(\delta)}\right), \end{aligned} \tag{42}$$

$$I_{12} = -E\left[\frac{\partial^2 L}{\partial \beta \partial \lambda}\right] = \frac{n}{\lambda\beta}\left[\frac{\Gamma'(\delta+1)}{\Gamma(\delta)} - \delta\log(\lambda)\right], \tag{43}$$

$$I_{13} = -E\left[\frac{\partial^2 L}{\partial \beta \partial \delta}\right] = \frac{n}{\beta}\left[\log(\lambda) - \frac{\Gamma'(\delta)}{\Gamma(\delta)}\right], \tag{44}$$

$$I_{22} = -E\left[\frac{\partial^2 L}{\partial \lambda^2}\right] = \frac{n\delta}{\lambda^2}, \quad I_{23} = -E\left[\frac{\partial^2 L}{\partial \lambda \partial \delta}\right] = -\frac{n}{\lambda}, \tag{45}$$

and

$$I_{33} = -E\left[\frac{\partial^2 L}{\partial \delta^2}\right] = n\left\{\frac{\Gamma''(\delta)\Gamma(\delta) - [\Gamma'(\delta)]^2}{[\Gamma(\delta)]^2}\right\}. \tag{46}$$

6.1 Asymptotic Confidence Intervals

In this section, we present the asymptotic confidence intervals for the parameters of the GIW distribution. The expectations in the Fisher Information Matrix (FIM) can be obtained numerically. Let $\hat{\Theta} = (\hat{\lambda}, \hat{\beta}, \hat{\delta})$ be the maximum likelihood estimate of $\Theta = (\lambda, \beta, \delta)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\Theta} - \Theta) \xrightarrow{d} N_3(\underline{0}, I^{-1}(\Theta))$, where $I(\Theta)$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $I(\Theta)$ is replaced by the observed information matrix evaluated at $\hat{\Theta}$, that is $J(\hat{\Theta})$. The multivariate normal distribution $N_3(\underline{0}, J(\hat{\Theta})^{-1})$, where the mean vector $\underline{0} = (0, 0, 0)^T$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions.

The likelihood ratio (LR) test can be used to compare the fit of the GIW distribution with its sub-models for a given data set. In fact to test $\delta = 1$, the LR statistic $\omega = 2[\ln L(\hat{\lambda}, \hat{\beta}, \hat{\delta}) - \ln L(\tilde{\lambda}, \tilde{\beta}, 1)]$, where $\hat{\lambda}, \hat{\beta}$, and $\hat{\delta}$ are the unrestricted estimates, and $\tilde{\lambda}$ and $\tilde{\beta}$ are the restricted estimates. The LR test rejects the null hypothesis H_0 if $\omega > \chi^2_\eta$, where χ^2_η denotes the upper $100\eta\%$ point of the χ^2 distribution with 1 degree of freedom.

7 Applications

In this section, we illustrate the usefulness of the GIW distribution applied to a real data set. We fit the density functions of the gamma inverse Weibull (GIW),inverse Weibull (IW), gamma inverse exponential (GIE), and gamma inverse Rayleigh (GIR). Estimates of the parameters of the distributions, standard errors (in parentheses), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC) are given in Table 1 for the first data set, and Table 2 for the second data set.

The first data set from Bjerkedal (1960) represents the survival times, in days of guinea pigs injected with different doses of tubercle bacilli. The data set consists of 72 observations and are listed below: 12, 15, 22, 24 ,24, 32, 32, 33, 34, 38, 38, 43,44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68,70, 70, 72, 73, 75, 76,76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376.

The second data set consists of the number of million of revolutions before failure of each of 23 ball bearings in a life testing experiment, see Lawless, (1982, p. 228). The observations are listed below: 17.88, 28.92, 33.00, 41.52, 42.12, 45.6, 48.8, 51.84, 51.96, 54.12, 55.56, 67.8, 68.44, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 105.84, 127.92, 128.04, 173.4.

Estimates of the parameters of GIW distribution (standard error in parentheses), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC) are given in Table 1 for the first data set and in Table 2 for the second data set.

Plots of the fitted densities and the histogram of the data are given in Figure 4 for the guinea pigs data, and Figure 5 for the ball bearings data.

