The Short Proof Of The Fermat Primes Problem.

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ABSTRACT AND DEFINITIONS

A *Fermat prime* (see [1] or [2] or [3] or [4] or [5] or [6] or [7] or [8] or [9]) is a prime number of the form $F_n = 2^{2^n} + 1$, where n is an integer ≥ 0. Fermat primes are characterized via divisibility in [6] and [7]. It is known (see [4] or [5] or [8] or [9]) that for every $j \in \{0, 1, 2, 3, 4\}$, F_j is a Fermat prime. The Fermat primes problem stipulates that there are infinitely many Fermat primes. That being so, in this paper, we give the short analytic simple proof of the Fermat primes problem, by reducing this problem into an equation of three unknowns and by using elementary combinatoric coupled with elementary computation, elementary arithmetic calculus, elementary divisibility and trivial complex calculus. Moreover, our paper clearly shows that divisibility helps to characterize Fermat primes as we did in [6] and [7], and elementary computation coupled with elementary arithmetic calculus and trivial complex calculus help to give the simple proof of the Fermat primes problem.

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PROLOGUE. In Section.*1*, we introduce definitions that are not standard and we present some elementary properties deduced from these definitions. In Section.*2*, we prove some properties linked to trivial complex calculus, elementary divisibility, trivial computation, elementary arithmetic calculus, and we reduce the Fermat primes problem into an equation of three unknowns [few elementary properties of Section.*2* will remain unproved and will be proved in Section.*2'* (Epilogue)]. In Section.*3*, using a simple proposition proved in Section.*1*, and some elementary properties of Section.*2*, we give the short proof of the Fermat primes problem. In Section.*2'* (Epilogue), we end this article by proving elementary properties we let unproved in Section.*2*.

1. *INTRODUCTION*

In this section, we introduce definitions that are not standard and we present some elementary properties deduced from these definitions.

Definitions 1.0. For every integer $n \geq 2$, we define $\mathcal{F}(n)$, f_n, and $f_{n.1}$ as follows:
$\mathcal{F}(n) = \{x; 1 < x < 2n \, and \, x \, is \, a \, Fermat \, prime\}$, $f_n = \max_{f \in \mathcal{F}(n)} f$, and

$$f_{n.1} = 2 f_n^{f_n} \prod_{f \in \mathcal{F}(n)} f$$

[observing (see Abstract And Definitions) that 3 and 5 are Fermat prime numbers, then it becomes immediate to deduce that for every integer $n \geq 3$, $\{3, 5\} \subseteq \mathcal{F}(n)$ and $f_n \geq 5$ and $f_{n.1} \geq 2 \times 5^5 \times 3 \times 5 > 93749$].

Using the previous definitions and denotations, let us remark.

Remark 1.1.*Let n be an integer $\geq F_3$ (see Abstract And Definitions for F_3); look at $\mathcal{F}(n)$, f_n, and $f_{n.1}$ introduced in Definitions* 1.0. *Then we have the following five simple properties.*
*(*1.1.0.*)* $-1 + F_3 < f_n < f_{n.1}$; *$f_{n.1}$ is even;* $f_{n.1} > 2 f_n^{f_n} > F_3^{F_3}$; *and*

$$f_{n.1} = 2 f_n^{f_n} \prod_{f \in \mathcal{F}(n)} f.$$

(1.1.1.) **If** *$f_n < n$, then: $f_n = f_{n-1}$ and $f_{n.1} = f_{n-1.1}$.*
(1.1.2.) **If** *$f_{n.1} \leq 2n$, then $f_n < n$ and $f_{n.1} = f_{n-1.1}$.*
(1.1.3) (The direct using of Fermat prime). **If** *the Fermat prime f_n is of the form $f_n < n$, then $f_n = f_{n-1}$ and $f_{n.1} = f_{n-1.1}$.*
(1.1.4) (The implicite using of Fermat prime). **If** *$f_{n.1} \leq 2n$, then the Fermat prime f_n is of the form $f_n < n$ and $f_{n.1} = f_{n-1.1}$.*
Proof. Property (1.1.0) is trivial [**Indeed**, it suffices to use the definition of f_n and $f_{n.1}$, and the fact that $F_3 \in \mathcal{F}(n)$ (note that F_3 is a Fermat prime (use Abstract and Definitions), and observe that n is an integer $\geq F_3$)]. Property (1.1.1) is immediate [**Indeed**, **if** $f_n < n$, clearly $n > F_3$ (use the definition of f_n and observe that $F_3 \in \mathcal{F}(n)$, since n is an integer $\geq F_3$), and so $f_n < n < 2n - 2$ (since $n > F_3$ (by the previous) and $f_n < n$ (via the hypotheses)); consequently

$$f_n < 2n - 2 \qquad (1.1).$$

Inequality *(1.1)* immediately implies that $\mathcal{F}(n) = \mathcal{F}(n-1)$ and therefore

$$f_n = f_{n-1} \qquad (1.2).$$

Equality *(1.2)* immediately implies that $f_{n.1} = f_{n-1.1}$. Property (1.1.1) follows]. Property (1.1.2) is trivial [**Indeed**, if $f_{n.1} \leq 2n$, then using the previous inequality and the definition of $(f_{n.1}, f_n)$ and the fact that $n \geq F_3$, it becomes trivial to deduce that

$$f_n < n \qquad (1.3).$$

So $f_{n.1} = f_{n-1.1}$ (use *(1.3)* and property (1.1.1)). Property (1.1.2) follows]. Property (1.1.3) is the trivial reformulation of property (1.1.1) and property (1.1.4) is an immediate reformulation of property (1.1.2). Remark 1.1 follows. □

Using the definition of $f_{n.1}$ (use Definitions 1.0) , then the following remark and proposition become immediate.

Remark 1.2. **If** $\lim_{n\to+\infty} f_{n.1} = +\infty$*, then there are infinitely many Fermat primes.*
Proof. Immediate [indeed, it suffices to use the definition of $f_{n.1}$ (use Definitions 1.0)]. □

Proposition 1.3.

If *for every integer $n \geq F_3$, $f_{n.1} > n$ or $2n - 1$ is a Fermat prime or $2n + 1$ is a Fermat prime,*

then there are infinitely many Fermat primes.
Proof. Clearly $\lim_{n\to+\infty} f_{n.1} = +\infty$; therefore there are infinitely many Fermat primes [use the previous equality and apply Remark 1.2]. □

Proposition 1.3 clearly says that: **if** *for every integer $n \geq F_3$, $f_{n.1} > n$ or $2n - 1$ is a Fermat prime or $2n + 1$ is a Fermat prime , then there are infinitely many Fermat primes*; this is what we will do in Section.*3*, by using only Proposition 1.3, elementary combinatoric, elementary complex calculus, elementary divisibility, trivial computation, elementary arithmetic calculus and reasoning by reduction to absurd via the reduction of the Fermat primes problem into an equation of three unknowns. Proposition 1.3 is stronger than all the investigations that have been done on the Fermat primes problem in the past. Morerover, the reader can easily see that Proposition 1.3 is easy and is completely different from all the investigations that have been done on the Fermat primes problem in the past. So, in Section.*3*, when we will give the analytic simple proof of the Fermat primes problem, we will not need strong investigations that have been done on the previous problem in the past.

2. SOME PROPERTIES LINKED TO TRIVIAL COMPLEX CALCULUS, ELEMENTARY COMPUTATION AND ELEMENTARY ARITHMETIC CALCULUS; THE REDUCTION OF THE FERMAT PRIMES PROBLEM INTO AN EQUATION OF THREE UNKNOWNS

In this section, the definitions of $\mathcal{F}(n)$, f_n, and $f_{n.1}$ (use Definitions 1.0) are crucial.

Recalls 2.1 $\Big($ *Real numbers*, $\mathcal{R}$, *complex numbers* and $\mathcal{C}$ $\Big)$. Recall that $\mathcal{R}$ is the set all *real numbers* and θ is a *complex number* if $\theta = x + iy$, where $x \in \mathcal{R}$, $y \in \mathcal{R}$ and $i^2 = -1$; $\mathcal{C}$ is the set of all *complex numbers*. That being said, we have the following three remarks which will help us when we will reduce the Fermat primes problem into an elementary equation of three unknowns.

Remark.2.1.0. *Let n be an integer $\geq F_3$ (recall [use Abstract and Definitions] that $F_3 = 2^{2^3} + 1$) and let $f_{n.1}$ (use Definitions* 1.0*); consider (X_n, Y_n, Z_n, x, y, k) where*

$$X_n = \sum_{j=1}^{4} X_{n.j} \qquad (2.0)$$

and where

$$X_{n.1} = 0 \qquad (2.1),$$

$$X_{n.2} = -8n(16f_{n.1}^{-3} + 50f_{n.1}^{-2} + 11if_{n.1}^{-3} - 13i + 5if_{n.1}) + 8f_{n.1} - 8i - 16 \qquad (2.2),$$

$$X_{n.3} = -8n(34if_{n.1}^{-2} - 70if_{n.1}^{-3} - 11i + 5 + 6if_{n.1}) \qquad (2.3),$$

and

$$X_{n.4} = -8n(11if_{n.1} + 7 - 50f_{n.1}^{-1} + 23if_{n.1}^{-3} - 54f_{n.1}^{-3}) \qquad (2.4);$$

$$Y_n = 32n - 8f_{n.1} + 8i + 16 \qquad (2.5),$$

$$Z_n = -32n - 8f_{n.1} + 8i + 16 \qquad (2.6),$$

$$x = -8f_{n.1} + 8i + 16,\ y = 32n \,and\, k = 16nf_{n.1}^{-3} - 16nf_{n.1}^{-2} + 16nf_{n.1}^{-1} \qquad (2.7).$$

Then

$$x + y - Y_n = 0 \,and\, x - y - Z_n = 0$$

and

$$x + 3iyf_{n.1}^{-1} + k(6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^{2} - 18i) + X_n = 0;\ k \in \mathcal{R}.$$

Proof. Let (X_n, Y_n, Z_n, x, y, k) explicited above and consider (Y_n, Z_n, x, y); using *(2.5)* and (2.6) and the first two equalities of *(2.7)* , we easily check (by elementary computation) that

$$x + y - Y_n = 0 \,and\, x - y - Z_n = 0 \qquad (2.8).$$

That being so, look at (X_n, Y_n, Z_n, x, y, k) explicited above and let (X_n, x, y, k); using the three equalities of *(2.7)* and *(2.j)* (where $0 \leq j \leq 4$), it becomes very easy to check (by elementary computation and the fact that $i^2 = -1$) that

$$x + 3iyf_{n.1}^{-1} + k(6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^{2} - 18i) + X_n = 0;\ k \in \mathcal{R} \,and\, X_n = \sum_{j=1}^{4} X_{n.j} \qquad (2.9)$$

$\Big($ use the three equalities of *(2.7)* (for x and y and k); and *(2.j)* (for $X_{n.j}$ where $1 \leq j \leq 4$); and *(2.0)* (for X_n)$\Big)$. Remark.2.1.0 immediately follows $\Big($ use *(2.8)* and *(2.9)* $\Big)$.

