Traces in Complex Hyperbolic Geometry

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Abstract

In this work, we review a paper by Parker [4]. In [4] Parker derived several trace identities for $M(3,\mathbb{C})$. In this paper we improve on the results obtained by Parker [4]. We discuss the relation between Wen's theorem and Lawton [1] and Will's work on the trace of the commutator [8]. We also present the merits on how to parametrise pair of pants groups via traces and cross-ratio. In the last section we use a trace formula which is due to Pratoussevitch [6] to compute traces of matrices generated by complex reflections in the sides of complex hyperbolic triangle groups.

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1 Introduction

In his survey paper published as [4], Parker studied the connection between the geometry of M and traces of Γ, where M is a complex hyperbolic orbifold written as $\mathbf{H}^2_{\mathbb{C}}/\Gamma$ and Γ is a discrete, faithful representation of $\pi_1(M)$ to $\mathrm{Isom}(\mathbb{H}^2_{\mathbf{C}})$. He did that by first considering the case where Γ is a free group on two generators and secondly, he discussed formulae of Pratoussevitch [6] in the case where Γ is a triangle group generated by complex reflections in three complex lines. Several geometrical information connecting traces and complex hyperbolic space could be seen in Parker [4].
Pratoussevitch [6] also presented several formulas for the traces of elements in complex hyperbolic triangle groups generated by complex reflections. In this paper we improve on the traces identities found by Parker [4] on $M(3,\mathbb{C})$. One new contribution in this paper lies on the relation between Wen's theorem and Lawton [1] and Will's work on the trace of the commutator [8].

The paper is organised as follows: In section 2 we recall the basic notions of complex hyperbolic geometry, specifically for complex hyperbolic space. Section 3 contains the geometry of two generator subgroups of SU(2, 1). Finally, in section 4 we discuss the application of a trace formula which is due to Pratoussevitch [6].

2 Preliminaries

In this section we recall the basic notions of complex hyperbolic geometry which may be needed later on, specifically for complex hyperbolic space. The main reference to this section is Parker [4].

2.1 Hermitian matrices

Let $A = (a_{ij})$ be a $k \times l$ complex matrix. The Hermitian transpose of A is the $l \times k$ complex matrix $A^* = (\overline{a}_{ji})$ formed by complex conjugating each entry of A and then taking the transpose.

A $k \times k$ complex matrix H is said to be Hermitian if it equals its own Hermitian transpose i.e. $H = H^*$. An example is

$$H = \begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & -2 \end{bmatrix} = H^*$$

Notice that the diagonal entries must be real, they have to be unchanged by the process of conjugation.

2.2 Hermitian forms on $\mathbb{C}^{p,q}$

For each $k \times k$ Hermitian matrix H we can associate a Hermitian form

$$\langle \cdot, \cdot \rangle : \mathbb{C}^k \times \mathbb{C}^k \to \mathbb{C} \textit{ given by } \langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* H \mathbf{z}$$

(notice the change in the order) where $\mathbf{w}$ and $\mathbf{z}$ are vectors in $\mathbb{C}^k$. Note that the $\langle \cdot, \cdot \rangle$ is the Hermitian form and is always with respect to a particular Hermitian matrix H.

2.3 Cayley transform

Given two Hermitian forms H and H' of the same signature we can move from one Hermitian form to another using a Cayley transform . If the vectors

$\mathbf{z} = (z_1, z_2, z_3)^t$ and $\mathbf{w} = (w_1, w_2, w_3)^t$ are in $\mathbb{C}^{2,1}$. The first Hermitian form is defined to be :

$$\langle \mathbf{z}, \mathbf{w} \rangle_1 = z_1\overline{w}_1 + z_2\overline{w}_2 - z_3\overline{w}_3 \; from \; \langle \mathbf{z}, \; \mathbf{w} \rangle_1 = \mathbf{w}^* H_1 \mathbf{z} \; where$$

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

the Hermitian matrix. The second Hermitian form is defined to be:

$$\langle \mathbf{z}, \mathbf{w} \rangle_2 = z_1\overline{w}_3 + z_2\overline{w}_2 + z_3\overline{w}_1 \; from \; \langle \mathbf{z}, \; \mathbf{w} \rangle_2 = \mathbf{w}^* H_2 \mathbf{z} \; where$$

$$H_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

the Hermitian matrix. The following Cayley transform interchanges the first and second Hermitian forms

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

2.4 Three models of complex hyperbolic space

There are three standard models of complex hyperbolic space, namely: the projective model in $\mathbb{P}^n_{\mathbb{C}}$, the unit ball model in $\mathbb{C}^n$ and the Siegel domain model in $\mathbb{C}^n$. We call a vector $z \in \mathbb{C}^{2,1}$ negative, null or positive according as $\langle z, z \rangle$ is negative, zero or positive.

Definition : The projective model of complex hyperbolic space is defined to be the collection of negative lines in $\mathbb{C}^{2,1}$, that is, $\mathbf{H}^2_{\mathbb{C}} = \mathbb{P}V_-$.

We can get the other two models from the projective model by taking a standard lift on $z = (z_1, z_2) \in \mathbb{C}^2$ to $\mathbb{C}^{2,1}$ and define $z_3 = 1$ for the first and second Hermitian forms.

Taking the first Hermitian form for $\langle z, z \rangle_1 < 0$ for $z = (z_1, z_1)^t \in \mathbb{C}^{2,1}$

$$\langle z, z \rangle_1 = z_1\overline{z}_1 + z_2\overline{z}_2 - 1 < 0 \Rightarrow |z_1|^2 + |z_2|^2 < 1.$$

Thus the point $z = (z_1, z_2)$ is in the unit ball in $\mathbb{C}$ forming the unit ball model of complex hyperbolic space. The boundary of the unit ball is the sphere S^3 given by

$$|z_1|^2 + |z_2|^2 = 1.$$

Taking the second Hermitian form we obtain $z \in \mathbf{H}^2_{\mathbb{C}}$ provided

$$\langle z, z\rangle_2 = z_1 + z_2\overline{z}_2 + \overline{x}_1 < 0 \Rightarrow 2\Re(z_1) + |z_2|^2 = 0.$$

Thus $z = (z_1, z_2)$ is in a domain in $\mathbb{C}^2$ whose boundary is the paraboloid defined by

$$2\Re(z_1) + |z_2|^2 = 0.$$

This domain is called the Siegel domain and forms the Siegel domain model of $\mathbf{H}^2_{\mathbb{C}}$.

3 Two Generator Groups and Fenchel-Nielsen Coordinates

This section talks about the geometry of two generator subgroups of $SU(2, 1)$. In this section, we pay much attention to the case where the generators and their products are all loxodromic.

3.1 Trace identities in $M(3, \mathbb{C})$

In this section we derive some trace identities for 3×3 matrices. The first lemma follows by writing $\mathrm{tr}(A)$, $\mathrm{tr}(A^2)$ and $\mathrm{tr}(A^3)$ as homogeneous polynomials in the eigenvalues of A and then solving for the coefficients of the characteristic polynomial.

Lemma 1: Let $A \in M(3, \mathbb{C})$. Then the characteristic polynomial of A (*ie.* $ch_A(x)$) is

$$x^3 - \mathrm{tr}(A)x^2 + \frac{\mathrm{tr}(A)^2 - \mathrm{tr}(A^2)}{2}x - \frac{\mathrm{tr}(A)^3 - 3\mathrm{tr}(A)\mathrm{tr}(A^2) + 2\mathrm{tr}(A^3)}{6}.$$

For any $A \in M(3, \mathbb{C})$ define $ch(A)$ to be the following matrix (here I is the 3×3 identity matrix):

$$\begin{aligned} ch(A) = \; & A^3 - \mathrm{tr}(A)A^2 + \frac{1}{2}(\mathrm{tr}(A)^2 - \mathrm{tr}(A^2))A \\ & - \frac{1}{6}(\mathrm{tr}(A)^3 - \mathrm{tr}(A)\mathrm{tr}(A^2) + 2\mathrm{tr}(A^3))I. \end{aligned}$$

Then by the Cayley-Hamilton theorem, $ch(A) = 0$, the 3×3 zero matrix. Parker [4] used a process known as trilinearisation on this identity to obtain the following:

Proposition 2: Let $A, B, C \in M(3, \mathbb{C})$. Then

$$\begin{aligned}O = {} & ABC + ACB + BAC + BCA + CAB + CBA \\ & - \mathrm{tr}(A)(BC + CB) - \mathrm{tr}(B)(AC + CA) - \mathrm{tr}(C)(AB + BA) \\ & + (\mathrm{tr}(B)\mathrm{tr}(C) - \mathrm{tr}(BC))A + (\mathrm{tr}(A)\mathrm{tr}(C) - \mathrm{tr}(AC))B \\ & + (\mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB))C - (\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(C) + \mathrm{tr}(ABC) \\ & + \mathrm{tr}(CBA))I + (\mathrm{tr}(A)\mathrm{tr}(BC) + \mathrm{tr}(B)\mathrm{tr}(AC) + \mathrm{tr}(C)\mathrm{tr}(AB))I.\end{aligned}$$

Corollary 3: For any $A, B, \in M(3, \mathbb{C})$ we have:

$$\begin{aligned}O = {} & ABA^{-1} + B + A^{-1}BA \\ & - \mathrm{tr}(A)(BA^{-1} + A^{-1}B) - \mathrm{tr}(A^{-1})(AB + BA) + \mathrm{tr}(A)\mathrm{tr}(A^{-1})B \\ & + (\mathrm{tr}(B)\mathrm{tr}(A^{-1}) - \mathrm{tr}(BA^{-1}))A + (\mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB))A^{-1} \\ & - (\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(A^{-1}) + \mathrm{tr}(B) - \mathrm{tr}(A)\mathrm{tr}(BA^{-1}) - \mathrm{tr}(A^{-1})\mathrm{tr}(AB))I.\end{aligned}$$

$$\begin{aligned}O = {} & ABA + A^2B + BA^2 \\ & - \mathrm{tr}(A)(AB + BA) - \frac{1}{2}\mathrm{tr}(B)A^2 + (\mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB))A \\ & + \frac{1}{2}(\mathrm{tr}(A)^2 - \mathrm{tr}(A^2))B - \frac{1}{2}(\mathrm{tr}(A)^2 - \mathrm{tr}(A^2))\mathrm{tr}(B)I \\ & + (\mathrm{tr}(A)\mathrm{tr}(AB) - \mathrm{tr}(A^2B))I.\end{aligned}$$

See Parker [4] for the proof.

