**Solving Second Order Delay Differential Equations directly by a Four-step**

**Multi-Hybrid Block Method**

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**Abstract**

The aim of this paper is to compute the numerical solution of special second order delay differential equations directly by a four-step multi-hybrid block method. The methods were generated using collocation and interpolation approach by means of a combination of power series and exponential function at some selected grid and off-grid points. The developed schemes and its first derivatives was combined to form block methods to concurrently solve special second order delay differential equations directly without reducing it to the system of first order. The basic properties of the methods such as order, error constants, consistency and convergence were examined. The developed methods were applied to solve some second order delay differential equations, the methods also solves application problem in other to test for the efficiency and accuracy of the methods. The results are displays in the tables.

**Keywords:** Multi-Hybrid Block Method, Second Order Delay Differential Equations, Four-step, Convergence

**1.0 Introduction**

Differential equations in the company of a time delay are used to model a phenomenon which does not only depend on the current state of a system but also the earlier states. This category of equation is called delay differential equations (DDEs) [1]. It is differential equations in which the derivative at any time depends on the solution at earlier times. The special second order DDE can be written in the form

 (1)

Where is the initial function and  is the delay term. Most of the methods for solving special second order ODEs can be adopted for solving special second order delay differential equation. There are two different ways to calculate the delay term in the developed method. See [2-4] .

DDEs have become significant tools to explore the complexities of the real-world problems concerning infectious diseases, biotic population, neuronal networks, and population dynamics.

Methods such as Runge-Kutta (RK), Runge-Kutta Nystrom (RKN), hybrid techniques, and multistep are widely used for solving DDEs. Adegboyega [5] derived a class of Numerical Integrators of Adams Moulton type for solution of Delay Differential equations. Ismail et al. [6] proposed a RK method and Hermite interpolation to solve first-order DDEs. Machee et al. [7] proposed a Runge-kutta-Nystrom method for solving special second order delay differential equations. Taiwo and Odetunde [8], and Evans and Raslan [9] presented a decomposition method as an integrator for the solution of delay differential equations. Several authors also derived block linear multistep method (LMM) to solve DDEs; and such work can be seen in [10-12]. San et al. [10] developed a coupled block method for solving delay differential equations. Zanariah and Hoo [11] examined the numerical solution of DDEs by the block method. Hoo et al. [12] constructed Adams-Moulton Method for directly solving second-order DDEs. Mechee et al. [8] in their paper has adapted RKN for directly solving second-order DDEs. Suleiman and Ishak [13] investigated the numerical solution and the stability of a multistep method for solving DDEs. The work of Familua et. al [14], A class of numerical integrators of order 13 for solving special second order delay differential equations was presented, the analysis of the method was examined. It was found to be consistence, convergent and zero stable. The numerical results show that the method is more accurate than the method compared with in the literature.

Conversely, all the studies formerly mentioned method are having one limitation or the other, such as instability in nature, low order of accuracy and poor accuracy of the methods. Hence, we are motivated to derive a linear multistep with nine hybrid points through the combination of power series and exponential function as a basic function applied for solving DDEs. The method is implemented in block hybrid method because it is a faster numerical method to obtain the approximate solution at more than one point per step. Finally, we tested the new methods using DDEs test problems to indicate that it is better-quality and more efficient for solving special second order DDEs, the method was also applied to solve application problem form engineering namely, Mathieu equation.

**2.0 Derivation of Four-step Multi-Hybrid Method**

This work considers an approximate solution that combines power series and exponential function of the form;

 (2)

The second derivatives of (2) is given as,

 (3)

Here, the interval of integration is taken by partitioning the Four-step length  into eight, that is, have twelve sub-steps. Collocating (3) atand interpolating (2) at to yield a system of equations of the form:

AX=U (4)

Where and

****Solving (4) with the Aid of Maple 18 Mathematical Software for the values of  to obtain values for the parameters:  and substituting the values of the parameters into equation (2) and simplifying the result, to obtain a continuous scheme of the form:

**** (5)

Setting 

The coefficients of  and  are:

, 



































Evaluating (5) at and other non-interpolating points to obtain the discrete schemes

 (6)

 (7)

 (8)

 (9)

 (10)

 (11)

 (12)

 (13)

 (14)

 (15)

 (16)

 (17)

 (18)

 (19)

 (20)

Evaluating the first derivatives of (5) at  we obtained the discrete schemes



































Equations (6) – Equations (37) are combined together as an Integrators to solve second order delay differential equations directly.