Remark.2.1.1 **(Fundamental).** *Let n be an integer $\geq F_3$ (recall [use Abstract and Definitions] that $F_3 = 2^{2^3} + 1$) and let $f_{n.1}$ (use Definitions* 1.0*); consider (X_n, Y_n, Z_n, x, y, k) where*

$$X_n = \sum_{j=1}^{4} X_{n.j} \qquad (2.10)$$

and where

$$X_{n.1}=\frac{126301if_{n.1}^{-3}+2273418if_{n.1}^{-4}-23837f_{n.1}-357566f_{n.1}^{-1}+2010800f_{n.1}^{-2}+2399719f_{n.1}^{-4}-757806f_{n.1}^{-3}}{1331} \quad (2.11),$$

$$X_{n.2}=-4f_{n.1}(16f_{n.1}^{-3}+50f_{n.1}^{-2}+11if_{n.1}^{-3}-13i+5if_{n.1})+8f_{n.1}-8i-16 \quad (2.12),$$

$$X_{n.3}=\frac{289080}{1331} \quad (2.13),$$

and

$$X_{n.4}=5f_{n.1}(11if_{n.1}+7-50f_{n.1}^{-1}+23if_{n.1}^{-3}-54f_{n.1}^{-3})-\frac{1003112i}{1331} \quad (2.14);$$

$$Y_n=(1-if_{n.1})^2+f_{n.1}^2-1 \quad (2.15),$$

$$Z_n=4(\ (f_{n.1}-1)^2-1-f_{n.1}^2+f_{n.1}\ )+(2i+1)(4if_{n.1}+4-8i\ ) \quad (2.16),$$

$$x=10-6f_{n.1}+if_{n.1},\ y=-10+6f_{n.1}-3if_{n.1},\ and\, k=-\frac{35}{11}-\frac{618f_{n.1}^{-1}}{121}+\frac{7494f_{n.1}^{-2}}{121}+\frac{126301f_{n.1}^{-4}}{1331} \quad (2.17).$$

Then

$$x+y-Y_n=0\, and\, x-y-Z_n=0$$

and

$$x+3iyf_{n.1}^{-1}+k(6f_{n.1}-19-if_{n.1}+11if_{n.1}^2-18i)+X_n=0;\ k\in\mathcal{R}.$$

Proof. Let (X_n, Y_n, Z_n, x, y, k) explicited above and consider (Y_n, Z_n, x, y); using *(2.15)* and (*2.16*) and the first two equalities of *(2.17)* , we easily check (by elementary computation and the fact that $i^2=-1$) that

$$x+y-Y_n=0\, and\, x-y-Z_n=0 \quad (2.18).$$

That being so, look at (X_n, Y_n, Z_n, x, y, k) explicited above and let (X_n, x, y, k); using the three equalities of *(2.17)* and *(2.j)* (where $10\le j\le 14$), it becomes very easy to check (by elementary computation and the fact that $i^2=-1$) that

$$x+3iyf_{n.1}^{-1}+k(6f_{n.1}-19-if_{n.1}+11if_{n.1}^2-18i)+X_n=0;\ k\in\mathcal{R}\, and\, X_n=\sum_{j=1}^{4}X_{n.j} \quad (2.19)$$

$\Big($ use the three equalities of *(2.17)* (for x and y and k); and *(2.j+10)* (for $X_{n.j}$ where $1\le j\le 4$); and *(2.10)* (for X_n)$\Big)$. Remark.2.1.1 immediately follows $\Big($ use *(2.18)* and *(2.19)* $\Big)$.

Remark.2.1.2 **(Fundamental)**. *Let n be an integer $\ge F_3$ (recall [use Abstract and Definitions] that $F_3=2^{2^3}+1$) and let $f_{n.1}$ (use Definitions* 1.0*); consider (X_n, Y_n, Z_n, x, y) where*

$$X_n=\sum_{j=1}^{4}X_{n.j} \quad (2.20)$$

and where

$$X_{n.1}=\frac{126301if_{n.1}^{-3}+2273418if_{n.1}^{-4}-23837f_{n.1}-357566f_{n.1}^{-1}+2010800f_{n.1}^{-2}+2399719f_{n.1}^{-4}-757806f_{n.1}^{-3}}{1331} \quad (2.21),$$

$$X_{n.2}=0 \quad (2.22),$$

$$X_{n.3}=4f_{n.1}(34if_{n.1}^{-2}-70if_{n.1}^{-3}-11i+5+6if_{n.1})+\frac{289080}{1331} \quad (2.23),$$

and

$$X_{n.4}=9f_{n.1}(11if_{n.1}+7-50f_{n.1}^{-1}+23if_{n.1}^{-3}-54f_{n.1}^{-3})-\frac{1003112i}{1331} \quad (2.24);$$

$$Y_n=(1-4i-if_{n.1})^2+f_{n.1}^2-1 \quad (2.25),$$

$$Z_n=4(\ (f_{n.1}+1)^2-1-f_{n.1}^2+f_{n.1}\ )+4if_{n.1}+4-8i \quad (2.26),$$

$$x=2f_{n.1}+if_{n.1}-6-8i\ and\ y=-10-10f_{n.1}-3if_{n.1} \quad (2.27).$$

Then

$$x+y-Y_n=0\,and\,x-y-Z_n=0$$

and

$$there\,exists\,not\,k\in\mathcal{R}\,such\,that\,x+3iyf_{n.1}^{-1}+k(6f_{n.1}-19-if_{n.1}+11if_{n.1}^2-18i)+X_n=0.$$

Proof. Indeed let (X_n,Y_n,Z_n,x,y) explicited above and consider (Y_n,Z_n,x,y); using *(2.25)* and *(2.26)* and the two equalities of *(2.27)*, then we easily check (by elementary computation and the fact that $i^2=-1$) that

$$x+y-Y_n=0\,and\,x-y-Z_n=0 \qquad (2.28).$$

That being said, to prove Remark.2.1.2, it suffices to show that

$$there\,exists\,not\,k\in\mathcal{R}\,such\,that\,x+3iyf_{n.1}^{-1}+k(6f_{n.1}-19-if_{n.1}+11if_{n.1}^2-18i)+X_n=0$$

Fact.0.

$$there\,exists\,not\,k\in\mathcal{R}\,such\,that\,x+3iyf_{n.1}^{-1}+k(6f_{n.1}-19-if_{n.1}+11if_{n.1}^2-18i)+X_n=0.$$

Otherwise $\Big($ we reason by reduction to absurd$\Big)$

$$Let\,k\in\mathcal{R}\,such\,that\,x+3iyf_{n.1}^{-1}+k(6f_{n.1}-19-if_{n.1}+11if_{n.1}^2-18i)+X_n=0 \qquad (2.29).$$

It is immediate to see that *(2.29)* says that

$$x+3iyf_{n.1}^{-1}+X_n=-k(6f_{n.1}-19-if_{n.1}+11if_{n.1}^2-18i);\; k\in\mathcal{R}\,and\,X_n=\sum_{j=1}^{4}X_{n.j} \qquad (2.30)$$

$\Big($ use *(2.20)* for X_n $\Big)$. Now let (X_n,Y_n,Z_n,x,y) explicited above; consider (X_n,x,y) and look at *(2.30)*; using elementary computation and elementary divisibility coupled with the fact that $i^2=-1$ and $k\in\mathcal{R}$, it becomes very easy to check that *(2.30)* immediately implies that

$$k=k_{n.1}=k_{n.2} \qquad (2.31),$$

where

$$k_{n.1}=\frac{-23837f_{n.1}-357566f_{n.1}^{-1}+2010800f_{n.1}^{-2}+2399719f_{n.1}^{-4}-757806f_{n.1}^{-3}+289080}{1331(19-6f_{n.1})}+\frac{85f_{n.1}-447-486f_{n.1}^{-3}}{19-6f_{n.1}} \qquad (2.32),$$

and

$$k_{n.2}=\frac{2273418f_{n.1}^{-4}+126301f_{n.1}^{-3}-1003112}{1331(f_{n.1}-11f_{n.1}^2+18)}+\frac{123f_{n.1}^2+106f_{n.1}^{-1}-38-43f_{n.1}-73f_{n.1}^{-2}}{f_{n.1}-11f_{n.1}^2+18} \qquad (2.33)$$

$\Big($ via *(2.30)*, use the two equalities of *(2.27)* (for x and y); and *(2.j+20)* (for $X_{n.j}$ where $1\le j\le 4$); and *(2.20)* (for X_n)$\Big)$. That being so, using *(2.31)* and (2.32) and (2.33), then we immediately deduce (via elementary computation and the fact that $k_{n.1}=k_{n.2}$) that

$$10996722f_{n.1}^{-2}-11643588f_{n.1}^{-3}-7115526+7762392f_{n.1}^{-1}=0 \qquad (2.34).$$

Equality (2.34) is clearly impossible $\Big($ since $f_{n.1}>F_3^{F_3}$ (use property (1.1.0) of Remark 1.1) and therefore

$$10996722f_{n.1}^{-2}-11643588f_{n.1}^{-3}-7115526+7762392f_{n.1}^{-1}<0$$

$\Big)$.So assuming that

$$there\,exists\;k\in\mathcal{R}\,such\,that\,x+3iyf_{n.1}^{-1}+k(6f_{n.1}-19-if_{n.1}+11if_{n.1}^2-18i)+X_n=0$$

gives rise to a serious contradiction; therefore

$$there\,exists\,not\,k\in\mathcal{R}\,such\,that\,x+3iyf_{n.1}^{-1}+k(6f_{n.1}-19-if_{n.1}+11if_{n.1}^2-18i)+X_n=0.$$

Fact.0. follows and Remark.2.1.2 immediately follows.

We will use the previous three Remarks to introduce the notion of *tackle* which will help us to reduce the Fermat primes problem into an elementary equation of three unknowns; this notion is fundamental and crucial for the short complete simple proof of the Fermat primes problem.

Definition 2.2 **(Fundamental)** $\Big($ *tackle* $\Big)$. Recall $\Big($ use Recalls 2.1 $\Big)$ that $\mathcal{R}$ is the set all *real numbers* and $\mathcal{C}$ is the set of all *complex numbers*. Clearly

$$\mathcal{C}^2 = \{(x, y);\ x \in \mathcal{C}\, and\, y \in \mathcal{C}\}\ and\ \mathcal{C}^2 \times \mathcal{R} = \{(x, y, k); (x, y) \in \mathcal{C}^2\, and\, k \in \mathcal{R}\}$$

and

$$\mathcal{C}^3 = \{(x, y, z); (x, y) \in \mathcal{C}^2\, and\, z \in \mathcal{C}\}.$$

Now let n be an integer $\geq F_3$ and look at $f_{n.1}$ (use Definitions 1.0); we say that $(\ \phi(n), \nu(n), \epsilon(n)\ ) \in \mathcal{C}^3$ *tackles* $(1, 3if_{n.1}^{-1})$ *around* $6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i$, **if** there exists $(x, y, k) \in \mathcal{C}^2 \times \mathcal{R}$ such that

$$x + 3iyf_{n.1}^{-1} + k(6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i) + \phi(n) = 0\ where\ x + y - \nu(n) = 0\, and\, x - y - \epsilon(n) = 0.$$

We will see that the definition of tackle introduced above helps to reduce the Fermat primes problem into an elementary equation of three unknowns. Before, let us define:

Definitions 2.3 **(Fundamental)**. Let n be an integer $\geq F_3$ (recall [use Abstract and Definitions] that $F_3 = 2^{2^3} + 1$) and let $f_{n.1}$ (see Definitions 1.0); now consider $(\phi_n, \nu_n, \epsilon_n\ )$, where

$$\phi_n = \sum_{j=1}^{4} \phi_{n.j} \qquad (2.35)$$

and where

$$\phi_{n.1} = \frac{126301if_{n.1}^{-3} + 2273418if_{n.1}^{-4} - 23837f_{n.1} - 357566f_{n.1}^{-1} + 2010800f_{n.1}^{-2} + 2399719f_{n.1}^{-4} - 757806f_{n.1}^{-3}}{1331} \qquad (2.36),$$

$$\phi_{n.2} = ((2n+1)^2 - 1 - f_{n.1}^2 - 2f_{n.1})(16f_{n.1}^{-3} + 50f_{n.1}^{-2} + 11if_{n.1}^{-3} - 13i + 5if_{n.1}) + \frac{(f_{n.1} - 2n)}{2}(8f_{n.1} - 8i - 16) \qquad (2.37),$$

$$\phi_{n.3} = ((2n+1)^2 - 1 - f_{n.1}^2 + 2f_{n.1})(34if_{n.1}^{-2} - 70if_{n.1}^{-3} - 11i + 5 + 6if_{n.1}) + \frac{289080}{1331} \qquad (2.38),$$

and

$$\phi_{n.4} = ((2n+1)^2 - 1 - f_{n.1}^2 + 7f_{n.1})(11if_{n.1} + 7 - 50f_{n.1}^{-1} + 23if_{n.1}^{-3} - 54f_{n.1}^{-3}) - \frac{1003112i}{1331} \qquad (2.39);$$

$$\nu_n = (\ if_{n.1} - 4in - 4i + 1\ )^2 - 1 + f_{n.1}^2 \qquad (2.40),$$

and

$$\epsilon_n = 4(\ (2n+1)^2 - 1 - f_{n.1}^2 + f_{n.1}\ ) + (\ if_{n.1} - 2in + 1\ )(\ 4if_{n.1} + 4 - 8i\ ) \qquad (2.41).$$

It is immediate that for every integer $n \geq F_3$, $(\ \phi_n, \nu_n, \epsilon_n\ )$ is well defined and gets sense. Now using the notion of *tackle* $\Big($ use Definition 2.2 $\Big)$, then the following Theorem immediately implies the Fermat primes problem.

Theorem.F. *Let n be an integer $\geq F_3$ and let $f_{n.1}$ (use Definitions* 1.0*); look at $(\ \phi_n, \nu_n, \epsilon_n\ )$ introduced in Definitions* 2.3. *Then at least one of the following four properties is satisfied by n.*

(A_0). $2n - 1$ is a Fermat prime.

(A_1). $f_{n.1} \geq 2n$ and $2n + 1$ is a Fermat prime.