Corollary 4 For any $A, B \in M(3, \mathbb{C})$ we have

$$\begin{aligned}\mathrm{tr}[A, B] + \mathrm{tr}[A^{-1}, B] = {} & \\ & \mathrm{tr}(A)\mathrm{tr}(A^{-1}) + \mathrm{tr}(B)\mathrm{tr}(B^{-1}) + \mathrm{tr}(A)\mathrm{tr}(A^{-1})\mathrm{tr}(B)\mathrm{tr}(B^{-1}) \\ & - 3 + \mathrm{tr}(AB)\mathrm{tr}(A^{-1}B^{-1}) - \mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(A^{-1}B^{-1}) \\ & - \mathrm{tr}(A^{-1})\mathrm{tr}(B^{-1})\mathrm{tr}(AB) + \mathrm{tr}(A^{-1}B)\mathrm{tr}(AB^{-1}) \\ & - \mathrm{tr}(A^{-1})\mathrm{tr}(B)\mathrm{tr}(AB^{-1}) - \mathrm{tr}(A)\mathrm{tr}(B^{-1})\mathrm{tr}(A^{-1}B).\end{aligned}$$

3.2 Traces identities in $SL(3, \mathbb{C})$

Suppose A is in $SL(3, \mathbb{C})$, then the characteristic polynomial of A may be defined by lemma 5.

Lemma 5: Let $A \in SL(3, \mathbb{C})$. The characteristic polynomial of A is

$$\mathrm{ch}_A(x) = x^3 - \mathrm{tr}(A)x^2 + \mathrm{tr}(A^{-1})x - 1.$$

Lemma 6: Let $A \in SU(2, 1)$. Then

1. $\mathrm{tr}(A^2) = (\mathrm{tr}(A))^2 - 2\mathrm{tr}(A^{-1})$;
2. $\mathrm{tr}(A^3) = (\mathrm{tr}(A))^3 - 3\mathrm{tr}(A)\mathrm{tr}(A^{-1}) + 3$.

Proposition 7: Let $A, B \in SL(3,\mathbb{C})$. Then $\mathrm{tr}[A,B]\mathrm{tr}[B,A]$ may be expressed as a polynomial function of the traces of $A, B, AB, A^{-1}B$ and their inverses.

Proof. Write $A = MN$ and $B = NM$ in the expression for corollary 4. This gives

$$\begin{aligned}
&\mathrm{tr}[MN{,}NM] + \mathrm{tr}[N^{-1}M^{-1}, NM]\\
&\qquad = \mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1}) + \mathrm{tr}(NM)\mathrm{tr}(N^{-1}M^{-1})\\
&\qquad + \mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(NM)\mathrm{tr}(N^{-1}M^{-1})\\
&\qquad - 3 + \mathrm{tr}(MNNM)\mathrm{tr}(M^{-1}N^{-1}N^{-1}M^{-1})\\
&\qquad - \mathrm{tr}(MN)\mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1}N^{-1}M^{-1})\\
&\qquad - \mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(N^{-1}M^{-1})\mathrm{tr}(MNNM)\\
&\qquad + \mathrm{tr}(M^{-1}N^{-1}NM)\mathrm{tr}(MNN^{-1}M^{-1})\\
&\qquad - \mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(NM)\mathrm{tr}(MNN^{-1}M^{-1})\\
&\qquad - \mathrm{tr}(MN)\mathrm{tr}(N^{-1}M^{-1})\mathrm{tr}(M^{-1}N^{-1}NM)\\
&\\
&\qquad = 2\mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1}) + \mathrm{tr}(MN)^2\mathrm{tr}(M^{-1}N^{-1})^2\\
&\qquad - 3 + \mathrm{tr}(M^2N^2)\mathrm{tr}(M^{-2}N^{-2}) - \mathrm{tr}(MN)^2\mathrm{tr}(M^{-2}N^{-2})\\
&\qquad - \mathrm{tr}(M^{-1}N^{-1})^2\mathrm{tr}(M^2N^2) + \mathrm{tr}[M,N]\mathrm{tr}[N,M]\\
&\qquad - \mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1})(\mathrm{tr}[M,N] + \mathrm{tr}[M^{-1},N]) - - - - - (1)
\end{aligned}$$

Using corollary 4, $\mathrm{tr}[M,N] + \mathrm{tr}[M^{-1},N]$ can be expressed in terms of the traces of $M, N, MN, M^{-1}N$ and their inverses. That is

$$\begin{aligned}
&\mathrm{tr}[M{,}N] + \mathrm{tr}[M^{-1}, N] =\\
&\qquad \mathrm{tr}(M)\mathrm{tr}(M^{-1}) + \mathrm{tr}(N)\mathrm{tr}(N^{-1}) + \mathrm{tr}(M)\mathrm{tr}(M^{-1})\mathrm{tr}(N)\mathrm{tr}(N^{-1})\\
&\qquad - 3 + \mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1}) - \mathrm{tr}(M)\mathrm{tr}(N)\mathrm{tr}(M^{-1}N^{-1})\\
&\qquad - \mathrm{tr}(M^{-1})\mathrm{tr}(M^{-1})\mathrm{tr}(MN) + \mathrm{tr}(M^{-1}N)\mathrm{tr}(MN^{-1})\\
&\qquad - \mathrm{tr}(M^{-1})\mathrm{tr}(N)\mathrm{tr}(MN^{-1}) - \mathrm{tr}(M)\mathrm{tr}(N^{-1})\mathrm{tr}(M^{-1}N). - - - - - (2)
\end{aligned}$$

If M and N are in $SL(3,\mathbb{C})$ we can use their characteristic polynomials to write

$$M^2 = \mathrm{tr}(M)M - \mathrm{tr}(M^{-1})I + M^{-1}, \quad N^2 = \mathrm{tr}(N)N - \mathrm{tr}(N^{-1})I + N^{-1}$$
$$M^{-2} = M - \mathrm{tr}(M)I + \mathrm{tr}(M^{-1})M^{-1}, \quad N^{-2} = N - \mathrm{tr}(N)I + \mathrm{tr}(N^{-1})N^{-1}$$

Hence

$$\begin{aligned} M^2N^2 &= (\mathrm{tr}(M)M - \mathrm{tr}(M^{-1})I + M^{-1})(\mathrm{tr}(N)N - \mathrm{tr}(N^{-1})I + N^{-1}) \\ &= \mathrm{tr}(M)\mathrm{tr}(N)MN - \mathrm{tr}(M)\mathrm{tr}(N^{-1})M + \mathrm{tr}(M)MN^{-1} \\ &- \mathrm{tr}(M^{-1})\mathrm{tr}(N)N + \mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1})I - \mathrm{tr}(M^{-1})N^{-1} \\ &- \mathrm{tr}(N^{-1})M^{-1} + M^{-1}N^{-1} \end{aligned}$$

Taking traces gives

$$\begin{aligned} \mathrm{tr}(M^2N^2) &= \mathrm{tr}(M)\mathrm{tr}(N)\mathrm{tr}(MN) - \mathrm{tr}(M)^2\mathrm{tr}(N^{-1}) \\ &+ \mathrm{tr}(M)\mathrm{tr}(MN^{-1}) - \mathrm{tr}(M^{-1})\mathrm{tr}(N)^2 \\ &+ \mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1}) + \mathrm{tr}(M^{-1}N^{-1}) - - - - - - - - -(3) \end{aligned}$$

Using similar argument gives the following:

$$\begin{aligned} \mathrm{tr}(M^2N^{-2}) &= \mathrm{tr}(M)\mathrm{tr}(MN) - \mathrm{tr}(M)^2\mathrm{tr}(N) \\ &+ \mathrm{tr}(M)\mathrm{tr}(N^{-1})\mathrm{tr}(MN^{-1}) \\ &+ \mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1}) - \mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1})^2 \\ &+ \mathrm{tr}(M^{-1}N) + \mathrm{tr}(N^{-1})\mathrm{tr}(M^{-1}N^{-1}) - - - - - - -(4) \end{aligned}$$

$$\begin{aligned} \mathrm{tr}(M^{-2}N^2) &= \mathrm{tr}(N)\mathrm{tr}(MN) + \mathrm{tr}(MN^{-1}) - \mathrm{tr}(M)\mathrm{tr}(N)^2 \\ &+ \mathrm{tr}(M)\mathrm{tr}(N^{-1}) + \mathrm{tr}(M^{-1})\mathrm{tr}(N)\mathrm{tr}(M^{-1}N) \\ &- \mathrm{tr}(M^{-1})^2\mathrm{tr}(N^{-1}) + \mathrm{tr}(M^{-1})\mathrm{tr}(M^{-1}N^{-1}) - - - -(5) \end{aligned}$$

$$\begin{aligned} \mathrm{tr}(M^{-2}N^{-2}) &= \mathrm{tr}(MN) + \mathrm{tr}(M)\mathrm{tr}(N) + \mathrm{tr}(N^{-1})\mathrm{tr}(MN^{-1}) \\ &- \mathrm{tr}(M)\mathrm{tr}(N^{-1})^2 + \mathrm{tr}(M^{-1})\mathrm{tr}(M^{-1}N) \\ &- \mathrm{tr}(M^{-1})^2\mathrm{tr}(N) + \mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1})\mathrm{tr}(M^{-1}N^{-1}) - - - - - (6) \end{aligned}$$

