Fixed Point Theorem with Contractive Condition in a M-Complete Fuzzy Metric Spaces

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Abstract

In this paper, we prove a common fixed point theorem with contactive condition in M - complete fuzzy metric spaces and we give an application of an integral type.

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1 Introduction

In 1965, the concept of fuzzy sets was introduced by Zadeh [17]. Since then many authors have expansively developed the theory of fuzzy sets and applications. In 1975, Kramosil and Michalek [9] first introduced the concept of a fuzzy metric space, which can be regarded as a generalization of the statistical (probabilistic) metric space and it provides an important basis for the construction of fixed point theory in fuzzy metric spaces. After that, Wenzhi [16] and many others initiated the study of probabilistic 2-metric spaces which is a real valued function of a point triples on a set X, whose abstract properties were suggested by the area function in Euclidean spaces. Afterwards, Grabiec [7] defined the completeness of the fuzzy metric space or what is known as a G-complete fuzzy metric space in [8], and extended the Banach contraction theorem to G-complete fuzzy metric spaces. Following Grabiec's work, Fang [3] further established some new fixed point theorems for contractive type mappings in G-complete fuzzy metric spaces. Soon after, Mishra et al. [10] also obtained several common fixed point theorems for asymptotically commuting maps in the same space, which generalize several fixed point theorems in metric, fuzzy, Menger and uniform spaces. Besides these works based on the G-complete fuzzy metric space, George and Veeramani [5] modified the definition of the Cauchy sequence introduced by Grabiec [7] because even R is not complete with Grabiec's completeness definition. George and Veeramani [5] slightly modified the notion of a fuzzy metric space introduced by Kramosil and Michalek [9] and then defined a Hausdorff and first countable topology on this fuzzy metric space which has important applications in quantum particle physics in connection with string and E-infinity theory. Since then, the notion of a complete fuzzy metric space presented by George and Veeramani [5], which is now known as an M-complete fuzzy metric space (as in [15]) has emerged as another characterization of completeness, and some fixed point theorems have also been constructed on the basis of this metric space.

Recently, Fang [4] gave some common fixed point theorems under 'φ - contractions' for compatible and weakly compatible mappings in Menger probabilistic metric spaces. Moreover, Rao et al. [11] have proved two unique common coupled fixed point theorems for self maps in symmetric G-fuzzy metric spaces. Recently, Shen et al. [14] have proposed a new class of self-maps by altering the distance between two points in fuzzy environment, in which the φ-function was used, and on the basis of this kind of self-map, they have proved some fixed point theorems in M-complete fuzzy metric spaces and compact fuzzy metric spaces.

From the above analysis, we can see that there are many studies related to fixed point theory based on the above two kinds of complete fuzzy metric spaces, namely: G-complete and M-complete fuzzy metric spaces. Note that every G-complete fuzzy metric space is M-complete; and the construction of fixed point theorems in M-complete fuzzy metric spaces seems to be more valuable.

The purpose of this work is to propose a new class of self-maps, by using a φ- function. More importantly, we prove the existence and uniqueness of a common fixed point for these self-maps in M-complete fuzzy metric spaces. For more details about results of fixed point in fuzzy metric spaces, the reader can see for instance [18-26].

2 Preliminaries

We begin with some basic concepts on fuzzy metric spaces.

Definition 1 ([12], [5]). *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is called a continuous t-norm if it satisfies the following conditions:*
$(TN-1)$ $*$ *is commutative and associative*
$(TN-2)$ $*$ *is continuous;*
$(TN-3)$ $a * 1 = a$ *for every* $a \in [0,1]$;
$(TN-4)$ $a * b \leq c * d$ *whenever* $a \leq c$, $b \leq d$ *and* $a, b, c, d \in [0,1]$.

Definition 2 *([5]). A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a nonempty set, $*$ is a continuous $t-$norm and M is a fuzzy set on $X \times X \times (0, \infty)$, satisfying the following conditions, for all $x, y, z \in X, s, t > 0$:*
$(FM-1)$ $M(x, y, t) > 0$;
$(FM-2)$ $M(x, y, t) = 1$ *if and only if* $x = y$;
$(FM-3)$ $M(x, y, t) = M(y, x, t)$;
$(FM-4)$ $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$;
$(FM-5)$ $M(x, y, \cdot) : (0, \infty) \to (0, 1]$ *is continuous.*

Note that $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t.