(A_2). $f_{n.1} \geq 2n + 4$.

(A_3). $(\ \phi_n, \nu_n, \epsilon_n\ )$ tackles $(1, 3if_{n.1}^{-1})$ around $6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i$ $\Big($ use Definition 2.2 *for the meaning of tackle $\Big)$.*

We will simply prove **Theorem.F** in Section.*3*. But before, let us remark.

Remark.2.3.1 **(fundamental: the using of Remark.**2.1.1**)** . *Let n be an integer $\geq F_3$ and*

let $f_{n.1}$ (use Definitions 1.0); now look at ($\phi_n, \nu_n, \epsilon_n$) introduced in Definitions 2.3. **If** $f_{n.1} = 2n + 2$, *then* ($\phi_n, \nu_n, \epsilon_n$) *tackles* $(1, 3if_{n.1}^{-1})$ *around* $6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i$ $\Big($ *use Definition 2.2 for the meaning of tackle* $\Big)$.
This Remark is explicite using of Remark.2.1.1. We will prove Remark.2.3.1 in Epilogue (Section.*2'*).

Remark.2.3.2 **(fundamental: reduction of the Fermat primes problem into a trivial equation of three unknowns ; the using of Remark.**2.1.2**)** . *Let n be an integer $\geq F_3$ and let $f_{n.1}$ (use Definitions 1.0); now look at ($\phi_n, \nu_n, \epsilon_n$) introduced in Definitions 2.3.* **If** $f_{n.1} = 2n$, *then* ($\phi_n, \nu_n, \epsilon_n$) *does not tackle* $(1, 3if_{n.1}^{-1})$ *around* $6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i$ $\Big($ *use Definition 2.2 for the meaning of tackle* $\Big)$.
We will prove Remark.2.3.2 in Epilogue (Section.*2'*) and we will see that Remark.2.3.2 reduces the Fermat primes problem into a simple equation of three unknowns. Indeed we will see in Epilogue (Section.*2'*) that Remark.2.3.2 clearly says that, if $f_{n.1} = 2n$, we will have a simple equation of three unknowns which implies that ($\phi_n, \nu_n, \epsilon_n$) does not tackle $(1, 3if_{n.1}^{-1})$ around $6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i$. We will use Remark.2.3.2 in Theorem 3.6 (Section.**3**) to immediately deduce the Fermat primes problem. Now using Definitions 2.3, then we have the following elementary Proposition.

Proposition 2.4.*Let n be an integer $\geq 1 + F_3$ and let $f_{n.1}$ (use Definitions 1.0); now look at* ($\phi_n, \nu_n, \epsilon_n$) *introduced in Definitions 2.3, and via* ($\phi_n, \nu_n, \epsilon_n$), *consider* ($\phi_{n-1}, \nu_{n-1}, \epsilon_{n-1}$) $\Big($ *this consideration gets sense, since $n \geq 1 + F_3$, and therefore $n - 1 \geq F_3$* $\Big)$. **If** $f_{n.1} \leq 2n$, *then* ($\phi_{n-1} - \phi_n, \nu_{n-1} - \nu_n, \epsilon_{n-1} - \epsilon_n$) *tackles* $(1, 3if_{n.1}^{-1})$ *around* $6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i$.
This Proposition is explicite using of Remark.2.1.0. We will prove Proposition 2.4 in Epilogue (Section.*2'*).

Having made the previous, we are now ready to give the analytic simple proof of the Fermat primes problem.

3.*THE SHORT ANALYTIC PROOF OF THE FERMAT PRIMES PROBLEM*

In this Section, the definitions of $\mathcal{F}(n)$, f_n and $f_{n.1}$ (use Definitions 1.0), the definition of F_3 (recall [see Abstract and Definitions] that $F_3 = 2^{2^3} + 1$), the definition of tackle (use Definition 2.2) and the definition of ($\phi_n, \nu_n, \epsilon_n$) (use Definitions 2.3), are fundamental and crucial.

Now the following Theorem immediately implies the Fermat primes problem.

Theorem 3.1. *Let n be an integer $\geq F_3$ and let $f_{n.1}$ (use Definitions 1.0); look at ($\phi_n, \nu_n, \epsilon_n$) introduced in Definitions 2.3. Then at least one of the following four properties is satisfied by n.*
(A_0). $2n - 1$ is a Fermat prime.
(A_1). $f_{n.1} \geq 2n$ and $2n + 1$ is a Fermat prime.
(A_2). $f_{n.1} \geq 2n + 4$.
(A_3). ($\phi_n, \nu_n, \epsilon_n$) tackles $(1, 3if_{n.1}^{-1})$ around $6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i$.

It is immediate that Theorem 3.1 is exactly **Theorem.F** stated in Section.**2** just after Definitions 2.3. We are going to prove simply Theorem 3.1. But before, let us remark.

Remark 3.2. *Let n be an integer $\geq F_3$ and let $f_{n.1}$. We have the following four elementary properties.*
*(*3.2.0.*)* **If** $f_{n.1} \geq 2n + 4$, *then Theorem* 3.1 *is satisfied by n.*
*(*3.2.1.*)* **If** $f_{n.1} = 2n + 2$, *then Theorem* 3.1 *is satisfied by n.*
*(*3.2.2.*)* **If** $n \leq 10 + F_3$, *then Theorem* 3.1 *is satisfied by n.*
*(*3.2.3.*) Let the Fermat prime f_n* $\Big($ *use Definitions* 1.0$\Big)$. **If** $f_n \geq 2n - 10$, *then Theorem* 3.1 *is satisfied by n.*

Proof. Property (3.2.0) is trivial $\Big($ **indeed** if $f_{n.1} \geq 2n + 4$, then property A_2 of Theorem 3.1 is clearly satisfied by $n\Big)$. Property (3.2.1) is immediate $\Big($**indeed** let n be an integer $\geq F_3$, observing (via the hypotheses) that $f_{n.1} = 2n + 2$, then

$$(\phi_n, \nu_n, \epsilon_n) \textit{ tackles } (1, 3if_{n.1}^{-1}) \textit{ around } 6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i \qquad (3.1)$$

$\Big($ use Remark 2.3.1$\Big)$. *(3.1)* clearly says that property A_3 of Theorem 3.1 is satisfied; therefore Theorem 3.1 is satisfied by n. Property (3.2.1) follows$\Big)$. Property (3.2.2) is immediate $\Big($ **indeed**, observing (by using property (1.1.0) of Remark 1.1) that $f_{n.1} > F_3^{F_3}$, and remarking (via the hypotheses) that $n \leq 10 + F_3$ (recall [see Abstract and Definitions] that $F_3 = 2^{2^3} + 1$) , then, using the previous two inequalities, it becomes trivial to deduce that

$$f_{n.1} > F_3^{F_3} > 2(10 + F_3) + 6 > 2n + 4 \qquad (3.2);$$

so

$$f_{n.1} > 2n + 4 \qquad (3.3)$$

$\Big($ use *(3.2)* $\Big)$. Theorem 3.1 is clearly satisfied by $n\Big($ use inequality *(3.3)* and property (3.2.0)$\Big)$. Property (3.2.3) is simple $\Big($ **indeed** if $f_n \geq 2n - 10$, then using the preceding inequality and the definition of (f_n,$f_{n.1}$), we immediately deduce that

$$f_{n.1} > f_n^{f_n} > -1 + (2n - 10)^{2n-10} \qquad (3.4).$$

Remarking that $n \geq F_3$ and using *(3.4)*, then we easily deduce that

$$f_{n.1} > -1 + (2n - 10)^{2n-10} > 2n + 4 \qquad (3.5);$$

so

$$f_{n.1} > 2n + 4 \qquad (3.6)$$

$\Big($ use *(3.5)* $\Big)$. Theorem 3.1 is clearly satisfied by n $\Big($ use inequality *(3.6)* and property (3.2.0)$\Big)$.□

Using Remark 3.2, let us Remark.

Remark 3.3. *Suppose that Theorem* 3.1 *is false; then there exists an integer* $n \geq F_3$ *such that* n *does not satisfied Theorem* 3.1. (*Proof.* Immediate.□)

From Remark 3.3, let us define:

Definitions 3.4 *(Fundamental).* **(i).** We say that n is a *counter-example to Theorem* 3.1, if $n \geq F_3$ and if n does not satisfied Theorem 3.1 (observe that if Theorem 3.1 is false, then such a n exists, via Remark 3.3) . **(ii).** We say that n is a *minimum counter-example to Theorem* 3.1, if n is a counter-example to Theorem 3.1 with n minimum (observe that if Theorem 3.1 is false, then such a n exists, by using **(i)**).

The previous simple remarks and definitions made, we now prove simply Theorem 3.1.

Proof of Theorem 3.1. Otherwise $\Big($ we reason by reduction to absurd,$\Big)$, let n be a minimum counter-example to Theorem 3.1 $\Big($such a n exists (use Remark 3.3 and Definitions 3.4)$\Big)$. We observe the following.

*Observation.*3.1.*i*. *Look at* $(n, f_{n.1})$ $\Big($*recall* n *is a minimum counter-example to Theorem* 3.1$\Big)$. *Then* $n > 10 + F_3$ *and* $f_{n.1} \leq 2n + 2$.

Clearly $n > 10 + F_3$ $\Big($Otherwise $n \leq 10 + F_3$ and Theorem 3.1 is satisfied by n [use the previous inequality and apply property (3.2.2) of Remark 3.2]; a contradiction, since n does not satisfy Theorem 3.1$\Big)$; and clearly $f_{n.1} \leq 2n + 2$ $\Big($Otherwise

$$f_{n.1} > 2n + 2;$$

noticing that $f_{n.1}$ and $2n + 2$ are even [$f_{n.1}$ is even (use the definition of $f_{n.1}$) and $2n + 2$ is trivially even], then we immediately deduce that the previous inequality implies that $f_{n.1} \geq 2n + 2 + 2$; so $f_{n.1} \geq 2n + 4$ and Theorem 3.1 is satisfied by n [use the preceding inequality and apply property (3.2.0) of Remark 3.2]; we have a contradiction since n does not satisfy Theorem 3.1$\Big)$.

Observation.3.1.i follows.

*Observation.*3.1.*ii*. *Look at* n $\Big($*recall* n *is a minimum counter-example to Theorem* 3.1$\Big)$. *Then*

$$(\phi_n, \nu_n, \epsilon_n) \textit{ does not tackle } (1, 3if_{n.1}^{-1}) \textit{ around } 6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i.$$

Immediate, since n is a counter-example to Theorem 3.1 and in particular, n does not satisfy property A_3 of Theorem 3.1.

Observation.3.1.*iii*. *Look at n and let* $f_{n.1}$. *Then*

$$f_{n.1} \leq 2n \ \ and \ \ f_{n.1} = f_{n-1.1}.$$

Firstly, we are going to show that $f_{n.1} \leq 2n$. *Fact:* $f_{n.1} \leq 2n$. Otherwise

$$f_{n.1} > 2n \qquad (3.7);$$

remarking that $f_{n.1}$ and $2n$ are even $\Big($ $f_{n.1}$ is even [use the definition of $f_{n.1}$] and $2n$ is trivially even$\Big)$, then inequality *(3.7)* immediately implies that $f_{n.1} \geq 2n+2$. Note $\Big($ by Observation.3.1.*i* $\Big)$ that $f_{n.1} \leq 2n+2$. Now using the previous two inequalities, then we immediately deduce that $f_{n.1} = 2n+2$; so Theorem 3.1 is satisfied by n $\Big($ use the previous equality and apply property (3.2.1) of Remark 3.2$\Big)$, and we have a contradiction, since n does not satisfied Theorem 3.1. So

$$f_{n.1} \leq 2n \qquad (3.8).$$

Now we show that $f_{n.1} = f_{n-1.1}$. Indeed using *(3.8)* and property (1.1.2) of Remark 1.1, then we immediately deduce that $f_{n.1} = f_{n-1.1}$. Observation.3.1.*iii* follows.
Observation.3.1.*iv*. *Let* $(\phi_n, \nu_n, \epsilon_n)$ $\Big($ use Definitions 2.3 $\Big)$ and via $(\phi_n, \nu_n, \epsilon_n)$, consider $(\phi_{n-1}, \nu_{n-1}, \epsilon_{n-1})$ $\Big($ this consideration gets sense, since $n > 10 + F_3$ [use Observation.3.1.*i*], and so $n-1 > 9 + F_3 > 1 + F_3$ $\Big)$. Then

$$(\,\phi_{n-1} - \phi_n, \nu_{n-1} - \nu_n, \epsilon_{n-1} - \epsilon_n\,)\ tackles\ (1, 3if_{n.1}^{-1})\ around\ 6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i.$$