Thus it suffices to express the trace of $[MN, NM]$ and $[N^{-1}M^{-1}, NM]$ in terms of these other traces. To do this, first we write

$$[MN, NM] = MN^2MN^{-1}M^{-2}N^{-1}$$

$$[NM, MN] = NM^2NM^{-1}N^{-2}M^{-1}$$

and substitute for N^2, N^{-2}, M^2 and M^{-2} as above to have

$$\begin{aligned}
&[MN,NM]\\
&\quad = M\left(\mathrm{tr}(N)N - \mathrm{tr}(N^{-1})I + N^{-1}\right)MN^{-1}\left(M - \mathrm{tr}(M)I + \mathrm{tr}(M^{-1})M^{-1}\right)N^{-1}\\
&\quad = \mathrm{tr}(N)MNMN^{-1}MN^{-1} - \mathrm{tr}(N)\mathrm{tr}(M)MNMN^{-1}N^{-1}\\
&\quad + \mathrm{tr}(N)\mathrm{tr}(M^{-1})MNMN^{-1}M^{-1}N^{-1} - \mathrm{tr}(N^{-1})MMN^{-1}MN^{-1}\\
&\quad + \mathrm{tr}(N^{-1})\mathrm{tr}(M)MMN^{-1}N^{-1} - \mathrm{tr}(N^{-1})\mathrm{tr}(M^{-1})MMN^{-1}M^{-1}N^{-1}\\
&\quad + MN^{-1}MN^{-1}MN^{-1} - \mathrm{tr}(M)MN^{-1}MN^{-1}N^{-1}\\
&\quad + \mathrm{tr}(M^{-1})MN^{-1}MN^{-1}M^{-1}N^{-1} - - - - - - - - - - - - - - (7)
\end{aligned}$$

and

$$\begin{aligned}
&[NM, MN]\\
&\quad = N(\mathrm{tr}(M)M - \mathrm{tr}(M^{-1})I + M^{-1})NM^{-1}(N - \mathrm{tr}(N)I + \mathrm{tr}(N^{-1})N^{-1})M^{-1}\\
&\quad = \mathrm{tr}(M)NMNMNM^{-1} - \mathrm{tr}(M)\mathrm{tr}(N)NMNM^{-2}\\
&\quad + \mathrm{tr}(M)\mathrm{tr}(N^{-1})NMNM^{-1}N^{-1}M^{-1} - \mathrm{tr}(M^{-1})N^2M^{-1}NM^{-1}\\
&\quad + \mathrm{tr}(M^{-1})\mathrm{tr}(N)N^2M^{-2} - \mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1})N^2M^{-1}N^{-1}M^{-1}\\
&\quad + NM^{-1}NM^{-1}NM^{-1} - \mathrm{tr}(N)NM^{-1}NM^{-2}\\
&\quad + \mathrm{tr}(N^{-1})NM^{-1}NM^{-1}N^{-1}M^{-1}. - - - - - - - - - - - - - (8)
\end{aligned}$$

Finally we use corollary 3 to substitute for expressions such as MNM, MNM^{-1} in equations 7 and 8.

Putting equations 2, 3, 4, 5, 6, 7 and 8 into equation 1, eventually yields the polynomial:

$$\begin{aligned}
&|\mathrm{tr}[M, N]|^2 = -5\mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1}) + 3 - \mathrm{tr}(M)^2\mathrm{tr}(N)^2\mathrm{tr}(MN)\\
&\quad - \mathrm{tr}(M)\mathrm{tr}(N)\mathrm{tr}(MN)\mathrm{tr}(N^{-1})\mathrm{tr}(MN^{-1}) + \mathrm{tr}(M)^2\mathrm{tr}(N)\mathrm{tr}(N^{-1})^2\mathrm{tr}(MN)\\
&\quad - \mathrm{tr}(M)\mathrm{tr}(N)\mathrm{tr}(MN)\mathrm{tr}(M^{-1})\mathrm{tr}(M^{-1}N) + \mathrm{tr}(M)^2\mathrm{tr}(N)^2\mathrm{tr}(MN)\mathrm{tr}(M^{-1})^2\\
&\quad - \mathrm{tr}(M)\mathrm{tr}(N)\mathrm{tr}(MN)\mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1})\mathrm{tr}(M^{-1}N^{-1}) - \mathrm{tr}(M)^3\mathrm{tr}(N)^3\\
&\quad + \mathrm{tr}(M)^2\mathrm{tr}(N^{-1})\mathrm{tr}(MN) + \mathrm{tr}(M)^3\mathrm{tr}(N^{-1})\mathrm{tr}(N) + \mathrm{tr}(M)^2\mathrm{tr}(N^{-1})^2\mathrm{tr}(MN^{-1})\\
&\quad + \mathrm{tr}(M)^2\mathrm{tr}(N^{-1})\mathrm{tr}(M^{-1})\mathrm{tr}(M^{-1}N) - \mathrm{tr}(M)^2\mathrm{tr}(N^{-1})\mathrm{tr}(M^{-1})^2\mathrm{tr}(N)\\
&\quad + \mathrm{tr}(M)^2\mathrm{tr}(N^{-1})^2\mathrm{tr}(M^{-1}N^{-1}) - \mathrm{tr}(M)\mathrm{tr}(MN^{-1})\mathrm{tr}(MN)\\
&\quad - \mathrm{tr}(M)^3\mathrm{tr}(MN^{-1})\mathrm{tr}(N) + \mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(N)\mathrm{tr}(N^{-1})\\
&\quad - \mathrm{tr}(M)\mathrm{tr}(N^{-1})\mathrm{tr}(MN^{-1})^2 + \mathrm{tr}(M)^2\mathrm{tr}(N^{-1})^2\mathrm{tr}(MN^{-1})\\
&\quad - \mathrm{tr}(M)\mathrm{tr}(M^{-1})\mathrm{tr}(MN^{-1})\mathrm{tr}(M^{-1}N) + \mathrm{tr}(M)\mathrm{tr}(M^{-1})^2\mathrm{tr}(N)\mathrm{tr}(MN^{-1})
\end{aligned}$$

$$
\begin{aligned}
&- \mathrm{tr}(M)\mathrm{tr}(MN^{-1})\mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1})\mathrm{tr}(M^{-1}N^{-1}) + \mathrm{tr}(M^{-1})\mathrm{tr}(N)^2\mathrm{tr}(MN)\\
&+ \mathrm{tr}(M^{-1})\mathrm{tr}(M)\mathrm{tr}(N)^3 + \mathrm{tr}(M^{-1})\mathrm{tr}(N)^2\mathrm{tr}(N^{-1})\mathrm{tr}(MN^{-1})\\
&- \mathrm{tr}(M^{-1})\mathrm{tr}(N)^2\mathrm{tr}(M)\mathrm{tr}(N^{-1})^2 + \mathrm{tr}(M^{-1})^2\mathrm{tr}(N)^2\mathrm{tr}(M^{-1}N)\\
&- \mathrm{tr}(M^{-1})^3\mathrm{tr}(N)^3 + \mathrm{tr}(M^{-1})^2\mathrm{tr}(N)^2\mathrm{tr}(N^{-1})\mathrm{tr}(M^{-1}N^{-1})\\
&- \mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1})\mathrm{tr}(MN) - \mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1})\mathrm{tr}(M)\mathrm{tr}(N)\\
&- \mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1})^2\mathrm{tr}(MN^{-1}) + \mathrm{tr}(M^{-1})\mathrm{tr}(M)\mathrm{tr}(N^{-1})^3\\
&- \mathrm{tr}(M^{-1})^2\mathrm{tr}(N^{-1})\mathrm{tr}(M^{-1}N) + \mathrm{tr}(M^{-1})^2\mathrm{tr}(N^{-1})\mathrm{tr}(N)\\
&- \mathrm{tr}(M^{-1})^2\mathrm{tr}(N^{-1})^2\mathrm{tr}(M^{-1}N^{-1}) - \mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(MN)\\
&- \mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(M)\mathrm{tr}(N) - \mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(N^{-1})\mathrm{tr}(MN^{-1})\\
&+ \mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(M)\mathrm{tr}(N^{-1})^2 - \mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(M^{-1})\mathrm{tr}(M^{-1}N)\\
&+ \mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(M^{-1})^2\mathrm{tr}(N) - \mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1})\mathrm{tr}(M^{-1}N^{-1})^2\\
&+ \mathrm{tr}(MN)^3 + \mathrm{tr}(MN)^2\mathrm{tr}(N^{-1})\mathrm{tr}(MN^{-1}) - \mathrm{tr}(MN)^2\mathrm{tr}(M)\mathrm{tr}(N^{-1})^2\\
&+ \mathrm{tr}(MN)^2\mathrm{tr}(M^{-1})\mathrm{tr}(M^{-1}N) - \mathrm{tr}(MN)^2\mathrm{tr}(M^{-1})^2\mathrm{tr}(N)\\
&+ \mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1})\mathrm{tr}(MN)^2\mathrm{tr}(M^{-1}N^{-1}) + \mathrm{tr}(M^{-1}N^{-1})^2\mathrm{tr}(M)\mathrm{tr}(N)\mathrm{tr}(MN)\\
&- \mathrm{tr}(M^{-1}N^{-1})^2\mathrm{tr}(M)^2\mathrm{tr}(N^{-1}) + \mathrm{tr}(M^{-1}N^{-1})^2\mathrm{tr}(M)\mathrm{tr}(MN^{-1})\\
&- \mathrm{tr}(M^{-1}N^{-1})^2\mathrm{tr}(M^{-1})\mathrm{tr}(N)^2 + \mathrm{tr}(M^{-1}N^{-1})^2\mathrm{tr}(M^{-1})\mathrm{tr}(N^{-1})\\
&+ \mathrm{tr}(M^{-1}N^{-1})^3 + \mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(M)\mathrm{tr}(M^{-1})\\
&+ \mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(M)\mathrm{tr}(M^{-1})\mathrm{tr}(N)\mathrm{tr}(N^{-1})\\
&- \mathrm{tr}(MN)\mathrm{tr}(M)\mathrm{tr}(N)\mathrm{tr}(M^{-1}N^{-1})^2 - \mathrm{tr}(M^{-1})^2\mathrm{tr}(MN)^2\mathrm{tr}(M^{-1}N^{-1})\\
&+ \mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(M^{-1}N)\mathrm{tr}(MN^{-1})\\
&- \mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(M^{-1})\mathrm{tr}(N)\mathrm{tr}(MN^{-1})\\
&- \mathrm{tr}(MN)\mathrm{tr}(M^{-1}N^{-1})\mathrm{tr}(M)\mathrm{tr}(N^{-1})\mathrm{tr}(M^{-1}N)\\
&+ \mathrm{tr}[M,N]\mathrm{tr}[N^{-1},M^{-1}] + \mathrm{tr}[N,M]\mathrm{tr}[M^{-1},N^{-1}].
\end{aligned}
$$