The LR statistic of the hypothesis H_0: $IW(\lambda, \beta, 1)$ against H_a: $GIW(\lambda, \beta, \delta)$, is $\omega = 791.3 - 780.5 = 10.8$. The p-value is $1.02 \times 10^{-3} < 0.001$. Therefore, we reject H_0 in favor of H_a. There is a significant difference between IE and IW distributions with $\omega = 13.7$ and p-value=0.000214. Thus, reject H_0 in favor of H_a. A test of H_0: GIE vs H_a : GIW shows that $\omega = 4.7$ and p-value=0.03016. Thus, we reject H_0 in favor of H_a. The values of the statistics AIC, AICC and BIC show that the GIW distribution is a "better" fit for the guinea pig survival times data.

Table 1: Estimates of Models for Guinea Pigs Data

	Estimates			Statistics			
Model	λ	β	δ	$-2\log L$	AIC	$AICC$	BIC
GIW(λ,β,δ)	159.15 (340.99)	0.1569 (.2535)	80.9857 (261.02)	780.5	786.5	786.9	793.3
IW$(\lambda,\beta,1)$	283.84 (125.63)	1.4148 (0.1173)	1	791.3	795.3	795.5	799.9
GIR$(\lambda,2,\delta)$	1349.19 (277.20)	2	0.6167 (0.0865)	799.8	803.8	804.0	808.4
IR$(\lambda,2,1)$	2187.94 (257.85)	2	1	813.5	815.5	815.5	817.7
GIE$(\lambda,1,\delta)$	130.13 (22.7656)	1	2.1652 (0.3368)	785.2	789.2	789.4	793.8
IE$(\lambda,1,1)$	60.0975 (7.0826)	1	1	805.3	807.3	807.4	809.6

For the second data set, the LR statistic for the hypothesis H_0: $GIE(\lambda,1,\delta)$ against H_a: $GIW(\lambda,\beta,\delta)$, is $\omega = 228.3 - 226.5 = 1.8$. The p-value is 0.1797. Therefore, there is no significant difference between GIE and GIW distributions. There is also no significant difference between GIR and IR distributions. There is a significant difference between GIE and IE distributions with $\omega = 15.2$ and p-value=0.0000967. A test of H_0: IW vs H_a : GIW shows that $\omega = 5.1$ and p-value=0.02393. Thus, we reject H_0 in favor of H_a. However, the values of the statistics AIC, AICC and BIC are smaller and show that the GIE distribution is a “better” fit for the ball bearings data.

8 Concluding Remarks

We have presented and developed the mathematical properties of a new class of distributions called the gamma-inverse Weibull (GIW) distribution including the hazard and reverse hazard functions, moments, entropy, mean deviations, Lorenz and Bonferroni curves, Fisher information and maximum likelihood estimates. Applications of the proposed model to real data in order to demonstrate the usefulness and applicability of the class of distributions are also

Table 2: Estimates of Models for Ball Bearings Data

	Estimates			Statistics			
Model	λ	β	δ	$-2\log L$	AIC	$AICC$	BIC
GIW(λ, β, δ)	268.48 (380.32)	0.1626 (0.1715)	137.23 (289.59)	226.5	232.5	233.7	235.9
IW($\lambda, \beta, 1$)	1240.59 (1231.77)	1.8344 (0.2693)	1	231.6	235.6	7236.2	237.8
GIR($\lambda, 2, \delta$)	2218.59 (740.02)	2	0.9886 (0.2564)	231.9	235.9	236.5	238.2
IR($\lambda, 2, 1$)	2244.37 (276.98)	2	1	231.9	233.9	234.1	235.1
GIE($\lambda, 1, \delta$)	202.51 (61.3048)	1	3.6783 (1.0392)	228.3	232.3	232.9	234.6
IE($\lambda, 1, 1$)	55.0551 (11.4798)	1	1	243.5	245.5	245.6	246.6

presented.