Indeed observing $\Big($ by Observation.3.1.*iii* $\Big)$ that $f_{n.1} \leq 2n$ and noticing $\Big($by Observation.3.1.*i* $\Big)$ that $n > 10 + F_3$, then using the previous two inequalities, we easily deduce that all the hypotheses of Proposition 2.4 are satisfied and therefore the conclusion of Proposition 2.4 is satisfied; consequently $(\,\phi_{n-1} - \phi_n, \nu_{n-1} - \nu_n, \epsilon_{n-1} - \epsilon_n\,)$ tackles $(1, 3if_{n.1}^{-1})$ around $6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i$. Observation.3.1.*iv* follows.
Observation.3.1.*v*. *Look at n (recall n is a minimum counter-example) and consider* $n-1$ $\Big($this consideration gets sense, since $n > 10 + F_3$ [use Observation.3.1.*i*], and so $n-1 > 9 + F_3 > 1 + F_3$$\Big)$.

$$Then\ property\ A_0\ of\ Theorem\ 3.1\ is\ not\ satisfied\ by\ n-1.$$

Otherwise $\Big($ we reason by reduction to absurd $\Big)$

$$2(n-1) - 1\ is\ a\ Fermat\ prime \qquad (3.9)$$

$\Big($ such a $2(n-1)-1$ exists, since property A_0 of Theorem 3.1 is satisfied by $n-1$ $\Big)$. Now let the Fermat prime f_n $\Big($ use Definitions 1.0 $\Big)$; observing $\Big($ by Observation.3.1.*i* $\Big)$ that $n > 10 + F_3$ and using *(3.9)*, then we easily deduce that

$$f_n \geq 2n - 3 > 2n - 10 \qquad (3.10).$$

Clearly

$$f_n > 2n - 10 \qquad (3.11)$$

$\Big($ use *(3.10)*$\Big)$. Now using *(3.11)* and property (3.2.3) of Remark 3.2, we deduce that Theorem 3.1 is satisfied by n and we have a contraction, since n does not satisfy Theorem 3.1. Observation.3.1.*v* follows.
Observation.3.1.*vi*. *Let n (recall n is a minimum counter-example to Theorem* 3.1*) and consider* $n-1$ $\Big($this consideration gets sense, since $n > 10 + F_3$ [use Observation.3.1.*i*], and so $n-1 > 9 + F_3 > F_3$ $\Big)$.

$$Then\ property\ A_1\ of\ Theorem\ 3.1\ is\ not\ satisfied\ by\ n-1.$$

Otherwise $\Big($ we reason by reduction to absurd $\Big)$

$$f_{n-1.1} \geq 2(n-1) \textit{ and } 2(n-1)+1 \textit{ is a Fermat prime} \qquad (3.12)$$

$\Big($ such a $2(n-1)+1$ exists, since property A_1 of Theorem 3.1 is satisfied by $n-1$ $\Big)$. Now let the Fermat prime f_n $\Big($ use Definitions 1.0 $\Big)$; observing $\Big($by Observation.3.1.i $\Big)$ that $n > 10 + F_3$ and using *(3.12)*, then we immediately deduce that

$$f_n \geq 2(n-1)+1 > 2n-10 \qquad (3.13).$$

Clearly

$$f_n > 2n-10 \qquad (3.14)$$

$\Big($ use *(3.13)*$\Big)$. Now using *(3.14)* and property (3.2.3) of Remark 3.2, we deduce that Theorem 3.1 is satisfied by n and we have a contradiction, since n does not satisfy Theorem 3.1. Observation.3.1.vi follows.

Observation.3.1.vii. *Look at* $(n, f_{n.1})$ *(recall* n *is a minimum counter-example to Theorem* 3.1*) and consider* $n-1$ $\Big($this consideration gets sense, since $n > 10 + F_3$ [use Observation.3.1.i], and so $n-1 > 9 + F_3 > F_3$$\Big)$.

Then property A_2 *of Theorem* 3.1 *is not satisfied by* $n-1$.

Otherwise $\Big($ we reason by reduction to absurd $\Big)$

$$f_{n-1.1} \geq 2(n-1)+4 \qquad (3.15)$$

$\Big($ inequality *(3.15)* exists, since property A_2 of Theorem 3.1 is satisfied by $n-1$ $\Big)$. It is immediate to see that *(3.15)* clearly says

$$f_{n-1.1} \geq 2n+2 \qquad (3.16).$$

Observing $\Big($ by Observation.3.1.iii $\Big)$ that $f_{n.1} = f_{n-1.1}$ and using the preceding equality, then we immediately deduce that *(3.16)* clearly says that

$$f_{n.1} \geq 2n+2 \qquad (3.17).$$

Now observing $\Big($by Observation.3.1.i $\Big)$ that

$$f_{n.1} \leq 2n+2 \qquad (3.18),$$

then using *(3.17)* and *(3.18)*, we immediately deduce that

$$f_{n.1} = 2n+2 \qquad (3.19).$$

Using *(3.19)* and property (3.2.1) of Remark 3.2, we immediately deduce that Theorem 3.1 is satisfied by n; we have a contraction, since n does not satisfy Theorem 3.1. Observation.3.1.vii follows.

Observation.3.1.$viii$. *Look at* $(\phi_n, \nu_n, \epsilon_n)$ $\Big($ use Definitions 2.3 and recall n is a minimum counter-example to Theorem 3.1 $\Big)$, and via $(\phi_n, \nu_n, \epsilon_n)$, consider $(\phi_{n-1}, \nu_{n-1}, \epsilon_{n-1})$ $\Big($this consideration gets sense, since $n > 10 + F_3$ [use Observation.3.1.i], and so $n-1 > 9 + F_3 > F_3$ $\Big)$. Then

$$(\,\phi_{n-1}, \nu_{n-1}, \epsilon_{n-1}\,) \textit{ tackles } (1, 3if_{n.1}^{-1}) \textit{ around } 6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i.$$

Indeed look at n $\Big($ recall n is a minimum counter-example to Theorem 3.1 $\Big)$, and via n, consider $n-1$ $\Big($this consideration gets sense, since $n > 10 + F_3$ [use Observation.3.1.i], and so $n-1 > 9 + F_3 > F_3$ $\Big)$; then, by the minimality of n, we immediately deduce that $n-1$ is not a counter-example to Theorem 3.1; so

$$\textit{Theorem } 3.1 \textit{ is satisfied by } n-1 \qquad (3.20).$$

Clearly

$$property\ A_3\ of\ Theorem\ 3.1\ is\ satisfied\ by\ n-1 \qquad (3.21)$$

$\Big($ use *(3.20)* and Observation.3.1.v and Observation.3.1.vi and Observation.3.1.vii $\Big)$. So

$$(\phi_{n-1}, \nu_{n-1}, \epsilon_{n-1})\, tackles\, (1, 3if_{n-1.1}^{-1})\ around\, 6f_{n-1.1} - 19 - if_{n-1.1} + 11if_{n-1.1}^{2} - 18i \qquad (3.22)$$

$\Big($ use *(3.21)* and property A_3 of Theorem 3.1 $\Big)$. Now observing $\Big($ by Observation.3.1.iii $\Big)$ that $f_{n.1} = f_{n-1.1}$ and using the preceding equality, then we easily deduce that *(3.22)* clearly says that

$$(\,\phi_{n-1}, \nu_{n-1}, \epsilon_{n-1}\,)\ tackles\ \ (1, 3if_{n.1}^{-1})\ around\, 6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^{2} - 18i.$$

Observation.3.1.$viii$ follows.
Observation.3.1.ix. *Look at* n *and let* $(\phi_n, \nu_n, \epsilon_n)$ $\Big($ use Definitions 2.3 $\Big)$. Then

$$(\,\phi_n, \nu_n, \epsilon_n\,)\ tackles\ (1, 3if_{n.1}^{-1})\ around\, 6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^{2} - 18i.$$

Indeed using Observation.3.1.iv and the definition of *tackle* $\Big($ use Definition 2.2 $\Big)$, then we immediately deduce that

$$there\, exists\, (\,x, y, k\,) \in \mathcal{C}^2 \times \mathcal{R}\, such\, that\, x + 3iyf_{n.1}^{-1} + k(6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^{2} - 18i) + (\phi_{n-1} - \phi_n) = 0 \qquad (3.23)$$

where

$$x + y - (\nu_{n-1} - \nu_n) = 0\ and\ x - y - (\epsilon_{n-1} - \epsilon_n) = 0 \qquad (3.24).$$

That being so, using Observation.3.1.$viii$ and the definition of *tackle* $\Big($ use Definition 2.2 $\Big)$, then we immediately deduce that

$$there\ exists\ (x', y', k') \in \mathcal{C}^2 \times \mathcal{R}\ such\ that\, x' + 3iy'f_{n.1}^{-1} + k'(6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^{2} - 18i) + \phi_{n-1} = 0 \qquad (3.25)$$

where

$$x' + y' - \nu_{n-1} = 0\ and\ x' - y' - \epsilon_{n-1} = 0 \qquad (3.26).$$

Now using equality of (3.25), then we immediately deduce that equality of *(3.23)* clearly says that

$$x - x' + 3i(y - y')f_{n.1}^{-1} + (k - k')(6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^{2} - 18i) - \phi_n = 0 \qquad (3.27).$$

It is trivial to see that equality *(3.27)* clearly says that

$$x' - x + 3i(y' - y)f_{n.1}^{-1} + (k' - k)(6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^{2} - 18i) + \phi_n = 0 \qquad (3.28).$$

Using the two equalities of *(3.26)*, then we immediately deduce that the two equalities of *(3.24)* clearly say that

$$(x - x') + (y - y') + \nu_n = 0\ \ and\ \ (x - x') - (y - y') + \epsilon_n = 0 \qquad (3.29).$$

It is trivial to see that *(3.29)* clearly says that

$$(x' - x) + (y' - y) - \nu_n = 0\ \ and\ \ (x' - x) - (y' - y) - \epsilon_n = 0 \qquad (3.30).$$

Now observing that $(\,x, y, k\,) \in \mathcal{C}^2 \times \mathcal{R}$ $\Big($ use *(3.23)* $\Big)$ and since $(\,x', y', k'\,) \in \mathcal{C}^2 \times \mathcal{R}$ $\Big($ use *(3.25)* $\Big)$, then it becomes trivial to deduce that *(3.28)* and *(3.30)* clearly say that

$$there\, exists\, (x'', y'', k'') \in \mathcal{C}^2 \times \mathcal{R}\, such\, that\, x'' = x' - x\, and\, y'' = y' - y\, and\, k'' = k' - k \qquad (3.31)$$

where

$$x''+y''-\nu_n = 0,\ x''-y''-\epsilon_n = 0\, and\, x''+3iy''f_{n.1}^{-1}+k''(6f_{n.1}-19-if_{n.1}+11if_{n.1}^{2}-18i)+\phi_n = 0 \quad (3.32).$$

Clearly

$$(\phi_n, \nu_n, \epsilon_n)\ tackles\ (1, 3if_{n.1}^{-1})\ around\ 6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i$$

$\Big($ use *(3.31)* and *(3.32)* and the definition of tackle introduced in Definition 2.2 $\Big)$. Observation.3.1.*ix* follows.

These simple observations made, then it becomes trivial to see that Observation.3.1.*ix* clearly contradicts Observation.3.1.*ii*. Theorem 3.1 follows.□

Now the Fermat primes problem directly results via the following.

Corollary 3.5. *Let n be an integer $\geq F_3$ and let $f_{n.1}$ (use Definitions* 1.0*); look at $(\phi_n, \nu_n, \epsilon_n)$ introduced in Definitions* 2.3*. Then*

$$There\ exists\ j \in \{-1, 1\}\ such\ that\ 2n + j\ is\ a\ Fermat\ prime\ or\ f_{n.1} > 2n$$

or

$$(\phi_n, \nu_n, \epsilon_n)\ tackles\ (1, 3if_{n.1}^{-1})\ around\ 6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i.$$

Proof. Indeed let n be an integer $\geq F_3$ and let $f_{n.1}$ (use Definitions 1.0); look at $(\phi_n, \nu_n, \epsilon_n)$ introduced in Definitions 2.3. Then using Theorem 3.1, we immediately deduce that

$$at\ least\ one\ of\ properties\ A_0\ and\ A_1\ and\ A_2\ and\ A_3\ of\ Theorem\ 3.1\ is\ satisfied\ by\ n \qquad (3.33).$$

We are going to observe four cases.