Definition 3 *Let $(X, M, *)$ be a fuzzy metric space. Then:*

1. *A sequence $\{x_n\}$ in X is said to be convergent* ([5], [7]) *to a point x in X, denoted by $\lim_{n\to\infty} x_n = x$ (or $x_n \to x$), if and only if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$, i.e. for each $r \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - r$ for all $n \geq n_0$.*

2. *A sequence $\{x_n\}$ in X is called a Cauchy sequence* [7] *if and only if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for all $t > 0$ and $p > 0$.*

3. *A sequence $\{x_n\}$ in X is called an $M-$Cauchy sequence* ([5], [8]) *if and only if for each $\varepsilon \in (0, 1)$, $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_m, x_n, t) > 1 - \varepsilon$ for any $m, n > n_0$.*

4. *The fuzzy metric space $(X, M, *)$ is called complete* ([5], [7]) *if every Cauchy sequence is convergent.*

5. *The fuzzy metric space $(X, M, *)$ is called M-complete* ([5], [8]) *if every M-Cauchy sequence is convergent.*

Lemma 4 ([7]). *For all $x, y \in X$, $M(x, y, \cdot)$ is nondecreasing.*

Remark 5 *Since $*$ is continuous, it follows from $(FM-4)$ that the limit of a sequence in a fuzzy metric space is uniquely determined.*

Definition 6 ([13]). *A function M is continuous in fuzzy metric spaces if, whenever $x_n \to x$, $y_n \to y$, then $\lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t)$ for all $t > 0$.*

Lemma 7 ([6]). *Let $M(x, y,)$ be a fuzzy metric space. Then M is a continuous function on $X \times X \times (0, 1)$.*

3 Main Result

Theorem 8 *Let $(X, M, *)$ be an $M-$complete fuzzy metric space and $\varphi : [0, 1] \to [0, 1]$ be a strictly decreasing and left continuous mapping. Furthermore, let f and g be maps that satisfy the following condition:*

(i) $g(X) \subseteq f(X)$,
(ii) f *is continuous,*

$$\varphi\left(M(gx,gy,t)\right) \leq \varphi(\alpha M(fx,fy,t) + \beta M(gy,fy,t)) \tag{1}$$

where $x, y \in X$, $\beta \in \mathbb{R}_+$ *and* $\beta < 1 \leq \alpha$ *and* $t > 0$*, then* f *and* g *have a unique fixed point provided* f *and* g *commute.*

Proof. Let x_0 be a point in X. By hypothesis (i), we can find x_1 in X such that $fx_1 = g\ x_0$, by induction, we can define a sequence $\{x_n\}$ in X such that $fx_n = g\ x_{n-1}$. By induction again and by (1) we have

$$\left.\begin{array}{rcl} \varphi\left(M\left(fx_n, fx_{n+1}, t\right)\right) & = & \varphi\left(M\left(gx_{n-1}, gx_n, t\right)\right) \\ & \leq & \varphi(\alpha M\left(fx_{n-1}, fx_n, t\right) + \beta M(fx_n, fx_{n+1}, t) \end{array}\right\} \tag{2}$$

Since φ is strictly decreasing, then

$$M\left(fx_n, fx_{n+1}, t\right) > \alpha M\left(fx_{n-1}, fx_n, t\right) + \beta M(fx_n, fx_{n+1}, t)$$

which implies that

$$\frac{1-\beta}{\alpha} M\left(fx_n, fx_{n+1}, t\right) > M\left(fx_{n-1}, fx_n, t\right)$$

Since $0 \leq \beta < 1 < \alpha$, then, we have

$$M\left(fx_{n-1}, fx_n, t\right) < \frac{1-\beta}{\alpha} M\left(fx_n, fx_{n+1}, t\right) < M\left(fx_n, fx_{n+1}, t\right) \tag{3}$$

Setting $\gamma_n\left(t\right) = M\left(fx_n, fx_{n+1}, t\right)$. From (3), the sequence $\{\gamma_n\left(t\right)\}$ is strictly increasing and bounded, then $\gamma_n\left(t\right)$ converges to $\gamma\left(t\right)$ for all $t > 0$. Assume that $\gamma\left(t\right) \in]0, 1[$. Since $\gamma_n\left(t\right) > \gamma_{n-1}\left(t\right)$ for all $t > 0$, then

$$\varphi\left(\gamma_n\left(t\right)\right) \leq \varphi\big(\left(\alpha + \beta\right)\gamma_{n-1}\left(t\right)\big) \text{ for every } t > 0.$$