**3.0 Basic Properties of the Block Methods**

**3.1 Order and Error Constant of the Block Method**

Let the linear Operator defined on the method be  where



Expanding the form and in Taylor Series and comparing coefficients of h,

we obtained



**Theorem 1:** The linear operator and the associated block method are said to be of order p if  is called the error constant. It implies that the local truncation error is given by

 (40)

Expanding the block method (38) in Taylor Series expansion and comparing the coefficients of h, the order of the block is of order  with error constant

(41)

**3.2 Consistency**

Here, the developed method has been examined and found to have order p greater than one and it is also convergence Areo and Omole [15], Fatunla [14]. Hence, the method satisfies the necessary and sufficient conditions for consistency of a numerical method.

**3.3 Stability Domain of the Block Methods**

In line with the approach in Familua et al. [14] , the figure below shows the stability domain of the Four –step twelve off step points block method.

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**Figure 1:** The Stability domain of Four-step twelve off-step points

**4.0 Implementation and Numerical Examples**

The methods was adopted on some delay differential equation of special second order to access the accuracy and efficiency of the methods.

**Problem 1**: Consider the linear delay equation



Exact solution: 

**Problem 2**: Consider the non-linear delay equation with single delay term



Exact solution: 

**Problem 3:** Application Electrical Engineering problem namely, Matheiu’s Equation.

In this section we apply our developed method to solve a well-known equation in engineering, the Matheiu’s equation, which is defined as follows:

 (42)

Source: Morisson and Rand [16]

which is a nonlinear delay differential equation. where δ, a, b, c and T are parameters. δ is the frequency squared of the simple harmonic oscillator, and a is the amplitude of the parametric resonance, and b is the amplitude of delay which c is the amplitude of the cubic nonlinearity and T is the time delay. Equation (42) is a model for high speed milling, a kind of parametrically interrupted cutting as opposed to the self-interrupted cutting arising in an unstable turning process.

According to Morisson and Rand [16], various special cases of (42) have been studied, depending on which parameters is zero. when δ = a = b = 1 and c = 0 we obtained the following Linear Matheiu’s equation:

 (43)

where T = τ = h/10 is the delay term, the exact solution does not exist.

when δ = a = b = c = 1 we obtained the following Nonlinear Matheiu’s equation:

 (44)

where T = τ = h/10 is the delay term, the exact solution does not exist. Both the linear and nonlinear Mathieu equations are solved using the developed method and the results are presented in Table 3.

**Table 1: Showing results for Problem 1 using the proposed method**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **x** | **y-exact** | **y-computed** | **Error in new method** | **Error in [14]** |
| **0.1** | 0.095310179804324935 | 0.095310179804324852 | 8.32667268e-017 | 2.49800181e-16 |
| **0.2** | 0.182321556793954590 | 0.182321556793954650 | 5.55111512e-017 | 3.60822483e-16 |
| **0.3** | 0.262364264467491060 | 0.262364264467491120 | 5.55111512e-017 | 5.55111512e-16 |
| **0.4** | 0.336472236621212840 | 0.336472236621213010 | 0.00000000e+000 | 5.55111512e-16 |
| **0.5** | 0.405465108108164380 | 0.405465108108164500 | 0.00000000e+000 | 6.10622664e-16 |
| **0.6** | 0.470003629245735580 | 0.470003629245735630 | 1.11022302e-016 | 6.66133815e-16 |
| **0.7** | 0.530628251062170490 | 0.530628251062170490 | 1.11022302e-016 | 7.77156117e-16 |
| **0.8** | 0.587786664902119060 | 0.587786664902119280 | 0.00000000e+000 | 6.66133815e-16 |
| **0.9** | 0.641853886172394810 | 0.641853886172395030 | 0.00000000e+000 | 6.66133815e-16 |
| **1.0** | 0.693147180559945290 | 0.693147180559945620 | 1.11022302e-016 | 5.55111512e-16 |