*Case.*0. *If property A_0 of Theorem* 3.1 *is satisfied by n, then* $2n - 1$ *is a Fermat prime and* $f_{n.1} > 2n$. Clearly

$$2n - 1\ is\ a\ Fermat\ prime \qquad (3.34)$$

$\Big($ use Theorem 3.1 and observe that property A_0 of Theorem 3.1 is satisfied by n $\Big)$. Now let the Fermat prime f_n $\Big($ use Definitions 1.0$\Big)$, then using *(3.34)* and the definition of f_n , we immediately deduce that

$$f_n \geq 2n - 1 \qquad (3.35).$$

Now look at $f_{n.1}\Big($ see Definitions 1.0$\Big)$, then using *(3.35)* and the definition of $f_{n.1}$, we immediately deduce that

$$f_{n.1} > 2f_n^{f_n} > (2n - 1)^{2n-1} \qquad (3.36).$$

Remarking $\Big($via the hypothesis $\Big)$ that $n \geq F_3$ and using *(3.36)*, then we immediately deduce that

$$f_{n.1} > (2n - 1)^{2n-1} > 2n \qquad (3.37).$$

So $f_{n.1} > 2n$ $\Big($ use *(3.37)* $\Big)$ and $2n - 1$ is a Fermat prime $\Big($ use *(3.34)* $\Big)$. Case.0 follows.

*Case.*1. *If property A_1 of Theorem* 3.1 *is satisfied by n, then* $2n + 1$ *is a Fermat prime.* Clearly

$$f_{n.1} \geq 2n\ and\ 2n + 1\ is\ a\ Fermat\ prime \qquad (3.38)$$

$\Big($ use Theorem 3.1 and observe that property A_1 of Theorem 3.1 is satisfied by n $\Big)$. *(3.38)* immediately implies that $2n + 1$ is a Fermat prime. Case.1 follows.

*Case.*2. *If property A_2 of Theorem* 3.1 *is satisfied by n, then* $f_{n.1} > 2n$. Clearly

$$f_{n.1} \geq 2n + 4 \qquad (3.39)$$

$\Big($ use Theorem 3.1 and observe that property A_2 of Theorem 3.1 is satisfied by n $\Big)$. *(3.39)* immediately implies that $f_{n.1} > 2n$. Case.2 follows.

*Case.*3. *If property A_3 of Theorem* 3.1 *is satisfied by n, then $(\phi_n, \nu_n, \epsilon_n)$ tackles* $(1, 3if_{n.1}^{-1})$ *around* $6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i$. Indeed use Theorem 3.1 and observe that property A_3 of Theorem 3.1 is satisfied by n. Case.3 follows.

Now using *(3.33)* and Case.0 or *(3.33)* and Case.1 or *(3.33)* and Case.2 or *(3.33)* and Case.3, then we immediately deduce that

$$There\ exists\ j \in \{-1, 1\}\ such\ that\ 2n + j\ is\ a\ Fermat\ prime\ or\ f_{n.1} > 2n\ or$$

$$(\phi_n, \nu_n, \epsilon_n)\ tackles\ (1, 3if_{n.1}^{-1})\ around\ 6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i.$$

Corollary 3.5 follows. □

Theorem 3.6. *For every integer,* $n \geq F_3$, $f_{n.1} > 2n$ *or there exists* $j \in \{-1, 1\}$ *such that* $2n + j$ *is a Fermat prime* .

Proof. Otherwise $\Big($ we reason by reduction to absurd$\Big)$, let n be a minimun counter-example; clearly

$$f_{n.1} \leq 2n \textit{ and there exists not } j \in \{-1,1\} \textit{ such that } 2n + j \textit{ is a Fermat prime} \qquad (3.40).$$

Now let let $(\phi_n, \nu_n, \epsilon_n)$ (use Definitions 2.3); then using *(3.40)* and Corollary 3.5, it follows that

$$(\phi_n, \nu_n, \epsilon_n) \textit{ tackles } (1, 3if_{n.1}^{-1}) \textit{ around } 6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i \qquad (3.41).$$

We observe the following.

Observation.3.6.1. $n > 10 + F_3$.

Otherwise $n \leq 10 + F_3$; now observing (by using property (1.1.0) of Remark 1.1) that $f_{n.1} > F_3^{F_3}$ and using the previous two inequalities, then it becomes trivial to deduce that $f_{n.1} > F_3^{F_3} > 2(10 + F_3) + 8 > 2n + 4$; so $f_{n.1} > 2n + 4$ and the previous inequality contradicts *(3.40)*. Observation.3.6.1 follows.

Observation.3.6.2. $f_{n.1} = f_{n-1.1}$.

Indeed, remarking (by *(3.40)*) that $f_{n.1} \leq 2n$, then, using the preceding inequality and property (1.1.2) of Remark 1.1, we immediately deduce that $f_{n.1} = f_{n-1.1}$. Observation.3.6.2 follows

Observation.3.6.3. *Let the Fermat prime* f_n $\Big($ *use Definitions* 1.0$\Big)$. *Then* $f_n < 2n - 10$.

Otherwise

$$f_n \geq 2n - 10 \qquad (3.42).$$

Now using *(3.42)* and the definition of $f_{n.1}$, we immediately deduce that

$$f_{n.1} > 2f_n^{f_n} > (2n - 10)^{2n-10} \qquad (3.43).$$

Remarking $\Big($by Observation.3.6.1 $\Big)$ that $n > 10 + F_3$ and using *(3.43)*, then we immediately deduce that

$$f_{n.1} > (2n - 10)^{2n-10} > 2n \qquad (3.44);$$

so

$$f_{n.1} > 2n \qquad (3.45)$$

$\Big($ use *(3.44)* $\Big)$ and inequality *(3.45)* contradicts *(3.40)*. Observation.3.6.3 follows.

Observation.3.6.4. *There exists not* $j \in \{-1, 1\}$ *such that* $2(n-1) + j$ *is a Fermat prime.*

Otherwise

$$\textit{let } j \in \{-1, 1\} \textit{ such that } 2(n-1) + j \textit{ is a Fermat prime} \qquad (3.46),$$

and let the Fermat prime f_n $\Big($ use Definitions 1.0$\Big)$. Then using *(3.46)* and the definition of f_n, we immediately deduce that

$$f_n \geq 2n - 3 \qquad (3.47).$$

Remarking $\Big($by Observation.3.6.1 $\Big)$ that $n > 10 + F_3$ and using (*3.47*), then we immediately deduce that $f_n > 2n - 10$; the preceding inequality contradicts Observation.3.6.3. Observation.3.6.4 follows.

Observation.3.6.5. $f_{n.1} = 2n$.

Indeed look at n, and via n, consider $n - 1$ $\Big($ this consideration gets sense, since $n > 10 + F_3$ (use Observation.3.6.1), and therefore $n - 1 > 9 + F_3 > F_3\Big)$. Then, by the minimality of n, $n - 1$ is not a counter-example to Theorem 3.6; consequently

$$f_{n-1.1} > 2(n-1) \textit{ or there exists } j \in \{-1,1\} \textit{ such that } 2(n-1) + j \textit{ is a Fermat prime} \qquad (3.48).$$

Clearly

$$f_{n-1.1} > 2(n-1) \qquad (3.49)$$

$\Big($ use *(3.48)* and Observation.3.6.4 $\Big)$ and inequality*(3.49)* clearly says that

$$f_{n-1.1} > 2n - 2 \qquad (3.49').$$

Note that

$$f_{n.1} = f_{n-1.1} \qquad (3.50)$$

$\Big($ use Observation.3.6.2 $\Big)$. Now using (*3.49'*) and *(3.50)*, then we immediately deduce that

$$f_{n.1} > 2n - 2 \qquad (3.51).$$

Noticing that $f_{n.1}$ and $2n-2$ are even $\Big($ $f_{n.1}$ is even [use the definition of $f_{n.1}$] and $2n-2$ is trivially even$\Big)$, then it becomes trivial to deduce that inequality *(3.51)* implies that $f_{n.1} \geq 2n-2+2$; the preceding inequality clearly says that

$$f_{n.1} \geq 2n \qquad (3.52).$$

Clearly $f_{n.1} = 2n$ $\Big($ use *(3.52)* and inequality of *(3.40)*$\Big)$. Observation.3.6.5 follows.

Observation.3.6.6 *(the using of Remark*.2.3.2 *of Section.***2***). Look at* $f_{n.1}$ *and consider* $(\phi_n, \nu_n, \epsilon_n)$ *(use Definitions* 2.3*). Then*

$$(\phi_n, \nu_n, \epsilon_n) \textit{ does not tackle } (1, 3if_{n.1}^{-1}) \textit{ around } 6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i.$$

Indeed observing (by Observation.3.6.5) that $f_{n.1} = 2n$ and using Remark.2.3.2 (Section.**2**), then we immediately deduce that

$$(\phi_n, \nu_n, \epsilon_n) \textit{ does not tackle } (1, 3if_{n.1}^{-1}) \textit{ around } 6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i.$$

Observation.3.6.6 follows.

These simple observations made, then it becomes trivial to see that Observation.3.6.6 contradicts *(3.41)*. Theorem 3.6 follows.□

Theorem 3.6 immediately implies the Fermat primes problem.

Theorem 3.7 *(The Proof of the Fermat primes problem). There are infinitely many Fermat primes.*
$\big[$*Proof.* Use Theorem 3.6 and Proposition 1.3. □$\big]$

$2'$. *EPILOGUE*

Our simple article clearly shows that divisibility helps to characterize Fermat primes as we did in [6] *and* [7]*, and elementary complex calculus coupled with elementary arithmetic calculus and trivial computation help to give a simple analytic proof of problem posed by the Fermat primes.* Now we end this article by proving properties that we let unproved in Section.*2*.

Proof of Remark.2.3.1 *(the using of Remark*.2.1.1*).* Indeed, observing (via the hypotheses) that $f_{n.1} = 2n+2$ and using the preceding equality, we easily deduce that

$$f_{n.1} - 2n = 2;\ if_{n.1} - 4in - 4i + 1 = 1 - if_{n.1};\ 2n + 1 = f_{n.1} - 1;\ \textit{and } if_{n.1} - 2in + 1 = 2i + 1 \qquad (2.42).$$

Now look at $(\phi_n, \nu_n, \epsilon_n)$ introduced in Definitions 2.3 and let $(\phi_{n.2}, \phi_{n.3}, \phi_{n.4})$ explicited in Definitions 2.3. Clearly

$$\phi_{n.2} = -4f_{n.1}(16f_{n.1}^{-3} + 50f_{n.1}^{-2} + 11if_{n.1}^{-3} - 13i + 5if_{n.1}) + 8f_{n.1} - 8i - 16 \qquad (2.43)$$

$\Big($ use *(2.37)* of Definitions 2.3 and the third equality of *(2.42)* (observe [via the third equality of *(2.42)*] that $(f_{n.1}-1)^2 = f_{n.1}^2 - 2f_{n.1} + 1$) and the first equality of *(2.42)* (observe [via the first equality of *(2.42)*] that $\frac{(f_{n.1}-2n)}{2} = 1$) $\Big)$, and

$$\phi_{n.3} = \frac{289080}{1331} \qquad (2.44)$$

$\Big($ use *(2.38)* of Definitions 2.3 and the third equality of *(2.42)* (observe [via the third equality of *(2.42)*] that $(f_{n.1}-1)^2 = f_{n.1}^2 - 2f_{n.1} + 1$)$\Big)$, and

$$\phi_{n.4} = 5f_{n.1}(11if_{n.1} + 7 - 50f_{n.1}^{-1} + 23if_{n.1}^{-3} - 54f_{n.1}^{-3}) - \frac{1003112i}{1331} \qquad (2.45)$$

$\Big($ use *(2.39)* of Definitions 2.3 and the third equality of *(2.42)* (observe [via the third equality of *(2.42)*] that $(f_{n.1}-1)^2 = f_{n.1}^2 - 2f_{n.1} + 1$)$\Big)$. That being said let (ν_n, ϵ_n) explicited in Definitions 2.3; clearly

$$\nu_n = (1 - if_{n.1})^2 - 1 + f_{n.1}^2 \qquad (2.46)$$

$\Big($ use (2.40) of Definitions 2.3 and the second equality of *(2.42)* $\Big)$ and clearly

$$\epsilon_n = 4(\ (f_{n.1}-1)^2 - 1 - f_{n.1}^2 + f_{n.1}\ ) + (\ 2i+1\ )(\ 4if_{n.1} + 4 - 8i\ ) \qquad (2.47)$$

$\Big($ use *(2.41)* of Definitions 2.3 and the last two equalities of *(2.42)* $\Big)$. That being so, let (x', y', k') such that

$$x' = 10 - 6f_{n.1} + if_{n.1};\ y' = -10 + 6f_{n.1} - 3if_{n.1};\ \textit{and } k' = -\frac{35}{11} - \frac{618f_{n.1}^{-1}}{121} + \frac{7494f_{n.1}^{-2}}{121} + \frac{126301f_{n.1}^{-4}}{1331} \qquad (2.48).$$