Letting $n \to \infty$, since φ is left continuous and $1 < \alpha + \beta$, we have

$$\varphi\left(\gamma\left(t\right)\right) \leq \varphi\left(\left(\alpha + \beta\right)\gamma\left(t\right)\right) < \varphi\left(\gamma\left(t\right)\right) \text{ for every } t > 0$$

which is a contradiction. Thererfore $\lim_{n\to\infty} \gamma_{n-1}\left(t\right) = 1 = \gamma\left(t\right)$, that is the sequence $\{\gamma_n\left(t\right)\}$ converges to 1 for every $t > 0$. Next, we show that the sequence $\{fx_n\}$ is an M-Cauchy sequence. Assume that it is not, then there exist $0 < \varepsilon < 1$ and two sequences $\{p(n)\}$ and $\{q(n)\}$ such that

$$\left.\begin{array}{rcl} p(n) > q(n) & \geq & n, \\ M(fx_{p(n)}, fx_{q(n)}, t) & \leq & 1 - \varepsilon \\ M(fx_{p(n)-1}, fx_{q(n)-1}, t) & > & 1 - \varepsilon \\ M(fx_{p(n)-1}, fx_{q(n)}, t) & > & 1 - \varepsilon \end{array}\right\} \tag{4}$$

for each $n \in \mathbb{N} \cup \{0\}$, we set $S_n(t) = M(fx_{p(n)}, fx_{q(n)}, t)$, then we have

$$\left.\begin{array}{rcl} 1-\varepsilon \geq S_n(t) & = & M(fx_{p(n)}, fx_{q(n)}, t) \\ & \geq & M(fx_{p(n)-1}, fx_{p(n)}, \frac{t}{2}) * M(fx_{p(n)-1}, fx_{q(n)}, \frac{t}{2}) \\ & > & \gamma_{p(n)-1}(\frac{t}{2}) * (1-\varepsilon) \end{array}\right\} \tag{5}$$

since $\gamma_{p(n)-1}(\frac{t}{2}) \to 1$ as $n \to \infty$ for every $t > 0$. Supposing that $n \to \infty$, we note that the sequence $\{S_n(t)\}$ converges to $1-\varepsilon$ for any $t > 0$. Moreover by (1), we have

$$\varphi\left(M(fx_{p(n)}, fx_{q(n)}, t)\right) \leq \varphi\left(\alpha M(fx_{p(n)-1}, fx_{q(n)-1}, t) + \beta M(fx_{q(n)}, fx_{q(n)-1}, t\right) \tag{6}$$

According to the monotonicity of φ, we known that

$$M(fx_{p(n)}, fx_{q(n)}, t) > \alpha M(fx_{p(n)-1}, fx_{q(n)-1}, t) + \beta M(fx_{q(n)}, fx_{q(n)-1}, t)$$

for each n. Thus, on the basis of formula(6), we can obtain

$$\begin{array}{rcl} 1-\varepsilon & \geq & M(fx_{p(n)}, fx_{q(n)}, t) > \alpha M(fx_{p(n)-1}, fx_{q(n)-1}, t) + \beta M(fx_{q(n)}, fx_{q(n)-1}, t) \\ & > & \alpha(1-\varepsilon) + \beta\left[M(fx_{q(n)}, fx_{p(n)-1}, \dfrac{t}{2}) * M(fx_{p(n)-1}, fx_{q(n)-1}, \dfrac{t}{2})\right] \\ & > & \alpha(1-\varepsilon) + \beta\left[(1-\varepsilon) * (1-\varepsilon)\right] = (\alpha+\beta)(1-\varepsilon) > (1-\varepsilon) \end{array} \tag{7}$$