Note that the last two terms could be expanded by applying corollary 3 on equations 7 and 8. Again this polynomial is equivalent to the polynomial given in lemma 10.

3.3 Trace parameters for two generator groups of $SU(2,1)$

Let Y be a three holed sphere (sometimes called pair of pants). If the boundary curves are denoted by α, β, γ, then the fundamental group of Y is

$$\pi_1(Y) = \langle [\alpha], [\beta], [\gamma] : [\alpha\beta\gamma] = id \rangle.$$

In fact, π_1 is a free group generated by any two of $[\alpha], [\beta], [\gamma]$ where$[\alpha], [\beta]$ and $[\gamma]$ are the homotopy classes in π_1 representing the boundary curves.

We want to study representations (conjugacy class of homomorphism) $\rho : \pi_1(Y) \longrightarrow \Gamma_Y < SU(2,1)$. Let $A = \rho([\alpha]), B = \rho([\beta]), C = \rho([\gamma])$, then $\rho(\pi_1(Y)) = \Gamma_Y$ is a subgroup of $SU(2,1)$ generated by A, B, C with $ABC = I$ (Parker, 2012). In other words, $C = (AB)^{-1} = B^{-1}A^{-1}$. We can parametrise pair of pants via traces and cross-ratio as in proposition 13 and theorem 14.

Theorem 8: (Wen's theorem): Suppose that $A, B \in SU(2,1)$ and that $\langle A, B\rangle$ is Zariski dense. Then $\langle A, B\rangle$ is determined up to conjugation within $SU(2,1)$ by

$$\mathrm{tr}(A),\ \ \mathrm{tr}(B),\ \ \mathrm{tr}(AB),\ \ \mathrm{tr}(A^{-1}B),\ \ \mathrm{tr}[A,B].$$

Lemma 9: let A, B, C be element of $SU(2,1)$ so that $ABC = I$. Then

$$\begin{aligned}\mathrm{tr}(A^{-1}B) - \mathrm{tr}(A^{-1})\mathrm{tr}(B) &= \mathrm{tr}(B^{-1}C) - \mathrm{tr}(B^{-1})\mathrm{tr}(C)\\ &= \mathrm{tr}(C^{-1}A) - \mathrm{tr}(C^{-1})\mathrm{tr}(A).\end{aligned}$$

We now express equation 18 of Lawton[1] in terms of $\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(AB)$ etc.

Lemma 10: There exists a polynomial $Q \in \mathbb{R}$ so $Q - t_{(5)}t_{(-5)} \in ker(\Pi)$, where $t_{(5)}$ and $t_{(-5)}$ are generators of $\mathbb{R}, t_{(5)} = \mathrm{tr}[A,B], t_{(-5)} = \mathrm{tr}[B,A], \Pi$ is a surjective algebra morphism and in particular

$$\begin{aligned}Q =& 9 - 6\mathrm{tr}(A)\mathrm{tr}(A^{-1}) - 6\mathrm{tr}(B)\mathrm{tr}(B^{-1}) - 6\mathrm{tr}(B^{-1}A^{-1})\mathrm{tr}(AB)\\
&- 6\mathrm{tr}(A^{-1}B)\mathrm{tr}(AB^{-1}) + \mathrm{tr}(A)^3 + \mathrm{tr}(B)^3 + \mathrm{tr}(AB)^3 + \mathrm{tr}(A^{-1}B)^3\\
&+ \mathrm{tr}(A^{-1})^3 + \mathrm{tr}(B^{-1})^3 + \mathrm{tr}(B^{-1}A^{-1})^3 + \mathrm{tr}(AB^{-1})^3\\
&- 3\mathrm{tr}(A^{-1}B)\mathrm{tr}(B^{-1}A^{-1})\mathrm{tr}(A^{-1}) - 3\mathrm{tr}(A^{-1}B)\mathrm{tr}(AB)\mathrm{tr}(A)\\
&- 3\mathrm{tr}(AB^{-1})\mathrm{tr}(B)\mathrm{tr}(AB) - 3\mathrm{tr}(A^{-1}B)\mathrm{tr}(B^{-1})\mathrm{tr}(B^{-1}A^{-1})\\
&+ 3\mathrm{tr}(AB^{-1})\mathrm{tr}(B^{-1})\mathrm{tr}(A) + 3\mathrm{tr}(A^{-1}B)\mathrm{tr}(B)\mathrm{tr}(A^{-1})\\
&+ 3\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(AB) + 3\mathrm{tr}(A^{-1})\mathrm{tr}(B^{-1})\mathrm{tr}(B^{-1}A^{-1})\\
&+ \mathrm{tr}(B^{-1})\mathrm{tr}(A^{-1})\mathrm{tr}(B)\mathrm{tr}(A) + \mathrm{tr}(AB)\mathrm{tr}(B^{-1})\mathrm{tr}(B^{-1}A^{-1})\mathrm{tr}(B)\\
&+ \mathrm{tr}(AB^{-1})\mathrm{tr}(A^{-1})\mathrm{tr}(A^{-1}B)\mathrm{tr}(A) + \mathrm{tr}(AB^{-1})\mathrm{tr}(B^{-1})\mathrm{tr}(A^{-1}B)\mathrm{tr}(B)\\
&+ \mathrm{tr}(B^{-1}A^{-1})\mathrm{tr}(A^{-1})\mathrm{tr}(AB)\mathrm{tr}(A) + \mathrm{tr}(B^{-1}A^{-1})\mathrm{tr}(AB^{-1})\\
&+ \mathrm{tr}(AB)\mathrm{tr}(A^{-1}B) + \mathrm{tr}(AB^{-1})^2\mathrm{tr}(B^{-1}A^{-1})\mathrm{tr}(B^{-1})\\
&+ \mathrm{tr}(A^{-1}B)^2\mathrm{tr}(AB)\mathrm{tr}(B) + \mathrm{tr}(A^{-1})^2\mathrm{tr}(B^{-1})\mathrm{tr}(AB^{-1})\\
&+ \mathrm{tr}(A)^2\mathrm{tr}(B)\mathrm{tr}(A^{-1}B) + \mathrm{tr}(A)\mathrm{tr}(B^{-1})^2\mathrm{tr}(B^{-1}A^{-1})\\
&+ \mathrm{tr}(A^{-1})^2\mathrm{tr}(B^{-1}A^{-1})\mathrm{tr}(B) + \mathrm{tr}(A)^2\mathrm{tr}(AB)\mathrm{tr}(B^{-1})\end{aligned}$$