Let $(\phi_n, \nu_n, \epsilon_n, x', y', k')$ where (x', y', k') is explicted in *(2.48)* and where $(\phi_n, \nu_n, \epsilon_n)$ is introduced in Definitions 2.3 $\Big($ use *(2.35)* for ϕ_n; and *(2.40)* for ν_n; and *(2.41)* for ϵ_n; and *(2.48)* for $(x', y', k')\Big)$. Now look at (X_n, Y_n, Z_n, x, y, k) introduced in Remark.2.1.1; clearly

$$(X_n, Y_n, Z_n, x, y, k) = (\phi_n, \nu_n, \epsilon_n, x', y', k') \tag{2.49}$$

$\Big($ **firstly, we prove that** $X_n = \phi_n$. *Indeed* observe that

$$X_{n.j} = \phi_{n.j} \, for \, 1 \leq j \leq 4 \tag{2.49.0}$$

$[X_{n.1} = \phi_{n.1}$ (use *(2.11)* of Remark.2.1.1 and *(2.36)* of Definitions 2.3); $X_{n.2} = \phi_{n.2}$ (use *(2.12)* of Remark.2.1.1 and *(2.43)*); $X_{n.3} = \phi_{n.3}$ (use *(2.13)* of Remark.2.1.1 and *(2.44)*); and $X_{n.4} = \phi_{n.4}$ (use *(2.14)* of Remark.2.1.1 and *(2.45)*). The prevous four equalities immediately imply that $X_{n.j} = \phi_{n.j} \, for \, 1 \leq j \leq 4]$. *(2.49.0)* immediately implies that

$$X_n = \sum_{j=1}^{4} X_{n.j} = \phi_n = \sum_{j=1}^{4} \phi_{n.j} \, [use (2.10) \, of \, Remark.2.1.1 \, and \, (2.35) \, of \, Definitions \, 2.3 \; and \, (2.49.0)] \tag{2.49.1}$$

$$\textbf{Note that}\, Y_n = \nu_n \, [use (2.15) \, of \, Remark.2.1.1 \, and \, (2.46)] \, \textbf{and} \, Z_n = \epsilon_n \, [use (2.16) \, of \, Remark.2.1.1 \, and \, (2.47)] \tag{2.49.2}$$

$$\textbf{Finally, note that} \, (x, y, k) = (x', y', k') \, [use (2.17) \, of \, Remark.2.1.1 \, and \, (2.48)] \tag{2.49.3}$$

Now using *(2.49.1)* and *(2.49.2)* and *(2.49.3)*, we immediately deduce that $(X_n, Y_n, Z_n, x, y, k) = (\phi_n, \nu_n, \epsilon_n, x', y', k')\Big)$.
That being so, using *(2.49)* and Remark.2.1.1 , it becomes trivial to deduce that $(\phi_n, \nu_n, \epsilon_n, x', y', k')$ satisfies all the hypotheses of Remark.2.1.1; therefore $(\phi_n, \nu_n, \epsilon_n, x', y', k')$ satisfies the conclusion of Remark.2.1.1; so

$$x' + y' - \nu_n = 0 \, and \, x' - y' - \epsilon_n = 0 \tag{2.50}$$

and

$$x' + 3iy' f_{n.1}^{-1} + k'(6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i) + \phi_n = 0; \; k' \in \mathcal{R} \tag{2.51}$$

$\Big($ use Remark.2.1.1 where we replace (X_n, Y_n, Z_n, x, y, k) by $(\phi_n, \nu_n, \epsilon_n, x', y', k')\Big)$. Clearly $(\phi_n, \nu_n, \epsilon_n)$ tackles $(1, 3if_{n.1}^{-1})$ around $6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i$ $\Big($ use *(2.50)* and *(2.51)* and the notion of tackle introduced in Definition 2.2 [observe that $(x', y') \in \mathcal{C}^2$ and $k' \in \mathcal{R}$; so $(x', y', k') \in \mathcal{C}^2 \times \mathcal{R}$] $\Big)$. Remark.2.3.1 follows.

Proof of Remark.2.3.2 (reduction of the Fermat primes problem into a trivial equation of three unknowns ; the using of Remark.2.1.2). Indeed, observing (via the hypotheses) that $f_{n.1} = 2n$ and using the preceding equality, we easily deduce that

$$f_{n.1} - 2n = 0; \; if_{n.1} - 4in - 4i + 1 = 1 - 4i - if_{n.1}; \; 2n + 1 = f_{n.1} + 1; \; and \; if_{n.1} - 2in + 1 = 1 \tag{2.52}$$

Now look at $(\phi_n, \nu_n, \epsilon_n)$ introduced in Definitions 2.3 and let $(\phi_{n.2}, \phi_{n.3}, \phi_{n.4})$ explicited in Definitions 2.3. Clearly

$$\phi_{n.2} = 0 \tag{2.53}$$

$\Big($ use *(2.37)* of Definitions 2.3 and the third equality of *(2.52)* (observe [via the third equality of *(2.52)*] that $(f_{n.1} + 1)^2 = f_{n.1}^2 + 2f_{n.1} + 1$) and the first equality of *(2.52)* (observe [via the first equality of *(2.52)*] that $\frac{(f_{n.1} - 2n)}{2} = 0$) $\Big)$, and

$$\phi_{n.3} = 4f_{n.1}(34if_{n.1}^{-2} - 70if_{n.1}^{-3} - 11i + 5 + 6if_{n.1}) + \frac{289080}{1331} \tag{2.54}$$

$\Big($ use *(2.38)* of Definitions 2.3 and the third equality of *(2.52)* (observe [via the third equality of *(2.52)*] that $(f_{n.1} + 1)^2 = f_{n.1}^2 + 2f_{n.1} + 1)\Big)$, and

$$\phi_{n.4} = 9f_{n.1}(11if_{n.1} + 7 - 50f_{n.1}^{-1} + 23if_{n.1}^{-3} - 54f_{n.1}^{-3}) - \frac{1003112i}{1331} \tag{2.55}$$

$\Big($ use *(2.39)* of Definitions 2.3 and the third equality of *(2.52)* (observe [via the third equality of *(2.52)*] that $(f_{n.1} + 1)^2 = f_{n.1}^2 + 2f_{n.1} + 1$) $\Big)$. That being said let (ν_n, ϵ_n) explicited in Definitions 2.3; clearly

$$\nu_n = (1 - 4i - if_{n.1})^2 - 1 + f_{n.1}^2 \tag{2.56}$$

$\Big($ use (2.40) of Definitions 2.3 and the second equality of *(2.52)* $\Big)$ and clearly

$$\epsilon_n = 4(\, (f_{n.1} + 1)^2 - 1 - f_{n.1}^2 + f_{n.1} \,) + 4if_{n.1} + 4 - 8i \tag{2.57}$$

$\Big($ use *(2.41)* of Definitions 2.3 and the last two equalities of *(2.52)* $\Big)$. That being so, let (x', y') such that

$$x' = 2f_{n.1} + if_{n.1} - 6 - 8i \textit{ and } y' = -10 - 10f_{n.1} - 3if_{n.1} \qquad (2.58).$$

Let $(\,\phi_n, \nu_n, \epsilon_n, x', y'\,)$ where (x', y') is explicted in *(2.58)* and where $(\,\phi_n, \nu_n, \epsilon_n)$ is introduced in Definitions 2.3 $\Big($ use *(2.35)* for ϕ_n; and *(2.40)* for ν_n; and *(2.41)* for ϵ_n; and *(2.58)* for $(x', y')\Big)$. Now look at $(\,X_n, Y_n, Z_n, x, y\,)$ introduced in Remark.2.1.2; clearly

$$(\,X_n, Y_n, Z_n, x, y\,) = (\,\phi_n, \nu_n, \epsilon_n, x', y'\,) \qquad (2.59)$$

$\Big($ **Firstly, we prove that** $X_n = \phi_n$. *Indeed* observe that

$$X_{n.j} = \phi_{n.j} \, for \, 1 \leq j \leq 4 \qquad (2.59.0)$$

$[X_{n.1} = \phi_{n.1}$ (use *(2.21)* of Remark.2.1.2 and *(2.36)* of Definitions 2.3); $X_{n.2} = \phi_{n.2}$ (use *(2.22)* of Remark.2.1.2 and *(2.53)*); $X_{n.3} = \phi_{n.3}$ (use *(2.23)* of Remark.2.1.2 and *(2.54)*); and $X_{n.4} = \phi_{n.4}$ (use *(2.24)* of Remark.2.1.2 and *(2.55)*). The prevous four equalities immediately imply that $X_{n.j} = \phi_{n.j} \, for \, 1 \leq j \leq 4]$. *(2.59.0)* immediately implies that

$$X_n = \sum_{j=1}^{4} X_{n.j} = \phi_n = \sum_{j=1}^{4} \phi_{n.j} \, [use (2.20) \, of \, Remark.2.1.2 \, and \, (2.35) \, of \, Definitions \, 2.3 \, and \, (2.59.0)] \qquad (2.59.1).$$

$$\textbf{Note that } Y_n = \nu_n \, [use (2.25) \, of \, Remark.2.1.2 \, and \, (2.56)\,] \, \textbf{and } Z_n = \epsilon_n \, [use (2.26) \, of \, Remark.2.1.2 \, and \, (2.57)] \qquad (2.59.2).$$

$$\textbf{Finally, note that } (x, y) = (x', y') \, [use (2.27) \, of \, Remark.2.1.2 \, and \, (2.58)] \qquad (2.59.3).$$

Now using *(2.59.1)* and *(2.59.2)* and *(2.59.3)* , we immediately deduce that $(\,X_n, Y_n, Z_n, x, y\,) = (\,\phi_n, \nu_n, \epsilon_n, x', y'\,)\Big)$.
That being so, using *(2.59)* and Remark.2.1.2 , it becomes trivial to deduce that $(\,\phi_n, \nu_n, \epsilon_n, x', y'\,)$ satisfies all the hypotheses of Remark.2.1.2; therefore $(\,\phi_n, \nu_n, \epsilon_n, x', y'\,)$ satisfies the conclusion of Remark.2.1.2; so

$$x' + y' - \nu_n = 0; \; x' - y' - \epsilon_n = 0 \qquad (2.60)$$

$$and \, there \, exists \, not \, k' \in \mathcal{R} \, such \, that \, x' + 3iy'f_{n.1}^{-1} + k'(6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i) + \phi_n = 0 \qquad (2.61)$$

$\Big($ use Remark.2.1.2 where we replace $(\,X_n, Y_n, Z_n, x, y\,)$ by $(\,\phi_n, \nu_n, \epsilon_n, x', y'\,)\Big)$. Clearly $(\,\phi_n, \nu_n, \epsilon_n\,)$ does not tackle $(1, 3if_{n.1}^{-1})$ around $6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i$ $\Big($ use *(2.60)* and *(2.61)* and the notion of tackle introduced in Definition 2.2$\Big)$. Remark.2.3.2 follows.

Remark.2.3.2 reduces the Fermat primes problem into a simple equation of three unknowns. Indeed Remark.2.3.2 clearly says that, if $f_{n.1} = 2n$, we will have a simple equation of three unknowns which implies that $(\,\phi_n, \nu_n, \epsilon_n\,)$ does not tackle $(1, 3if_{n.1}^{-1})$ around $6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i$. We used Remark.2.3.2 in Theorem 3.6 (Section.*3*) to immediately deduce the Fermat primes problem.
Now it remains us to prove Proposition 2.4. To prove Proposition 2.4, we need four elementary remarks.