**Table 2: Showing results for Problem 2 using the proposed method (Non-Linear with single delay)**

|  |  |  |  |
| --- | --- | --- | --- |
| **X** | **y-exact** | **y-computed** | **Error in new method** |
| **0.1** | 0.904837418035959520 | 0.904837418035959740 | 2.22044605e-016 |
| **0.2** | 0.818730753077981820 | 0.818730753077981040 | 7.77156117e-016 |
| **0.3** | 0.740818220681717770 | 0.740818220681717100 | 6.66133815e-016 |
| **0.4** | 0.670320046035639330 | 0.670320046035642440 | 3.10862447e-015 |
| **0.5** | 0.606530659712633420 | 0.606530659712638750 | 5.32907052e-015 |
| **0.6** | 0.548811636094026390 | 0.548811636094029940 | 3.44169138e-015 |
| **0.7** | 0.496585303791409470 | 0.496585303791411690 | 2.16493490e-015 |
| **0.8** | 0.449328964117221560 | 0.449328964117222230 | 6.66133815e-015 |
| **0.9** | 0.406569659740599050 | 0.406569659740599720 | 6.10622664e-015 |
| **1.0** | 0.367879441171442220 | 0.367879441171441780 | 5.55111512e-015 |

**Table 3: Showing results of application problem for Linear and Non linear using h=0.1**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **t** | **y-computed**  **(Linear problem)** | **Time** | **y-computed**  **(Nonlinear problem)** | **Time** |
| **0.1** | 0.066915931967836559 | 0.1249 | 0.106402490682701220 | 0.3252 |
| **0.2** | 0.119736074898535010 | 0.1538 | 0.213539379413318100 | 0.3790 |
| **0.3** | 0.159083017071007220 | 0.1806 | 0.321194476666256670 | 0.4668 |
| **0.4** | 0.179398173160589980 | 0.2056 | 0.430884668392530420 | 0.5242 |
| **0.5** | 0.197257104598293180 | 0.2099 | 0.560806687140919410 | 0.5265 |
| **0.6** | 0.194783177415669940 | 0.2119 | 0.696368628862776170 | 0.5340 |
| **0.7** | 0.172336315345482910 | 0.2126 | 0.836901935996008040 | 0.5349 |
| **0.8** | 0.126430711712440080 | 0.2134 | 0.987798359457115360 | 0.5358 |
| **0.9** | 0.077043234605272862 | 0.2143 | 1.154259182652009400 | 0.5374 |
| **1.0** | 0.004808573520858618 | 0.2150 | 1.335531260425935500 | 0.5462 |



**Figure 2**: showing the comparison of linear and non linear Mathieu’s equation with h=0.1

**4.1 Discussion of Results**

The developed method of equations (6) - (37) were simultaneously implemented on three test problems. Examples 1 and 2 performed accurately with the exact solutions and also compete favourably with other author in the literature namely Familua et al. [14] as displayed in Table 1 and Table 2. In example 3 which is an application problem. The application problem is linear and non linear problem in nature as the parameter changes. The Results are shown in Table 3. Figure 1 shows the region in which the method is absolutely stable. Figure 2 shown comparisons of the linear and non linear application problems. The graph shows that the method performs better on linear problem than the non linear.

**4.2 Conclusion**

We want to conclude that this paper demonstrated a successful implementation of four-step twelve off-grid points for the solution of special second order DDEs. The method has great basic properties. The results were obtained in block forms which speed up the computational processes, less burden in the implementations and also increase the rate of convergence of the solutions. The results are displayed in Table 1-3. Hence the method is recommended for solving second order DDEs.

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