$$
\begin{aligned}
&+ \mathrm{tr}(AB^{-1})\mathrm{tr}(A)\mathrm{tr}(B)^2 + \mathrm{tr}(AB^{-1})\mathrm{tr}(B)\mathrm{tr}(B^{-1}A^{-1})^2\\
&+ \mathrm{tr}(A^{-1}B)\mathrm{tr}(B^{-1})\mathrm{tr}(AB^{-1})^2 + \mathrm{tr}(A^{-1})^2\mathrm{tr}(B^{-1}A^{-1})\mathrm{tr}(B)\\
&+ \mathrm{tr}(A)^2\mathrm{tr}(AB)\mathrm{tr}(B^{-1}) + \mathrm{tr}(AB^{-1})\mathrm{tr}(A)\mathrm{tr}(B)^2\\
&+ \mathrm{tr}(A^{-1}B)\mathrm{tr}(A^{-1})\mathrm{tr}(B^{-1})^2 + \mathrm{tr}(A^{-1}B)\mathrm{tr}(B^{-1}A^{-1})\mathrm{tr}(B)^2\\
&+ \mathrm{tr}(A)\mathrm{tr}(AB)\mathrm{tr}(AB^{-1})^2 + \mathrm{tr}(A^{-1})\mathrm{tr}(AB)\mathrm{tr}(A^{-1}B)^2\\
&+ \mathrm{tr}(A^{-1})\mathrm{tr}(AB^{-1})\mathrm{tr}(AB)^2 + \mathrm{tr}(A)\mathrm{tr}(A^{-1}B)\mathrm{tr}(B^{-1}A^{-1})^2\\
&- 2\mathrm{tr}(B^{-1}A^{-1})^2\mathrm{tr}(B^{-1})\mathrm{tr}(A^{-1}) - 2\mathrm{tr}(AB)^2\mathrm{tr}(B)\mathrm{tr}(A)\\
&- 2\mathrm{tr}(AB^{-1})^2\mathrm{tr}(A^{-1})\mathrm{tr}(B) - 2\mathrm{tr}(A^{-1}B)^2\mathrm{tr}(A)\mathrm{tr}(B^{-1})\\
&+ \mathrm{tr}(A^{-1})^2\mathrm{tr}(B^{-1})^2\mathrm{tr}(B^{-1}A^{-1}) + \mathrm{tr}(A)^2\mathrm{tr}(B)^2\mathrm{tr}(AB)\\
&+ \mathrm{tr}(AB^{-1})\mathrm{tr}(A^{-1})^2\mathrm{tr}(B)^2 + \mathrm{tr}(A^{-1}B)\mathrm{tr}(A)^2\mathrm{tr}(B^{-1})^2\\
&- \mathrm{tr}(AB^{-1})\mathrm{tr}(B^{-1})^2\mathrm{tr}(B)\mathrm{tr}(A) - \mathrm{tr}(A^{-1}B)\mathrm{tr}(B)^2\mathrm{tr}(B^{-1})\mathrm{tr}(A^{-1})\\
&- \mathrm{tr}(B^{-1}A^{-1})\mathrm{tr}(A)^2\mathrm{tr}(A^{-1})\mathrm{tr}(B) - \mathrm{tr}(AB)\mathrm{tr}(A^{-1})^2\mathrm{tr}(A)\mathrm{tr}(B^{-1})\\
&- \mathrm{tr}(B^{-1}A^{-1}\mathrm{tr}(B)^2\mathrm{tr}(B^{-1})\mathrm{tr}(A) - \mathrm{tr}(AB)\mathrm{tr}(B^{-1})^2\mathrm{tr}(B)\mathrm{tr}(A^{-1})\\
&- \mathrm{tr}(AB^{-1})\mathrm{tr}(B^{-1})\mathrm{tr}(A^{-1})\mathrm{tr}(A)^2 - \mathrm{tr}(A^{-1}B)\mathrm{tr}(B)\mathrm{tr}(A)\mathrm{tr}(A^{-1})^2\\
&- \mathrm{tr}(A^{-1})\mathrm{tr}(B^{-1})^3\mathrm{tr}(A) - \mathrm{tr}(A^{-1})\mathrm{tr}(B)^3\mathrm{tr}(A) - \mathrm{tr}(A^{-1})^3\mathrm{tr}(B^{-1})\mathrm{tr}(B)\\
&- \mathrm{tr}(A)^3\mathrm{tr}(B^{-1})\mathrm{tr}(B) - \mathrm{tr}(AB^{-1})\mathrm{tr}(B^{-1}A^{-1})\mathrm{tr}(B)\mathrm{tr}(A^{-1})\mathrm{tr}(B)\\
&- \mathrm{tr}(A^{-1}B)\mathrm{tr}(AB)\mathrm{tr}(B)\mathrm{tr}(A)\mathrm{tr}(B^{-1}) - \mathrm{tr}(A^{-1})\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(AB^{-1})\mathrm{tr}(AB)\\
&- \mathrm{tr}(A^{-1})\mathrm{tr}(A)\mathrm{tr}(B^{-1})\mathrm{tr}(A^{-1}B)\mathrm{tr}(B^{-1}A^{-1}) + \mathrm{tr}(B^{-1})\mathrm{tr}(A^{-1})^2\mathrm{tr}(A)^2\mathrm{tr}(B)\\
&+ \mathrm{tr}(A^{-1})\mathrm{tr}(B^{-1})^2\mathrm{tr}(B)^2\mathrm{tr}(A).
\end{aligned}
$$

Proposition 11: Suppose that A, B, C are elements of $SU(2,1)$ such that $ABC = I$. Let $a = \mathrm{tr}(A), b = \mathrm{tr}(B), c = \mathrm{tr}(C)$ and $d = \mathrm{tr}(A^{-1}B) - \mathrm{tr}(A^{-1})\mathrm{tr}(B)$. Then the equation in lemma 10 becomes

$$
\begin{aligned}
Q = {} & 9 - 6|a|^2 - 6|b|^2 - 6|c|^2 - 6(\overline{d} + a\overline{b})(d + \overline{a}b) + a^3 + b^3\\
&+ \overline{c}^3 + (\overline{d} + a\overline{b})^3 + \overline{a}^3 + \overline{b}^3 + c^3 + (d + \overline{a}b)^3 - 3(d + \overline{a}b)\overline{a}c\\
&- 3(\overline{d} + a\overline{b})a\overline{c} - 3(d + \overline{a}b)b\overline{c} - 3(\overline{d} + a\overline{b})\overline{b}c + 3(d + \overline{a}b)a\overline{b}\\
&+ 3(\overline{d} + a\overline{b})\overline{a}b + 3abc + 3\overline{a}\,\overline{b}\,\overline{c} + |a|^2|b|^2 + |b|^2|c|^2\\
&+ (d + \overline{a}b)(\overline{d} + a\overline{b})|a|^2 + (d + \overline{a}b)(\overline{d} + a\overline{b})|b|^2 + |a|^2|c|^2\\
&+ (d + \overline{a}b)(\overline{d} + a\overline{b})|c|^2 + (d + \overline{a}b)\overline{b}c + (\overline{d} + a\overline{b})b\overline{c} + \overline{a}^2\overline{b}(d + \overline{a}b)\\
&+ a^2b(\overline{d} + a\overline{b}) + a\overline{b}^2c + \overline{a}b^2\overline{c} + (d + \overline{a}b)a^2c + (\overline{d} + a\overline{b})\overline{a}^2\overline{c}\\
&+ (d + \overline{a}b)bc^2 + (\overline{d} + a\overline{b})\overline{b}\overline{c}^2 + \overline{a}^2bc + a^2\overline{b}\,\overline{c} + (d + \overline{a}b)ab^2
\end{aligned}
$$

$$
\begin{aligned}
&+ (\overline{d} + a\overline{b})b^2c + a\overline{c}(d + \overline{a}b)^2 + (\overline{d} + a\overline{b})\overline{a}\,\overline{b}^2 + (d + \overline{a}b)\overline{b}^2\overline{c} \\
&+ \overline{a}c(\overline{d} + a\overline{b})^2 + \overline{a}\,\overline{c}^2(d + \overline{a}b) + ac^2(\overline{d} + a\overline{b}) - 2\overline{a}\,\overline{b}c^2 \\
&- 2ab\overline{c}^2 - 2\overline{a}b(d + \overline{a}b)^2 - 2a\overline{b}(\overline{d} + a\overline{b})^2 + \overline{a}^2\overline{b}^2c + a^2b^2\overline{c} \\
&+ (d + \overline{a}b)\overline{a}^2b^2 + (\overline{d} + a\overline{b}) - (d + \overline{a}b)a\overline{b}|b|^2 - (\overline{d} + a\overline{b})\overline{a}b|b|^2 \\
&- a|a|^2bc - \overline{a}|a|^2\overline{b}\,\overline{c} - ab|b|^2c - \overline{a}\,\overline{b}|b|^2\overline{c} - (d + \overline{a}b)a|a|^2\overline{b} \\
&- (\overline{d} + a\overline{b})\overline{a}|a|^2b - |a|^2\overline{b}^3 - |a|^2b^3 - \overline{a}^3|b|^2 - a^3|b|^2 \\
&- (d + \overline{a}b)\overline{a}|b|^2c - (\overline{d} + a\overline{b})a|b|^2\overline{c} - |a|^2b(d + \overline{a}b)\overline{c} \\
&- |a|^2\overline{b}(\overline{d} + a\overline{b})c + |a|^4|b|^2 + |a|^2|b|^4.
\end{aligned}
$$

Proposition 12: Let $A, B, C \in SU(2,1)$ with $ABC = I$. Let

$$a = \mathrm{tr(A)},\ \mathrm{b = tr(B),\ c = tr(C),\ d = tr(A^{-1}B) - tr(A^{-1})tr(B)}.$$