Remark 2.5. *Let n be an integer $\geq F_3$ and let $f_{n.1}$* (use Definitions 1.0). **If** $f_{n.1} \leq 2n$, then $f_{n.1} = f_{n-1.1}$. *Proof.* Indeed observing (via the hypotheses) that

$$f_{n.1} \leq 2n \qquad (2.62),$$

clearly $f_{n.1} = f_{n-1.1}$ $\Big($use inequality (2.62) and property (1.1.2) of Remark 1.1 $\Big)$. Remark 2.5 follows.□

Remark 2.6.*Let n be an integer $\geq 1 + F_3$ and let $f_{n.1}$* (use Definitions 1.0). Look at $(\,\phi_{n.1}, \phi_{n.2}, \phi_{n.3}, \phi_{n.4}\,)$ introduced in Definitions 2.3, and via $(\,\phi_{n.1}, \phi_{n.2}, \phi_{n.3}, \phi_{n.4}\,)$, consider $(\,\phi_{n-1.1}, \phi_{n-1.2}, \phi_{n-1.3}, \phi_{n-1.4}\,)$ $\Big($ this consideration gets sense, since $n \geq 1 + F_3$, and therefore $n - 1 \geq F_3$ $\Big)$. **If** $f_{n.1} \leq 2n$, then we have the following four properties.
(2.6.1) $\phi_{n-1.1} - \phi_{n.1} = 0$.
(2.6.2) $\phi_{n-1.2} - \phi_{n.2} = -8n(16f_{n.1}^{-3} + 50f_{n.1}^{-2} + 11if_{n.1}^{-3} - 13i + 5if_{n.1}) + 8f_{n.1} - 8i - 16$.
(2.6.3) $\phi_{n-1.3} - \phi_{n.3} = -8n(34if_{n.1}^{-2} - 70if_{n.1}^{-3} - 11i + 5 + 6if_{n.1})$.
(2.6.4) $\phi_{n-1.4} - \phi_{n.4} = -8n(11if_{n.1} + 7 - 50f_{n.1}^{-1} + 23if_{n.1}^{-3} - 54f_{n.1}^{-3})$.
Proof. (2.6.1) Indeed let $\phi_{n-1.1}$; clearly

$$\phi_{n-1.1} = \frac{126301if_{n-1.1}^{-3} + 2273418if_{n-1.1}^{-4} - 23837f_{n-1.1} - 357566f_{n-1.1}^{-1}}{1331} + \phi'_{n-1.1} \qquad (2.62')$$

where

$$\phi'_{n-1.1} = \frac{2010800f_{n-1.1}^{-2} + 2399719f_{n-1.1}^{-4} - 757806f_{n-1.1}^{-3}}{1331} \qquad (2.62'')$$

$\Big($ use *(2.36)* of Definitions 2.3 $\Big)$. So

$$\phi_{n-1.1} = \frac{126301if_{n.1}^{-3} + 2273418if_{n.1}^{-4} - 23837f_{n.1} - 357566f_{n.1}^{-1} + 2010800f_{n.1}^{-2} + 2399719f_{n.1}^{-4} - 757806f_{n.1}^{-3}}{1331} \qquad (2.63)$$

$\Big($ use $(2.62')$ and $(2.62'')$ and notice $\Big[$ by observing that $f_{n.1} \le 2n$ and by using Remark 2.5 $\Big]$ that $f_{n.1} = f_{n-1.1}\Big)$. Clearly

$$\phi_{n-1.1} = \phi_{n.1} \qquad (2.64)$$

$\Big($ use *(2.36)* of Definitions 2.3 and *(2.63)* $\Big)$ and so $\phi_{n-1.1} - \phi_{n.1} = 0$ $\Big($ use $(2.64)\Big)$. Property (2.6.1) immediately follows.
(2.6.2) Indeed let $\phi_{n-1.2}$; clearly

$$\phi_{n-1.2} = ((2(n-1)+1)^2 - 1 - f_{n-1.1}^2 - 2f_{n-1.1})(16f_{n-1.1}^{-3} + 50f_{n-1.1}^{-2} + 11if_{n-1.1}^{-3} - 13i + 5if_{n-1.1}) + \gamma_n \qquad (2.65)$$

where

$$\gamma_n = \frac{(f_{n-1.1} - 2(n-1))}{2}(8f_{n-1.1} - 8i - 16) \qquad (2.65')$$

$\Big($ use *(2.37)* of Definitions 2.3 $\Big)$. So

$$\phi_{n-1.2} = ((2(n-1)+1)^2 - 1 - f_{n.1}^2 - 2f_{n.1})(16f_{n.1}^{-3} + 50f_{n.1}^{-2} + 11if_{n.1}^{-3} - 13i + 5if_{n.1}) + \frac{(f_{n.1} - 2(n-1))}{2}(8f_{n.1} - 8i - 16) \quad (2.66)$$

$\Big($ use *(2.65)* and $(2.65')$ and notice $\Big[$ by observing that $f_{n.1} \le 2n$ and by using Remark 2.5 $\Big]$ that $f_{n.1} = f_{n-1.1}\Big)$. Clearly

$$\phi_{n-1.2} = ((2n+1)^2 - 1 - f_{n.1}^2 - 2f_{n.1})(16f_{n.1}^{-3} + 50f_{n.1}^{-2} + 11if_{n.1}^{-3} - 13i + 5if_{n.1}) + \frac{(f_{n.1} - 2n)}{2}(8f_{n.1} - 8i - 16) + \phi'_{n-1.2} \quad (2.67)$$

where

$$\phi'_{n-1.2} = -8n(16f_{n.1}^{-3} + 50f_{n.1}^{-2} + 11if_{n.1}^{-3} - 13i + 5if_{n.1}) + 8f_{n.1} - 8i - 16 \qquad (2.67')$$

$\Big($ use the first member of *(2.66)* and observe [by elementary computation and by using the first member of *(2.66)*] that

$$((2(n-1)+1)^2 - 1 - f_{n.1}^2 - 2f_{n.1}) = ((2n+1)^2 - 1 - f_{n.1}^2 - 2f_{n.1}) - 8n,$$

and remark [by elementary computation and by using the second member of *(2.66)*] that

$$\frac{(f_{n.1} - 2(n-1))}{2} = \frac{(f_{n.1} - 2n)}{2} + 1$$

$\Big)$. So

$$\phi_{n-1.2} = \phi_{n.2} - 8n(16f_{n.1}^{-3} + 50f_{n.1}^{-2} + 11if_{n.1}^{-3} - 13i + 5if_{n.1}) + 8f_{n.1} - 8i - 16 \qquad (2.68)$$

$\Big($ use *(2.37)* of Definitions 2.3 and *(2.67)* and $(2.67')$ $\Big)$. Clearly

$$\phi_{n-1.2} - \phi_{n.2} = -8n(16f_{n.1}^{-3} + 50f_{n.1}^{-2} + 11if_{n.1}^{-3} - 13i + 5if_{n.1}) + 8f_{n.1} - 8i - 16$$

$\Big($ use $(2.68)\Big)$. Property (2.6.2) immediately follows.
(2.6.3) Indeed let $\phi_{n-1.3}$; clearly

$$\phi_{n-1.3} = ((2(n-1)+1)^2 - 1 - f_{n-1.1}^2 + 2f_{n-1.1})(34if_{n-1.1}^{-2} - 70if_{n-1.1}^{-3} - 11i + 5 + 6if_{n-1.1}) + \frac{289080}{1331} \qquad (2.69)$$

$\Big($ use *(2.38)* of Definitions 2.3 $\Big)$. So

$$\phi_{n-1.3} = ((2(n-1)+1)^2 - 1 - f_{n.1}^2 + 2f_{n.1})(34if_{n.1}^{-2} - 70if_{n.1}^{-3} - 11i + 5 + 6if_{n.1}) + \frac{289080}{1331} \qquad (2.70)$$

$\Big($ use *(2.69)* and notice $\Big[$ by observing that $f_{n.1} \le 2n$ and by using Remark 2.5 $\Big]$ that $f_{n.1} = f_{n-1.1}\Big)$. Clearly

$$\phi_{n-1.3} = ((2n+1)^2 - 1 - f_{n.1}^2 + 2f_{n.1})(34if_{n.1}^{-2} - 70if_{n.1}^{-3} - 11i + 5 + 6if_{n.1}) + \frac{289080}{1331} + \phi'_{n-1.3} \qquad (2.71)$$

where

$$\phi'_{n-1.3} = -8n(34if_{n.1}^{-2} - 70if_{n.1}^{-3} - 11i + 5 + 6if_{n.1}) \qquad (2.72)$$

$\Big($ use *(2.70)* and observe [by elementary computation] that $((2(n-1)+1)^2-1-f_{n.1}^2+2f_{n.1}) = ((2n+1)^2-1-f_{n.1}^2+2f_{n.1})-8n$ $\Big)$. So

$$\phi_{n-1.3} = \phi_{n.3} - 8n(34if_{n.1}^{-2} - 70if_{n.1}^{-3} - 11i + 5 + 6if_{n.1}) \tag{2.73}$$

$\Big($ use *(2.71)* and *(2.72)* and *(2.38)* of Definitions 2.3 $\Big)$ and clearly

$$\phi_{n-1.3} - \phi_{n.3} = -8n(34if_{n.1}^{-2} - 70if_{n.1}^{-3} - 11i + 5 + 6if_{n.1})$$

$\Big($ use *(2.73)* $\Big)$. Property (2.6.3) immediately follows.
(2.6.4) Indeed let $\phi_{n-1.4}$; clearly

$$\phi_{n-1.4} = ((2(n-1)+1)^2 - 1 - f_{n-1.1}^2 + 7f_{n-1.1})(11if_{n-1.1} + 7 - 50f_{n-1.1}^{-1} + 23if_{n-1.1}^{-3} - 54f_{n-1.1}^{-3}) - \frac{1003112i}{1331} \tag{2.74}$$

$\Big($ use *(2.39)* of Definitions 2.3 $\Big)$. So

$$\phi_{n-1.4} = ((2(n-1)+1)^2 - 1 - f_{n.1}^2 + 7f_{n.1})(11if_{n.1} + 7 - 50f_{n.1}^{-1} + 23if_{n.1}^{-3} - 54f_{n.1}^{-3}) - \frac{1003112i}{1331} \tag{2.75}$$

$\Big($ use *(2.74)* and notice $\Big[$ by observing that $f_{n.1} \leq 2n$ and by using Remark 2.5 $\Big]$ that $f_{n.1} = f_{n-1.1}\Big)$. Clearly

$$\phi_{n-1.4} = ((2n+1)^2 - 1 - f_{n.1}^2 + 7f_{n.1})(11if_{n.1} + 7 - 50f_{n.1}^{-1} + 23if_{n.1}^{-3} - 54f_{n.1}^{-3}) - \frac{1003112i}{1331} + \phi'_{n-1.4} \tag{2.76}$$

where

$$\phi'_{n-1.4} = -8n(11if_{n.1} + 7 - 50f_{n.1}^{-1} + 23if_{n.1}^{-3} - 54f_{n.1}^{-3}) \tag{2.77}$$

$\Big($ use *(2.75)* and observe [by elementary computation and by using *(2.75)*] that

$$((2(n-1)+1)^2 - 1 - f_{n.1}^2 + 7f_{n.1}) = ((2n+1)^2 - 1 - f_{n.1}^2 + 7f_{n.1}) - 8n$$

$\Big)$. So

$$\phi_{n-1.4} = \phi_{n.4} - 8n(11if_{n.1} + 7 - 50f_{n.1}^{-1} + 23if_{n.1}^{-3} - 54f_{n.1}^{-3}) \tag{2.78}$$

$\Big($ use *(2.76)* and *(2.77)* and *(2.39)* of Definitions 2.3 $\Big)$ and clearly

$$\phi_{n-1.4} - \phi_{n.4} = -8n(11if_{n.1} + 7 - 50f_{n.1}^{-1} + 23if_{n.1}^{-3} - 54f_{n.1}^{-3})$$

$\Big($ use *(2.78)* $\Big)$. Property (2.6.4) follows and Remark 2.6 immediately follows.□

Remark 2.7.*Let n be an integer $\geq 1 + F_3$ and let $f_{n.1}$ (use Definitions 1.0). Look at (ν_n, ϵ_n) introduced in Definitions 2.3, and via (ν_n, ϵ_n), consider ($\nu_{n-1}, \epsilon_{n-1}$) $\Big($ this consideration gets sense, since $n \geq 1 + F_3$, and therefore $n - 1 \geq F_3\Big)$.* **If** *$f_{n.1} \leq 2n$, then we have the following two properties.*
(2.7.1.) $\nu_{n-1} - \nu_n = 32n - 8f_{n.1} + 8i + 16$.
(2.7.2.) $\epsilon_{n-1} - \epsilon_n = -32n - 8f_{n.1} + 8i + 16$.
Proof. Indeed let ($\nu_{n-1}, \epsilon_{n-1}$); clearly

$$\nu_{n-1} = (\, if_{n-1.1} - 4i(n-1) - 4i + 1\,)^2 - 1 + f_{n-1.1}^2 \tag{2.79}$$

$\Big($ use *(2.40)* of Definitions 2.3 $\Big)$ and

$$\epsilon_{n-1} = 4(\ (2(n-1)+1)^2 - 1 - f_{n-1.1}^2 + f_{n-1.1}\ ) + (\, if_{n-1.1} - 2i(n-1) + 1\,)(\, 4if_{n-1.1} + 4 - 8i\ ) \tag{2.80}$$

$\Big($ use *(2.41)* of Definitions 2.3 $\Big)$. So

$$\nu_{n-1} = (\, if_{n.1} - 4i(n-1) - 4i + 1\,)^2 - 1 + f_{n.1}^2 \tag{2.81}$$

$\Big($ use *(2.79)* and notice $\Big[$ by observing that $f_{n.1} \leq 2n$ and by using Remark 2.5 $\Big]$ that $f_{n.1} = f_{n-1.1}\Big)$ and

$$\epsilon_{n-1} = 4(\ (2(n-1)+1)^2 - 1 - f_{n.1}^2 + f_{n.1}\ ) + (\, if_{n.1} - 2i(n-1) + 1\,)(\, 4if_{n.1} + 4 - 8i\ ) \tag{2.82}$$

$\Big($ use *(2.80)* and notice $\Big($ by observing that $f_{n.1} \leq 2n$ and by using Remark 2.5 $\Big)$ that $f_{n.1} = f_{n-1.1}\Big)$. That being said, we now prove easily property (2.7.1) and property (2.7.2).