Clearly, this leads to a contradiction. Hence $\{fx_n\}$ is an M-Cauchy sequence. By the completeness of X, $\{fx_n\}$ converges to y, so $gx_{n-1} = fx_n$ tends to y. It can be seen that from (1) and the left continuous of φ, that the continuity of f implies the continuity of g. So $g(f(x_n)) \to g(y)$. However, $g(f(x_n)) = f(g(x_n))$ by the commutativity of f and g. So $f(g(x_n))$ converges to $f(y)$. By the uniqueness of the limit , we get $f(y) = g(y)$. So $f(f(y)) = f(g(y))$ by commutativity

$$\begin{array}{rcl} \varphi\left(M(gy, g(gy), t)\right) & \leq & \varphi\left[\alpha M(fy, f(gy), t) + \beta M(g(gy), f(gy), t)\right] \\ & \leq & \varphi\left[\alpha M(gy, f(gy), t) + \beta M(g(fy), f(gy), t)\right] \\ & \leq & \varphi([\alpha+\beta] M(gy, f(gy), t) \\ & < & \varphi(M(gy, f(gy), t) \end{array}$$

then if $gy \neq g(gy)$, we have a contradiction, hence $gy = g(gy)$. Thus $gy = g(gy) = f(gy)$. So gy is a common fixed point of f and g. Now we prove the uniqueness of the common fixed point of f and g. If y and z are two common fixed points to f and g respectively, and if $y \neq z$, then

$$\begin{array}{rcl} \varphi\left(M(y, z, t)\right) & = & \varphi\left(M(gy, gz, t)\right) \\ & \leq & \varphi\left[\alpha M(fy, gz, t) + \beta M(gz, fz, t)\right] \\ & \leq & \varphi\left(\alpha M(y, z, t) + \beta M(z, z, t)\right) \\ & \leq & \varphi\left(\alpha M(y, z, t) + \beta\right) \\ & < & \varphi\left(\alpha M(y, z, t)\right) \end{array}$$

then $M(y, z, t) > \alpha M(y, z, t)$, which is a contradiction. So $y = z$. ■

Corollary 9 *Let $(X, M, *)$ be an $M-$complete fuzzy metric space and g a self - map of X and assume that $\varphi : [0,1] \to [0,1]$ a strictly decreasing and left continuous mapping such that .*

$$\varphi\left(M(gx, gy, t)\right) \leq \varphi(\alpha M(x, y, t) + \beta M(gy, fy, t))$$

where $x, y \in X$, $\beta \in \mathbb{R}_+$ and $\beta < 1 \leq \alpha$ and $t > 0$, then g has a unique fixed point .

4 APPLICATION

Let $Y = \{\chi : [0,1[\to [0,1[, \chi$ is a Lebesgue integrable mapping which is summable, nonnegative, and satisfies $\int_{1-\varepsilon}^{1} \chi(t)dt > 0$ for each $0 < \varepsilon < 1\}$

Theorem 10 *Let $(X, M, *)$ be an $M-$complete fuzzy metric space and T a self - map of X and assume that $\varphi : [0,1] \to [0,1]$ a strictly decreasing and left continuous mapping . Furthermore, let f and g be maps that satisfy the following condition:*
(i) $g(X) \subseteq f(X)$,
(ii) f is continuous,

$$\int_{1-\varphi(M(gx,gy,t))}^{1} \chi(s)ds \leq \int_{1-\varphi(\alpha M(gx,gy,t)+\beta M(gy,fy,t))}^{1} \chi(s)ds \text{ for } \chi \in Y \tag{8}$$

where $x, y \in X$, $\beta \in \mathbb{R}_+$ and $\beta < 1 \leq \alpha$ and $t > 0$, then f and g have a unique common fixed point.

Proof. for $\chi \in Y$, we consider the function $\Lambda : [0,1] \to [0,1]$, by $\Lambda(\varepsilon) = \int_{1-\varepsilon}^{1} \chi(s)ds$. Λ is continuous, Λ is strictly increasing. (8) becomes

$$\Lambda\left(\varphi\left(M(gx, gy, t)\right)\right) \leq \Lambda\left(\varphi((\alpha M(gx, gy, t) + \beta M(gy, fy, t))\right). \tag{9}$$

Setting, $\varphi_1 = \Lambda \circ \varphi$ and φ_1 is strictly decreasing, left continuous for any $t > 0$, then by theorem (8), f and g have a unique common fixed point. ■

5 Conclusion

In this work, we proposed a new class of self-maps by altering the distance between two points in fuzzy metric spaces. On this kind of self-map, we proved the existence and uniqueness of a common fixed point in M-complete fuzzy metric spaces and we applied the results on maps satisfying a contractive condition of an integral type.

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