Extreme Value Theory with an Application to Bank Failures through Contagion

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Abstract. This study attempts to quantify the shocks to a banking network and analyze the transfer of shocks through the network. We consider two sources of shocks: external shocks due to market and macroeconomic factors which impact the entire banking system, and idiosyncratic shocks due to failure of a single bank. The external shocks we considered in this study are due to exchange rate shocks. An ARMA/GARCH model is used to extract i.i.d. residuals for this purpose. The effect of external shocks will be estimated by using two methods: (i) bootstrap simulation of the time series of shocks that occurred to the banking system in the past, and (ii) using the extreme value theory (EVT) to model the tail part of the shocks. In the next step, the probability of the failure of banks in the system is studied by using the Monte Carlo simulation. We also introduce the importance sampling technique in the EVT modeling to increase the probability of failure in the simulation. We calibrate the model such that the network resembles the Canadian banking system.

Keywords: Monte Carlo Simulation; Extreme Value Theory; GARCH; Importance Sampling; Bank Contagion

JEL classification: G2, C1

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1. Introduction

This study attempts to quantify the shocks to a banking network and analyze the transfer of shocks through the network. We consider two sources of shocks: external shocks due to market and macroeconomic factors which impact the entire banking system, and idiosyncratic shocks due to failure of a single bank. The effect of external shocks will be estimated by using two methods: (i) bootstrap simulation of the time series of shocks that occurred to the banking system in the past, and (ii) using the extreme value theory (EVT) to model the tail part of the shocks. In the next step, the probability of the failure of banks in the system will be studied.

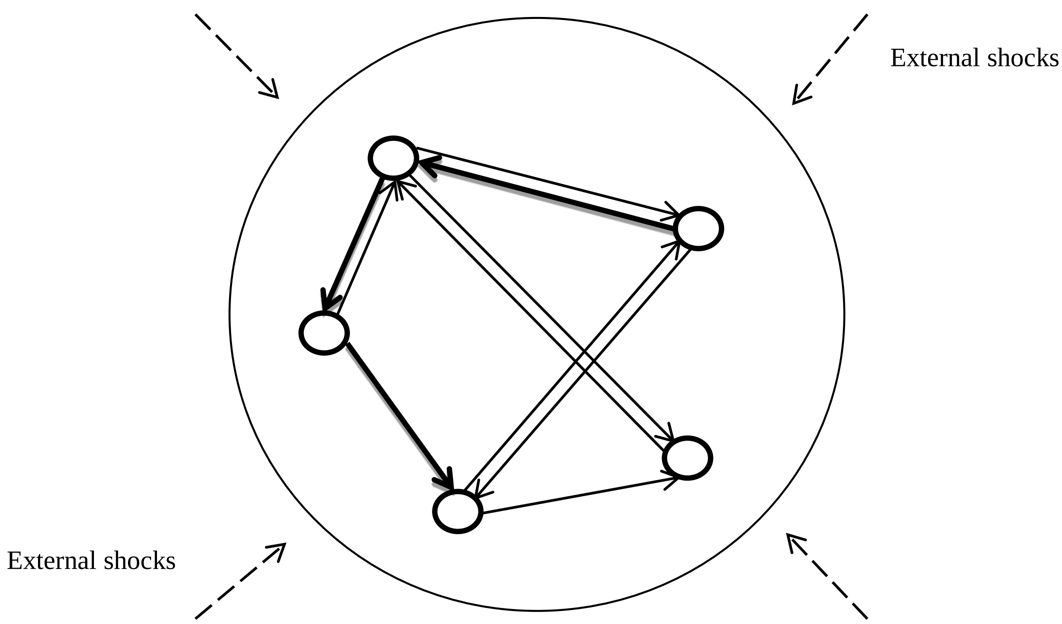
Graph 1 presents a schematic view of a banking network. In this graph, each node represents a bank, and the links between the banks represent interbank loans and their directions. The banking network is subject to external shocks. This banking network could represent a network of domestic banks or a network of international banks. In either case, the network receives shocks that could cause the failure of a bank, and potentially through contagion, the failure of the entire network. In this study, we calibrate the model such that the network resembles the Canadian banking system[3]. Then, we analyze the contagion in this context.