(2.7.1.) Indeed observing $\Big($ by elementary computation and the fact that $i^2 = -1\Big)$ that

$$(\,if_{n.1} - 4i(n-1) - 4i + 1\,)^2 - 1 + f_{n.1}^2 = (\,if_{n.1} - 4in - 4i + 1\,)^2 - 1 + f_{n.1}^2 + 32n - 8f_{n.1} + 8i + 16 \qquad (2.83)$$

then clearly

$$\nu_{n-1} = (\,if_{n.1} - 4in - 4i + 1\,)^2 - 1 + f_{n.1}^2 + 32n - 8f_{n.1} + 8i + 16 \qquad (2.84)$$

$\Big($ use *(2.81)* and *(2.83)*$\Big)$. So

$$\nu_{n-1} = \nu_n + 32n - 8f_{n.1} + 8i + 16 \qquad (2.85)$$

$\Big($ use *(2.84)* and *(2.40)* of Definitions 2.3 $\Big)$ and clearly

$$\nu_{n-1} - \nu_n = 32n - 8f_{n.1} + 8i + 16$$

$\Big($ use *(2.85)* $\Big)$. Property (2.7.1.) follows.

(2.7.2.) Indeed observing $\Big($ by elementary computation and the fact that $i^2 = -1\Big)$ that

$$4(\ (2(n-1)+1)^2 - 1 - f_{n.1}^2 + f_{n.1}\ ) = 4(\ (2n+1)^2 - 1 - f_{n.1}^2 + f_{n.1}\ ) - 32n \qquad (2.86)$$

and

$$(\,if_{n.1} - 2i(n-1) + 1\,)(\,4if_{n.1} + 4 - 8i\,) = (\,if_{n.1} - 2in + 1\,)(\,4if_{n.1} + 4 - 8i\,) + 2i(\,4if_{n.1} + 4 - 8i\,) \qquad (2.87),$$

then clearly

$$\epsilon_{n-1} = 4(\ (2n+1)^2 - 1 - f_{n.1}^2 + f_{n.1}\ ) + (\,if_{n.1} - 2in + 1\,)(\,4if_{n.1} + 4 - 8i\,) - 32n + 2i(\,4if_{n.1} + 4 - 8i\,) \qquad (2.88)$$

$\Big($ use *(2.82)* and *(2.86)* and *(2.87)* $\Big)$. So

$$\epsilon_{n-1} = \epsilon_n - 32n + 2i(\,4if_{n.1} + 4 - 8i\,) \qquad (2.89)$$

$\Big($ use *(2.88)* and (*2.41*) of Definitions 2.3 $\Big)$. Clearly

$$\epsilon_{n-1} - \epsilon_n = -32n - 8f_{n.1} + 8i + 16$$

$\Big($ use *(2.89)* and observe [by elementary computation and the fact that $i^2 = -1$] that

$$-32n + 2i(\,4if_{n.1} + 4 - 8i\,) = -32n - 8f_{n.1} + 8i + 16$$

$\Big)$. Property (2.7.2) follows and Remark 2.7 immediately follows. □

Remark 2.8.*Let* n *be an integer* $\geq 1 + F_3$ *and let* $f_{n.1}$ *(use Definitions* 1.0*). Look at* ϕ_n *introduced in Definitions* 2.3*, and via* ϕ_n*, consider* ϕ_{n-1} $\Big($ this consideration gets sense, since $n \geq 1 + F_3$, and therefore $n - 1 \geq F_3$ $\Big)$. Then

$$\phi_{n-1} - \phi_n = \sum_{j=1}^{4}(\phi_{n-1.j} - \phi_{n.j}).$$

Proof. Indeed let ϕ_n; clearly

$$\phi_{n-1} = \sum_{j=1}^{4}\phi_{n-1.j} \qquad (2.90)$$

$\Big($ use (*2.35*) of Definitions 2.3 $\Big)$. So

$$\phi_{n-1} - \phi_n = \sum_{j=1}^{4}\phi_{n-1.j} - (\sum_{j=1}^{4}\phi_{n.j}) \qquad (2.91)$$

$\Big($ use (*2.90*) and (*2.35*) of Definitions 2.3 $\Big)$ and clearly

$$\phi_{n-1} - \phi_n = \sum_{j=1}^{4}(\phi_{n-1.j} - \phi_{n.j})$$

$\Big($ use (2.91) $\Big)$. Remark 1.8 follows.□

Now using the previous four Remarks, then Proposition 2.4 becomes elementary to prove.

Proof of Proposition 2.4 *(the using of Remark*.2.1.0 *).* Indeed look ($\phi_{n-1}-\phi_n, \nu_{n-1}-\nu_n, \epsilon_{n-1}-\epsilon_n$). Clearly

$$\phi_{n-1}-\phi_n = \sum_{j=1}^{4}(\phi_{n-1.j}-\phi_{n.j}) \qquad (2.92)$$

$\Big($ use Remark 2.8$\Big)$, where

$$\phi_{n-1.1}-\phi_{n.1}=0 \qquad (2.93)$$

$\Big($ use property (2.6.1) of Remark 2.6$\Big)$, and

$$\phi_{n-1.2}-\phi_{n.2} = -8n(16f_{n.1}^{-3}+50f_{n.1}^{-2}+11if_{n.1}^{-3}-13i+5if_{n.1})+8f_{n.1}-8i-16 \qquad (2.94)$$

$\Big($ use property (2.6.2) of Remark 2.6$\Big)$, and

$$\phi_{n-1.3}-\phi_{n.3} = -8n(34if_{n.1}^{-2}-70if_{n.1}^{-3}-11i+5+6if_{n.1}) \qquad (2.95)$$

$\Big($ use property (2.6.3) of Remark 2.6$\Big)$, and

$$\phi_{n-1.4}-\phi_{n.4} = -8n(11if_{n.1}+7-50f_{n.1}^{-1}+23if_{n.1}^{-3}-54f_{n.1}^{-3}) \qquad (2.96)$$

$\Big($ use property (2.6.4) of Remark 2.6$\Big)$, and

$$\nu_{n-1}-\nu_n = 32n-8f_{n.1}+8i+16 \qquad (2.97)$$

$\Big($ use property (2.7.1) of Remark 2.7$\Big)$, and

$$\epsilon_{n-1}-\epsilon_n = -32n-8f_{n.1}+8i+16 \qquad (2.98)$$

$\Big($ use property (2.7.2) of Remark 2.7$\Big)$. That being so, let (x',y',k') such that

$$x'=-8f_{n.1}+8i+16;\ y'=32n;\ and\, k'=16nf_{n.1}^{-3}-16nf_{n.1}^{-2}+16nf_{n.1}^{-1} \qquad (2.99).$$

Now let ($\phi_{n-1}-\phi_n, \nu_{n-1}-\nu_n, \epsilon_{n-1}-\epsilon_n, x', y', k'$) where (x',y',k') is explicted in *(2.99)* and where

$$(\ \phi_{n-1}-\phi_n, \nu_{n-1}-\nu_n, \epsilon_{n-1}-\epsilon_n)$$

is explicited above $\Big($ use *(2.92)* for $\phi_{n-1}-\phi_n$; and *(2.97)* for $\nu_{n-1}-\nu_n$; and (2.98) for $\epsilon_{n-1}-\epsilon_n\Big)$. Now look at (X_n, Y_n, Z_n, x, y, k) introduced in Remark.2.1.0; clearly

$$(\ X_n, Y_n, Z_n, x, y, k\ ) = (\ \phi_{n-1}-\phi_n, \nu_{n-1}-\nu_n, \epsilon_{n-1}-\epsilon_n, x', y', k'\ ) \qquad (2.100)$$

$\Big($ **Firstly, we prove that** $X_n=\phi_{n-1}-\phi_n$. *Indeed* observe that

$$X_{n.j} = \phi_{n-1.j}-\phi_{n.j}\ for\, 1\le j\le 4 \qquad (2.100.0)$$

[$X_{n.1}=\phi_{n-1.1}-\phi_{n.1}$ (use *(2.1)* of Remark.2.1.0 and *(2.93)*); $X_{n.2}=\phi_{n-1.2}-\phi_{n.2}$ (use *(2.2)* of Remark.2.1.0 and *(2.94)*); $X_{n.3}=\phi_{n-1.3}-\phi_{n.3}$ (use *(2.3)* of Remark.2.1.0 and *(2.95)*); and $X_{n.4}=\phi_{n-1.4}-\phi_{n.4}$ (use *(2.4)* of Remark.2.1.0 and *(2.96)*). The prevous four equalities immediately imply that $X_{n.j}=\phi_{n-1.j}-\phi_{n.j}\ for\, 1\le j\le 4$]. *(2.100.0)* immediately implies that

$$X_n=\sum_{j=1}^{4}X_{n.j}=\phi_{n-1}-\phi_n=\sum_{j=1}^{4}(\phi_{n-1.j}-\phi_{n.j})\,[use(2.0)\,of\,Remark.2.1.0\,and\,(2.92)\,and\,(2.100.0)] \qquad (2.100.1).$$

Note that $Y_n=\nu_{n-1}-\nu_n\,[use(2.5)\,of\,Remark.2.1.0\,and\,(2.97)]$ **and** $Z_n=\epsilon_{n-1}-\epsilon_n\,[use(2.6)\ and\,(2.98)]$ $(2.100.2)$.

Finally, note that $(x,y,k)=(x',y',k')\,[use(2.7)\,of\,Remark.2.1.0\,and\,(2.99)]$ $(2.100.3)$.

Now using *(2.100.1)* and *(2.100.2)* and *(2.100.3)*, we immediately deduce that

$$(\ X_n, Y_n, Z_n, x, y, k\ ) = (\ \phi_{n-1}-\phi_n, \nu_{n-1}-\nu_n, \epsilon_{n-1}-\epsilon_n, x', y', k'\ )$$

$\Big)$. That being so, using *(2.100)* and Remark.2.1.0 , it becomes trivial to deduce that ($\phi_{n-1} - \phi_n, \nu_{n-1} - \nu_n, \epsilon_{n-1} - \epsilon_n, x', y', k'$) satisfies all the hypotheses of Remark.2.1.0; therefore ($\phi_{n-1} - \phi_n, \nu_{n-1} - \nu_n, \epsilon_{n-1} - \epsilon_n, x', y', k'$) satisfies the conclusion of Remark.2.1.0; so

$$x' + y' - (\nu_{n-1} - \nu_n) = 0 \, and \, x' - y' - (\epsilon_{n-1} - \epsilon_n) = 0 \tag{2.101}$$

and

$$x' + 3iy'f_{n.1}^{-1} + k'(6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i) + \phi_{n-1} - \phi_n = 0; \; k' \in \mathcal{R} \tag{2.102}$$

$\Big($ use Remark.2.1.0 where we replace (X_n, Y_n, Z_n, x, y, k) by ($\phi_{n-1} - \phi_n, \nu_{n-1} - \nu_n, \epsilon_{n-1} - \epsilon_n, x', y', k'$) $\Big)$. Clearly ($\phi_{n-1} - \phi_n, \nu_{n-1} - \nu_n, \epsilon_{n-1} - \epsilon_n$) tackles $(1, 3if_{n.1}^{-1})$ around $6f_{n.1} - 19 - if_{n.1} + 11if_{n.1}^2 - 18i$ $\Big($ use *(2.101)* and *(2.102)* and the notion of tackle introduced in Definition 2.2 [observe that $(x', y') \in \mathcal{C}^2$ and $k' \in \mathcal{R}$; so $(x', y', k') \in \mathcal{C}^2 \times \mathcal{R}$] $\Big)$. Proposition 2.4 immediately follows. □

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