Then

$$2\Re(\mathrm{tr[A,B]}) = \mathrm{|a|^2 + |b|^2 + |c|^2 + |d|^2 - abc - \overline{a}\overline{b}\overline{c} - 3}$$

and

$$
\begin{aligned}
&|\mathrm{tr[A,B]}|^2 = \\
&\quad |a|^2|b|^2|c|^2 + a^2b^2\overline{c}^2 + \overline{a}^2\overline{b}^2c + a^2\overline{b}^2c^2 + \overline{a}^2b\overline{c}^2 + \overline{a}b^2c^2 \\
&\quad + a\overline{b}^2\overline{c}^2 + |a|^2|b|^2 + |b|^2|c|^2 + |a|^2|c|^2 - ab\overline{c}^2 - 2\overline{a}\overline{b}c^2 \\
&\quad - 2a\overline{b}^2c - 2\overline{a}b^2\overline{c} - 2\overline{a}^2bc - 2a^2\overline{b}\overline{c} + a^3 + \overline{a}^3 + b^3 + \overline{b}^3 \\
&\quad + c^3 + \overline{c}^3 + 3abc + 3\overline{a}\overline{b}\overline{c} - 6|a|^2 - 6|b|^2 - 6|c|^2 \\
&\quad + d(|a|^2\overline{b}c + \overline{a}b|c|^2 + a|b|^2\overline{c} + \overline{a}\overline{b}^2 + a^2b + \overline{a}^2\overline{c} + ac^2 + \overline{b}\overline{c}^2 + b^2c) \\
&\quad + \overline{d}(|a|^2\overline{b}c + \overline{a}b|c|^2 + a|b|^2\overline{c} + \overline{a}\overline{b}^2 + a^2b + \overline{a}^2\overline{c} + ac^2 + \overline{b}\overline{c}^2 + b^2c) \\
&\quad + (d^2 - 3\overline{d})(\overline{a}b + \overline{b}c + a\overline{c}) + (\overline{d}^2 - 3d)(a\overline{b} + b\overline{c} + \overline{a}c) \\
&\quad + |d|^2(|a|^2 + |b|^2 + |c|^2 - 6) + d^3 + \overline{d}^3 + 9.
\end{aligned}
$$

Proof. Using $\mathrm{tr}(A^{-1}) = \overline{\mathrm{tr(A)}} = \overline{\mathrm{a}}$ etc and also $\mathrm{tr}(A^{-1}B) = d + \overline{a}b$ in the expression of corollary 4 gives the proof of the first part. (see Parker [4] for the details).

For the second part, we simplify the equation given in proposition 11 to have

$$
\begin{aligned}
|\mathrm{tr}[\mathrm{A},\mathrm{B}]|^2 &= 9 - 6|a|^2 - 6|b|^2 - 6|c|^2 - 6(\overline{d} + a\overline{b})(d + \overline{a}b) + a^3 + b^3 \\
&+ \overline{c}^3 + (\overline{d} + a\overline{b})^3 + \overline{a}^3 + \overline{b}^3 + c^3 + (d + \overline{a}b)^3 - 3(d + \overline{a}b)\overline{a}c \\
&- 3(\overline{d} + a\overline{b})a\overline{c} - 3(d + \overline{a}b)b\overline{c} - 3(\overline{d} + a\overline{b})\overline{b}c + 3(d + \overline{a}b)a\overline{b} \\
&+ 3(\overline{d} + a\overline{b})\overline{a}b + 3abc + 3\overline{a}\,\overline{b}\,\overline{c} + |a|^2|b|^2 + |b|^2|c|^2 \\
&+ (d + \overline{a}b)(\overline{d} + a\overline{b})|a|^2 + (d + \overline{a}b)(\overline{d} + a\overline{b})|b|^2 + |a|^2|c|^2 \\
&+ (d + \overline{a}b)(\overline{d} + a\overline{b})|c|^2 + (d + \overline{a}b)\overline{b}c + (\overline{d} + a\overline{b})b\overline{c} + \overline{a}^2\overline{b}(d + \overline{a}b) \\
&+ a^2b(\overline{d} + a\overline{b}) + a\overline{b}^2c + \overline{a}b^2\overline{c} + (d + \overline{a}b)a^2c + (\overline{d} + a\overline{b})\overline{a}^2\overline{c} \\
&+ (d + \overline{a}b)bc^2 + (\overline{d} + a\overline{b})\overline{b}\overline{c}^2 + \overline{a}^2bc + a^2\overline{b}\,\overline{c} + (d + \overline{a}b)ab^2 \\
&+ (\overline{d} + a\overline{b})b^2c + a\overline{c}(d + \overline{a}b)^2 + (\overline{d} + a\overline{b})\overline{a}\,\overline{b}^2 + (d + \overline{a}b)\overline{b}^2\overline{c} \\
&+ \overline{a}c(\overline{d} + a\overline{b})^2 + \overline{a}\,\overline{c}^2(d + \overline{a}b) + ac^2(\overline{d} + a\overline{b}) - 2\overline{a}\,\overline{b}c^2 \\
&- 2ab\overline{c}^2 - 2\overline{a}b(d + \overline{a}b)^2 - 2a\overline{b}(\overline{d} + a\overline{b})^2 + \overline{a}^2\overline{b}^2c + a^2b^2\overline{c} \\
&+ (d + \overline{a}b)\overline{a}^2b^2 + (\overline{d} + a\overline{b}) - (d + \overline{a}b)a\overline{b}|b|^2 - (\overline{d} + a\overline{b})\overline{a}b|b|^2 \\
&- a|a|^2bc - \overline{a}|a|^2\overline{b}\,\overline{c} - ab|b|^2c - \overline{a}\,\overline{b}|b|^2\overline{c} - (d + \overline{a}b)a|a|^2\overline{b} \\
&- (\overline{d} + a\overline{b})\overline{a}|a|^2b - |a|^2\overline{b}^3 - |a|^2b^3 - \overline{a}^3|b|^2 - a^3|b|^2 \\
&- (d + \overline{a}b)\overline{a}|b|^2c - (\overline{d} + a\overline{b})a|b|^2\overline{c} - |a|^2b(d + \overline{a}b)\overline{c} \\
&- |a|^2\overline{b}(\overline{d} + a\overline{b})c + |a|^4|b|^2 + |a|^2|b|^4 \\
\\
&= 9 - 6|a|^2 - 6|b|^2 - 6|c|^2 - 6|d|^2 - 6\overline{a}b\overline{d} - 6|a|^2|b|^2 - 6a\overline{b}d \\
&+ a^3 + b^3 + \overline{c}^3 + d^3 + \overline{a}^3 + \overline{b}^3 + c^3 + \overline{d}^3 + 3a\overline{bd}^2 + 3a^2\overline{b}^2\overline{d} \\
&+ a^3\overline{b}^3 + 3\overline{a}bd^2 + 3\overline{a}^2b^2d + \overline{a}^3b^3 - 3\overline{a}cd - 3\overline{a}^2bc - 3a\overline{c}\,\overline{d} \\
&- 3a^2\overline{b}\,\overline{c} - 3b\overline{c}d - 3\overline{a}b^2\overline{c} - 3\overline{b}c\overline{d} - 3a\overline{b}^2c + 3a\overline{b}d + 3|a|^2|b|^2 \\
&+ 3\overline{a}b\overline{d} + 3|a|^2|b|^2 + 3abc + 3\overline{a}\,\overline{b}\,\overline{c} + |a|^2|b|^2 + |b|^2|c|^2 + |a|^2|d|^2 \\
&+ a|a|^2\overline{b}d + \overline{a}|a|^2b\overline{d} + |a|^4|b|^2 + |b|^2|d|^2 + a\overline{b}|b|^2d + \overline{a}b|b|^2\overline{d} \\
&+ |a|^2|b|^4 + |c|^2|d|^2 + a\overline{b}|c|^2d + \overline{a}b|c|^2\overline{d} + |a|^2|b|^2|c|^2 + \overline{b}cd^2 \\
&+ 2\overline{a}|b|^2\overline{c}d + \overline{a}^2b|b|^2c + b\overline{c}\overline{d}^2 + 2a|b|^2\overline{c}\,\overline{d} + a^2|b|^2\overline{c}\,\overline{c} + \overline{a}^2\overline{b}d \\
&+ \overline{a}^3|b|^2 + a^2b\overline{d} + a^3|b|^2 + a\overline{b}^2c + \overline{a}b^2\overline{c} + a^2cd + a|a|^2bc + \overline{a}^2\overline{c}\,\overline{d} \\
&+ \overline{a}|a|^2\overline{b}\,\overline{c} + bc^2d + \overline{a}b^2c^2 + \overline{b}\,\overline{c}^2\overline{d} + a\overline{b}^2\,\overline{c}^2 + \overline{a}^2bc + a^2\overline{b}\,\overline{c}
\end{aligned}
$$

$$
\begin{aligned}
&+ ab^2d + |a|^2b^3 + \overline{a}\,\overline{b}^2\overline{d} + |a|^2\overline{b}^3 + \overline{b}^2\,\overline{c}d + \overline{a}|b|^2\overline{b}\,\overline{c} + b^2c\overline{d}\\
&+ ab|b|^2c + a\overline{c}d^2 + 2|a|^2b\overline{c}d + \overline{a}|a|^2b^2\overline{c} + \overline{a}c\overline{d}^2 + 2|a|^2\overline{b}c\overline{d}\\
&+ a|a|^2\overline{b}^2c + \overline{a}\,\overline{c}^2d + \overline{a}^2b\overline{c}^2 + ac^2d + a^2\overline{b}c^2 - 2\overline{a}\,\overline{b}c^2 - 2\overline{a}\,\overline{b}\,\overline{c}^2\\
&- 2\overline{a}bd^2 - 4\overline{a}^2b^2d - 2\overline{a}^3b^3 - 2a\overline{b}\,\overline{d}^2 - 4a^2\overline{b}^2\overline{d} - 2a^3\overline{b}^3 + \overline{a}^2\overline{b}^2c\\
&+ a^2b^2\overline{c} + \overline{a}^2b^2d + \overline{a}^3b^3 + a^2b^2\overline{d} + a^3\overline{b}^3 - a\overline{b}|b|^2d - |a|^2|b|^4\\
&- \overline{a}\,\overline{b}|b|^2\overline{d} - |a|^2|b|^4 - a|a|^2bc - \overline{a}|a|^2\overline{b}\,\overline{c} - a\overline{b}|b|^2c - \overline{a}\,\overline{b}|b|^2\overline{c}\\
&- a|a|^2\overline{b}d - |a|^4|b|^2 - \overline{a}|a|^2b\overline{d} - |a|^4|b|^2 - |a|^2\overline{b}^3 - |a|^2b^3\\
&- \overline{a}^3|b|^2 - a^3|b|^2 - \overline{a}|b|^2cd - \overline{a}^2|b|^2bc - a|b|^2\overline{c}d - a^2\overline{b}|b|^2\overline{c}\\
&- |a|^2b\overline{c}d - \overline{a}|a|^2b^2\overline{c} - |a|^2\overline{b}c\overline{d} - a|a|^2\overline{b}^2c + |a|^4|b|^2 + |a|^2|b|^4
\end{aligned}
$$