The contribution of this study to the literature is two-fold. First, instead of assuming random/exogenous idiosyncratic shocks to the banking system, we simulate shocks to the banking system as extremes of the stresses to the Canadian banking system using the bootstrap and by using extreme value theory. We present the advantages of using a parametric model suggested by the extreme value theory over the empirical distributions used in the previous studies. Second, we will combine importance sampling with this process to improve the estimate of the

[3] The main objective of this study is to show different methods that can be used to simulate the failure of a banking system. Therefore, more emphasis was given to the procedure and methods, and less to the accuracy of the numbers used to resemble the Canadian banking system. Paying attention to this issue is important when interpreting the results.

probability of failure. Since contagion is a rare event in the system, the usual Monte Carlo approach used in previous studies is not an efficient way to estimate its likelihood in the banking system.

Graph 1- Schematic view of a banking network



The structure of this paper is as follows. Section 2 presents a short introduction to the modes of failure of the banking system and related literature. Section 3 presents the theoretical backgrounds of the study which include GARCH/ARMA modeling of time series data, extreme value theory (EVT), and importance sampling. Section 4 presents the application of these theories to the US-Canada exchange rates. The objective of Section 4 is to use GARCH/ARMA to fit shocks to the banks leaving i.i.d. residuals. Section 5 presents the simulation results. The first part of this section presents a non-parametric Monte Carlo simulation of bank failure based on bootstrapping the i.i.d. residuals obtained in Section 4. The second part of Section 5 improves this simulation by using EVT and importance sampling techniques. Section 6 concludes.

2. Literature review of failures in the banking system

This section reviews the literature on failure of the banking system and the way the probability of failure is estimated. The section consists of two parts. In the first part, we explain the shocks to the banking system and the way failure happens. The second part discusses the process of failure and contagion, and the way their probabilities are estimated.

2.1. External shocks and bank failure

As mentioned earlier, bank failure may happen due to losses from market shocks or due to contagion as a consequence of other banks' failure. We follow the terminology of Elsinger et al. (2006) to call the former type of failure a fundamental default and the latter a contagious default. Identifying and capturing these two sources of failure are the main modeling challenges. Contagion defaults are not independent of fundamental ones since contagion is more likely to happen in situations where the banking system has already been weakened by external shocks. In what follows, we first identify the external shocks the market may impose to the system, and then, we will show how these shocks may be spread out in the system. Section 2.1 takes a closer look at the mechanism through which contagion happens.

In terms of market shocks, Illing and Liu (2003) have identified highly stressful financial events in the Canadian banking system. They define financial stress as "the force exerted on economic agents by uncertainty and changing expectations of loss in financial markets and institutions". In their study, financial stress is measured with an index called the Financial Stress

Index (FSI). The extreme values of FSI are called financial crises. According to them, the four main sources of stress in the Canadian banking system are as follows:

- Banking crises/stress
- Foreign exchange crises/stress
- Debt crises/stress
- Equity crises/stress

This study formulates the US-Canada exchange rate shocks by using time series and EVT techniques. The US-Canada exchange rate constitutes 86% of the Canadian exchange rate index. Therefore, it can be a proxy measure for a foreign exchange crisis. If the shocks are strong enough, they may cause a bank, and eventually, the entire banking system to fail. Sections 4 and 5 use the US-Canada exchange rates to show how bootstrap simulation and extreme value theory could be used to simulate failure in the banking system.

Considerable effort has been made by academia and the central banks to model banking crises and predict financial contagion in recent years. The aim of these studies is to better understand the nature of banking crises and mitigate their impacts at the national and international levels. Examples of this kind of research include Allen and Gale (2000, 2007), Eisenberg and Noe (2001), Santor (2003), Li (2009), May (2010), and Gai and Kapadia (2010). Moreover, Boss et al. (2004), Elsinger et al. (2006), and Gauthier (2010) are examples of empirically applying these theories to national banking systems.

Financial contagion means the transmission of financial shocks from one financial entity to other interdependent entities. The transfer of shocks among banks could normally occur through financial linkages. However, banking contagion is possible even when the banks are independent (Santor 2003). This study assumes contagion happens through interbank linkages, i.e. through interbank loans and borrowings.