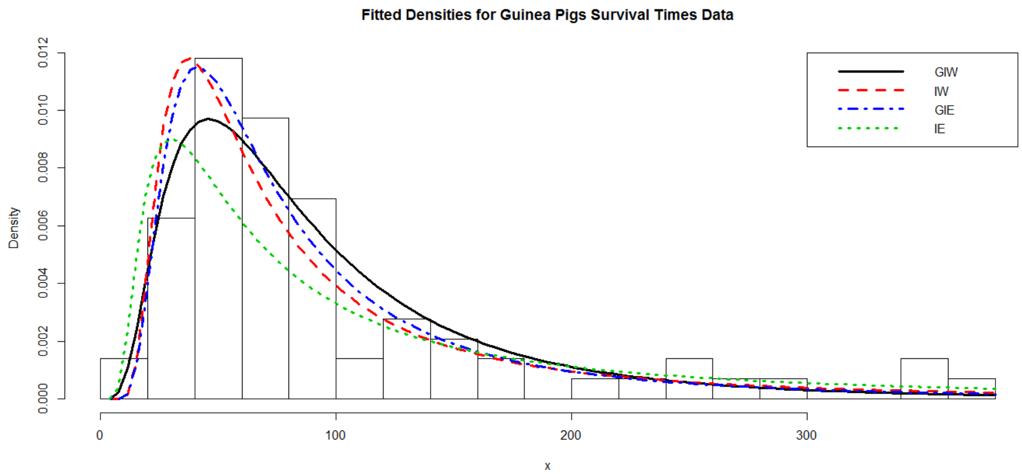

Figure 4: Histogram and Fitted Density for Guinea Pig Data

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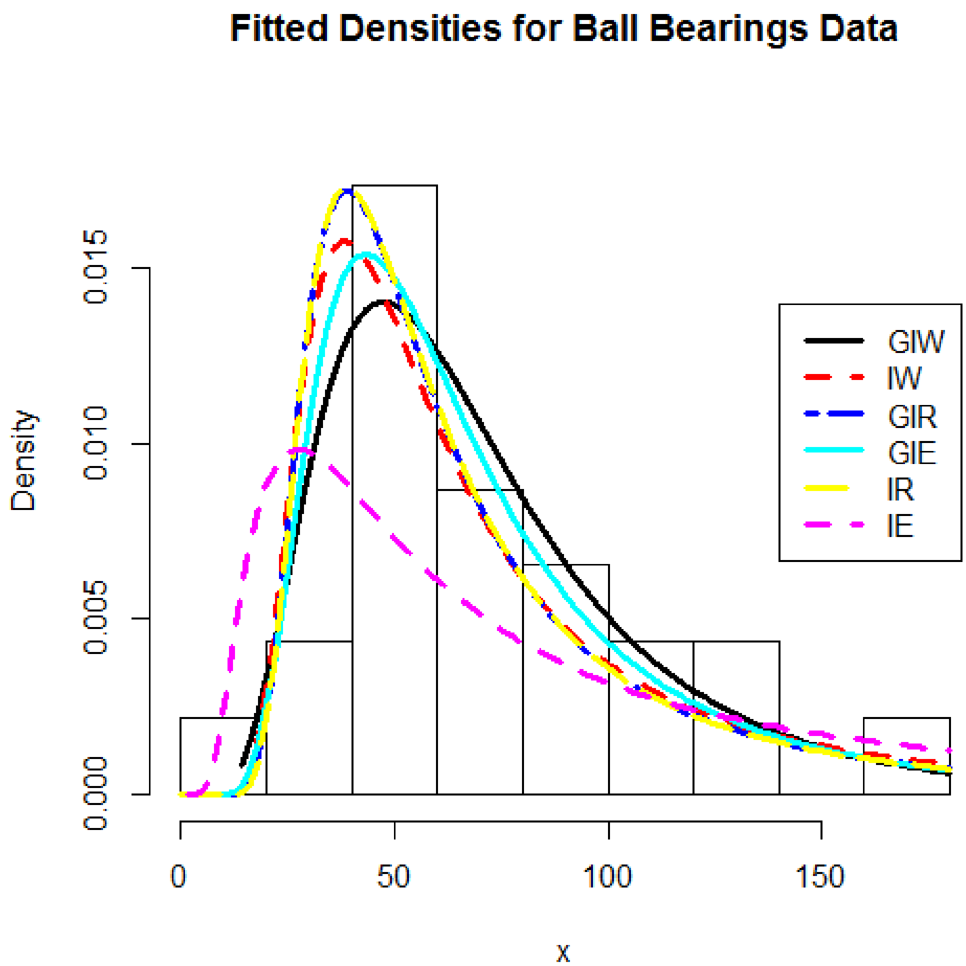

Figure 5: Histogram and Fitted Density for Ball Bearings Data