$$
\begin{aligned}
&= 9 - 6|a|^2 - 6|b|^2 - 6|c|^2 + a^3 + \overline{a}^3 + b^3 + \overline{b}^3 + c^3 + \overline{c}^3 + d^3\\
&+ \overline{d}^3 + |a|^2|b|^2 + |a|^2|c|^2 + |b|^2|c|^2 + |a|^2|b|^2|c|^2 + \overline{a}^2b\overline{c}^2 + a^2\overline{b}c^2\\
&+ |d|^2(|a|^2 + |b|^2 + |c|^2 - 6) + (d^2 - 3\overline{d})(\overline{a}b + \overline{b}c + a\overline{c})\\
&+ (\overline{d}^2 - 3d)(a\overline{b} + b\overline{c} + \overline{a}c) - 2\overline{a}^2bc - 2a^2\overline{b}\,\overline{c} - 2\overline{a}b^2\overline{c} - 2a\overline{b}^2c\\
&- 2\overline{a}\,\overline{b}c^2 - 2ab\overline{c}^2 + 3abc + 3\overline{a}\,\overline{b}\,\overline{c} + \overline{a}b^2\overline{c} + \overline{a}b^2c^2 + a\overline{b}^2\overline{c}^2 + a^2b^2c\\
&+ d(|a|^2\overline{b}c + \overline{a}b|c|^2 + a|b|^2\overline{c} + \overline{a}\overline{b}^2 + a^2b + \overline{a}^2\overline{c} + ac^2 + \overline{b}\overline{c}^2 + b^2c)\\
&+ \overline{d}(|a|^2\overline{b}c + \overline{a}b|c|^2 + a|b|^2\overline{c} + \overline{a}\overline{b}^2 + a^2b + \overline{a}^2\overline{c} + ac^2 + \overline{b}\overline{c}^2 + b^2c).
\end{aligned}
$$

We remark that when we write the formula of the real and modulos of $\mathrm{tr}[A,B]$ in terms of traces of A, B, AB and $A^{-1}B$ (see lemma 10) then there is a set symmetries generated by $(A,B) \to (B,A)$ and $(A,B) \to (A^{-1},B)$ etc. Some of these send [A, B] to itself, others to its inverse. Thus there are two solutions to the quadratic. However, when we write in terms of a,b,c,d (as in proposition 11) there is a three fold cyclic symmetry $a \to b \to c \to a$.

Now when we put the real and modulos of $\mathrm{tr}[A,B]$ together we have

Proposition 13: Let A, B, C elements of $SU(2,1)$ with $ABC = I$. Then if $\langle A, B, C\rangle$ is Zariski dense, it is determined up to conjugacy by

$$\mathrm{tr(A)},\ \mathrm{tr(B)},\ \mathrm{tr(C)},\ \mathrm{tr(A^{-1}B)} - \mathrm{tr(A^{-1})tr(B)},\ \mathrm{tr[A,B]}.$$

Remark: In proposition 13 Parker[4] attempts to parametrise pair of pants groups via traces. As seen in the discussion in [4], since $SU(2,1)$ has dimension four one cannot determine $\langle A, B\rangle$ up to conjugation. One would expect to only need to use four traces to describe $\langle A, B\rangle$. Ideally, one needs an extra one, tr[A, B] but the real part and absolute value of $\mathrm{tr}[A, B]$ are determined by other parameters. So Parker[4] considered a group with three generators $\langle A, B, C\rangle$ whose product is the identity instead of $\langle A, B\rangle$. The reason for doing that is to get a formulae with three fold symmetry.

3.4 Cross-ratios

In [4] Parker outlined the construction made by Parker and Platis [2] on cross-ratio and related them on the trace coordinates of the previous section. We also comment on how to parametrise pair of pants via trace and cross-ratio.

Theorem 14: Suppose that A and B are loxodromic elements of $SU(2,1)$ with distinct fixed points. Also, suppose that $\langle A, B\rangle$ does not preserve a complex line. Then the group $\langle A, B\rangle$ is determined up to conjugation in $SU(2,1)$ by: $\mathrm{tr}(A)$, $\mathrm{tr}(B), \mathbb{X}_1(A, B), \mathbb{X}_2(A, B)$ and $\mathbb{X}_3(A, B)$.

Proposition 15: Let A, B and C be loxodromic elements of $SU(2,1)$ with $ABC = I$. Then $\mathrm{tr}(C)$, $\mathbb{X}_1(A, C), \mathbb{X}_2(A, C)$ and $\mathbb{X}_3(A, C)$ may be expressed as real analytic functions of $\mathrm{tr}(A)$, $\mathrm{tr}(B), \mathbb{X}_1(A, B), \mathbb{X}_2(A, B)$ and $\mathbb{X}_3(A, B)$.

Remark: In theorem 14 Parker[4] again tries to parametrise pair of pants group by using traces of two elements and cross-ratios. Even with this, there is a problem of a sign. This time it is the sign of the imaginary part of $\mathbb{X}_3$. Furthermore, this ambiguity is the same as the ambiguity in the sign of the $\Im(\mathrm{tr}[A, B])$ (found in above remark). Now from proposition 4.15 in Parker [4], we can express $\mathbb{X}_1$ and $\mathbb{X}_2$ in terms of $\lambda, \mu, \mathrm{tr}(\mathrm{AB})$ and $\mathrm{tr}(A^{-1}B)$ which give the trace coordinates found in the previous section. So combining trace and cross-ratio we can parametrise pair of pants by considering the group $\langle A, B\rangle$. The merit of this method is that, we can still determine conjugation in $SU(2,1)$ with only two elements $A, B \in \mathrm{SU}(2,1)$.

4 Traces for Triangle Groups

4.1 Traces in general triangle groups

The material in this section is completely standard. In order for readers to understand them, they are advised to read sections 5.2, 5.3, 5,4 and 5.5 in Parker [4] very carefully.

In [6] Pratoussevitch found a formula for the trace of each element of $\Delta = \langle R_1, R_2, R_3 \rangle$, written as a word in R_1, R_2, R_3 and their inverses. In this section we use proposition 16 which is due to Pratoussevitch [6] to compute traces of groups that are generated by complex reflections. Note that R_1, R_2 and R_3 are matrices as in (5.14), (5.15) and (5.16) respectively Parker[4].

Proposition 16: Let $a = (a_1 ... a_r)$ be a cyclic word with $a_k \in \{1, 2, 3\}$. Let $\epsilon = (\epsilon_1 ... \epsilon_r)$ with $\epsilon_k = \{1, -1\}$. Let $E = \sum_{j=1}^{n} \epsilon_j$. Then

$$\operatorname{tr}(R_{a_1}^{\epsilon_1} \cdots R_{a_r}^{\epsilon_r}) = (e^{i\psi})^{-E/3} \left(3 + \sum_{S} \frac{(e^{i\psi} - 1)^z (-e^{i\psi})^n (e^{i\psi})^w}{(-e^{i\psi})^{m_-}} \rho^{p_1} \overline{\rho}^{n_1} \sigma^{p_2} \overline{\sigma}^{n_2} \tau^{p_3} \overline{\tau}^{n_3} \right)$$

where the sum is taken overall non-empty subsets of $S = \{k_1, ..., k_m\}$ of the set $\{1, ..., r\}$. Such a subset determines a subset $a_s = (a_{k_1}, ..., a_{k_m})$ of a and $\epsilon_s = (\epsilon_{k_1}, ..., \epsilon_{k_m})$ of ϵ. The numbers $p_j, n_j, w = p_j - n_j, z = z_1 + z_2 + z_3, n = n_1 + n_2 + n_3$ are determined from a_s. Finally, m_- is determined from ϵ_s.

Corollary 17: The trace of any element of Δ may be written as a power of $e^{i\psi/3}$ times as polynomial in $|\rho|^2, |\sigma|^2, |\tau|^2, \rho\sigma\tau$ and $\overline{\rho}\,\overline{\sigma}\,\overline{\tau}$ with coefficients in $\mathbb{Z}[e^{i\psi}, e^{-i\psi}]$. In particular, when ψ is a rotational multiple of π then the coefficient may be written in $\mathbb{Z}[e^{i\psi}]$.

We give an illustrative example of proposition 16.

Proposition 18: Let R_1, R_2 and R_3 be as above. Then for any distinct $j, k, l = \{1, 2, 3\}$ we have

$$\operatorname{tr}(R_1^{-1} R_2^{-1} R_3^{-1}) = 3 - (|\rho|^2 + |\tau|^2 + |\sigma|^2) - e^{-i\psi} \rho\tau\sigma,$$

$$\operatorname{tr}[R_1, R_2] = 3 + 2(\cos(\psi) - 1)|\rho|^2 + |\rho|^4,$$

$$\operatorname{tr}(R_1 R_2 R_3^{-1} R_2^{-1}) = 1 + \cos(\psi)(2 + |\sigma|^2) + |\rho\sigma - \overline{\tau}|^2.$$

a_s	ϵ_s	m_-	z	p_1	n_1	p_2	n_2	p_3	n_3	w	term
$\{1\}$	$\{-\}$	1	1	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)$
$\{2\}$	$\{-\}$	1	1	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)$
$\{3\}$	$\{-\}$	1	1	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)$
$\{1, 2\}$	$\{-,-\}$	2	0	1	1	0	0	0	0	0	$-e^{-i\psi}\|\rho\|^2$
$\{1, 3\}$	$\{-,-\}$	2	0	0	0	0	0	1	1	0	$-e^{-i\psi}\|\tau\|^2$
$\{2, 3\}$	$\{-,-\}$	2	0	0	0	1	1	0	0	0	$-e^{-i\psi}\|\sigma\|^2$
$\{1, 2, 3\}$	$\{-,-,-\}$	3	0	1	0	1	0	1	0	1	$-e^{-2i\psi}\rho\sigma\tau$

Proof. We first consider $R_1^{-1}R_2^{-1}R_3^{-1}$. We now enumerate all non-empty subsets, their index and winding number, and the contribution they make to the trace. For $R_1^{-1}R_2^{-1}R_3^{-1}$ the terms are given by the following table above:

Therefore

$$\begin{aligned}\mathrm{tr}(R_1^{-1}R_2^{-1}R_3^{-1}) &= e^{i\psi}[3 - e^{-i\psi}(e^{i\psi}-1) - e^{-i\psi}(e^{i\psi}-1) - e^{-i\psi}(e^{i\psi}-1)\\ &\quad - e^{-i\psi}|\rho|^2 - e^{-i\psi}|\tau|^2 - e^{-i\psi}|\sigma|^2 - e^{-2i\psi}\rho\sigma\tau]\\ &= 3e^{i\psi} - 3(e^{i\psi}-1) - (|\rho|^2+|\tau|^2+|\sigma|^2) - e^{-i\psi}\rho\sigma\tau\\ &= 3 - (|\rho|^2+|\tau|^2+|\sigma|^2) - e^{-i\psi}\rho\tau\sigma\end{aligned}$$

.

First note that $[R_1, R_2] = R_1R_2R_1^{-1}R_2^{-1}$. For $R_1R_2R_1^{-1}R_2^{-1}$ the terms are given by the following table:

Thus

$$\begin{aligned}\mathrm{tr}[R_1,R_2] &= 3 + e^{i\psi} - 1 + e^{i\psi} - 1 - e^{-i\psi}(e^{i\psi}-1) - e^{-\psi}(e^{i\psi}-1) - e^{i\psi}|\rho|^2\\ &\quad - e^{-i\psi}(e^{i\psi}-1)^2 + |\rho|^2 + |\rho|^2 - e^{-i\psi}(e^{i\psi}-1)^2 - e^{i\psi}|\rho|^2 + (e^{i\psi}-1)|\rho|^2\\ &\quad + (e^{i\psi}-1)|\rho|^2 - e^{-i\psi}(e^{i\psi}-1)|\rho|^2 - e^{-i\psi}(e^{i\psi}-1)|\rho|^2 + |\rho|^4\\ &= 1 + 2e^{i\psi} - 2e^{-i\psi}(e^{i\psi}-1)[1+(e^{i\psi}-1)]\\ &\quad + |\rho|^2[-2e^{i\psi} + 2 + 2(e^{i\psi}-1) - 2e^{-i\psi}(e^{i\psi}-1)] + |\rho|^4\\ &= 1 + 2e^{i\psi} - 2e^{i\psi} + 2 + |\rho|^2(2e^{\psi}-2) + |\rho|^4\\ &= 3 + 2(\cos(\psi)-1)|\rho|^2 + |\rho|^4.\end{aligned}$$

Similarly, we do the same thing for $R_1R_2R_3^{-1}R_2^{-1}$.

a_s	ϵ_s	m_-	z	p_1	n_1	p_2	n_2	p_3	n_3	w	term
$\{1\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{2\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{1\}$	$\{-\}$	1	1	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)$
$\{2\}$	$\{-\}$	1	1	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)$
$\{1,\, 2\}$	$\{+,+\}$	0	0	1	1	0	0	0	0	0	$-e^{i\psi}\|\rho\|^2$
$\{1,\, 1\}$	$\{+,-\}$	1	2	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)^2$
$\{1,\, 2\}$	$\{+,-\}$	1	0	1	1	0	0	0	0	0	$\|\rho\|^2$
$\{2,\, 1\}$	$\{+,-\}$	1	0	1	1	0	0	0	0	0	$\|\rho\|^2$
$\{2,\, 2\}$	$\{+,-\}$	1	2	1	1	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)^2$
$\{1,\, 2\}$	$\{-,-\}$	2	0	1	1	0	0	0	0	0	$-e^{i\psi}\|\rho\|^2$
$\{1,\, 2,\, 1\}$	$\{+,+,-\}$	1	1	1	1	0	0	0	0	0	$(e^{i\psi}-1)\|\rho\|^2$
$\{1,\, 2,\, 2\}$	$\{+,+,-\}$	1	1	1	1	0	0	0	0	0	$(e^{i\psi}-1)\|\rho\|^2$
$\{1,\, 1,\, 2\}$	$\{+,-,-\}$	2	1	1	1	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)\|\rho\|^2$
$\{2,\, 1,\, 2\}$	$\{+,-,-\}$	2	1	1	1	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)\|\rho\|^2$
$\{1,\, 2,\, 1,\, 2\}$	$\{+,+,-,-\}$	2	0	2	2	0	0	0	0	0	$\|\rho\|^4$

a_s	ϵ_s	m_-	z	p_1	n_1	p_2	n_2	p_3	n_3	w	term
$\{1\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{2\}$	$\{+\}$	0	1	0	0	0	0	0	0	0	$e^{i\psi}-1$
$\{3\}$	$\{-\}$	1	1	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)$
$\{2\}$	$\{-\}$	1	1	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)$
$\{1,\, 2\}$	$\{+,+\}$	0	0	1	1	0	0	0	0	0	$-e^{i\psi}\|\rho\|^2$
$\{1,\, 3\}$	$\{+,-\}$	1	0	0	0	0	0	1	1	0	$\|\tau\|^2$
$\{1,\, 2\}$	$\{+,-\}$	1	0	1	1	0	0	0	0	0	$\|\rho\|^2$
$\{2,\, 3\}$	$\{+,-\}$	1	0	0	0	1	1	0	0	0	$\|\sigma\|^2$
$\{2,\, 2\}$	$\{+,-\}$	1	2	1	1	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)^2$
$\{3,\, 2\}$	$\{-,-\}$	2	0	0	0	1	1	0	0	0	$-e^{i\psi}\|\sigma\|^2$
$\{1,\, 2,\, 3\}$	$\{+,+,-\}$	1	1	1	1	0	0	0	0	0	$-\rho\sigma\tau$
$\{1,\, 2,\, 2\}$	$\{+,+,-\}$	1	0	1	0	1	0	1	0	1	$(e^{i\psi}-1)\|\rho\|^2$
$\{1,\, 3,\, 2\}$	$\{+,-,-\}$	2	0	0	1	0	1	0	1	-1	$-\overline{\rho}\;\overline{\sigma}\;\overline{\tau}$
$\{2,\, 3,\, 2\}$	$\{+,-,-\}$	2	1	0	0	1	1	0	0	0	$-e^{-i\psi}(e^{i\psi}-1)\|\sigma\|^2$
$\{1,\, 2,\, 3,\, 2\}$	$\{+,+,-,-\}$	2	0	1	1	1	1	0	0	0	$\|\rho\|^2\|\sigma\|^2$

Hence

$$\begin{aligned}\operatorname{tr}(R_1R_2R_3^{-1}R_2^{-1}) &= 3 + e^{i\psi} - 1 + e^{i\psi} - 1 - e^{-i\psi}(e^{i\psi} - 1) - e^{-i\psi}(e^{i\psi} - 1) \\ &- e^{i\psi}|\rho|^2 + |\tau|^2 + |\rho|^2 + |\sigma|^2 - e^{-i\psi}(e^{i\psi} - 1)^2 - e^{i\psi}|\sigma|^2 \\ &- \rho\sigma\tau + (e^{i\psi} - 1)|\rho|^2 - \overline{\rho}\,\overline{\sigma}\,\overline{\tau} \\ &= 1 + 2e^{i\psi} - 1 + e^{-i\psi} - e^{i\psi} + 1 - |\sigma|^2 + e^{-i\psi}|\sigma|^2 + |\tau|^2 \\ &+ |\sigma|^2 - \rho\sigma\tau - \overline{\rho}\,\overline{\sigma}\,\overline{\tau} + |\rho|^2|\sigma|^2 \\ &= 1 + e^{i\psi} + e^{-i\psi}(1 + |\sigma|^2) + |\sigma|^2|\rho|^2 - \rho\sigma\tau \\ &- \overline{\rho}\,\overline{\sigma}\,\overline{\tau} + |\rho|^2|\sigma|^2 + |\tau|^2 \\ &= 1 + \cos(\psi)(2 + |\sigma|^2) + |\rho\sigma - \overline{\tau}|^2.\end{aligned}$$

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