Most studies rely on "counterfactual simulations" to estimate the likelihood of contagion arising from a default in repaying interbank loans. One reason is that troubled banks are often bailed out by central banks rather than letting them fail. This limits the use of other methods of study, including event analysis to estimate the probability of failure. The down side of counterfactual simulation is the implication of strong assumptions we have to make to define different scenarios. Upper (2007) has done a survey on these studies, their methodologies, and the results. He concludes that though contagions in banking systems are unlikely, their possibility cannot be fully ruled out. This needs authorities' attention since the cost of contagion default to society could be very high.

Moreover, Upper (2007) mentions that most studies focus only on the contagion that results from the failure of individual banks (for example due to a fraud). However, this represents only a small fraction of all bank failures. Most failures happen when several banks are hit by an external shock at the same time and become insolvent since their net value becomes negative. This is in contrast to the former models that consider only one source of risk, i.e. interbank linkages, and ignore other sources, e.g. macroeconomic factors. Elsinger et al. (2006) and Gauthier et al. (2010) are among the few studies that integrate both idiosyncratic (e.g. a fraud) and aggregate shocks (i.e. economy wide shocks) to analyze contagion in the banking system.

This study will follow a similar approach as Gai and Kapadia (2010) and May (2010). However, contrary to these models, we also consider both idiosyncratic and aggregate shocks to estimate the probability of failure. In this sense, our approach will be similar to Elsinger et al. (2006).

2.2. Contagion in banking system

This section establishes a simple system of a banking network and the way shocks are transferred within the system. This representation is based on Gai and Kapadia (2010) and May (2010). As before, we assume each node in Graph 1 represents a bank and arrows show the direction and magnitude of interbank loans. Graph 2 presents the structure of assets and liabilities of each bank.

Graph 2- Financial structure of a bank

<table>
<tr><th>Liabilities</th><th>Assets</th></tr>
<tr><td rowspan="2">Deposits d_i</td><td>External assets e_i</td></tr>
<tr><td rowspan="2">Interbank loans l_i</td></tr>
<tr><td rowspan="2">Interbank borrowing b_i</td></tr>
<tr><td>Reserves y_i</td></tr>
</table>

Assume interbank borrowing $\mathrm{b_i}$ comes from j other banks and interbank loans l_i goes to k other banks. If a bank fails (i.e. it goes bankrupt), it will impact the entire network by being unable to repay its debts. A bank is insolvent if its net value becomes negative, *i.e.*, $e_i + l_i + y_i < d_i + b_i$. We assume each bank keeps a proportion of its assets, y_i, as a reserve to protect itself against shocks. The external assets e_i could consist of q subclasses of assets with different interest rates and risk degrees. In this study, we divide external assets into foreign assets, loans, and other assets.

To analyze the impact of external shocks, we assume when an external asset is hit by the shock, its value will drop to $e_i^{'}$. Then, the bank survives if $e_i^{'} + l_i + y_i \geq d_i + b_i$. However, if the bank fails due to the shock to its external assets, a second (internal) shock will be generated through the interbank linkages: since the bank is bankrupt, the creditor banks lose an amount *f* of their interbank loans. The value of *f* depends on our recovery assumption (zero recovery or the resale price of the bankrupt bank's assets) and the amount of the creditor bank's loan to the failed bank. In this case, the creditor bank *j* survives if $e_j^{'} + l_j + y_j - f_j \geq d_j + b_j$. If the creditor bank does not survive, the shock will spread out to the network through the same interbank linkage mechanism to generate a second round of shocks. This process continues until the network gets to a steady-state position, i.e. all banks fail or no more banks fail.

We may make two assumptions when a bank fails. A simplifying assumption is zero recovery, meaning that the value of the insolvent bank will become zero and creditors will lose all of their loans to the bank. The other assumption is that the insolvent bank sells its assets, probably in a lower market price, to repay its creditor. In this study, we follow the second approach and assume the depositors are cleared first, and then, the rest of the asset is distributed proportionally to the creditor banks.

We should note that when a bank sells its assets, prices drop, which causes a depreciation of other banks' assets and net value. This feedback effect increases the probability of the bank's default as its net value decreases. Gauthier et al. (2010) assumed that illiquid assets at each bank can lose 2% at most in value even when banks sell all their holdings. Cyclical interdependence among the banks is another factor that could increase the probability of contagion. We ignore these two effects in this study.

Matrices are a convenient way to show interbank loans when it comes to the simulation of contagion (Upper 2007; Boss et al. 2004). If there are N banks in the system that may lend to each other, interbank loans can be represented by an $N \times N$ matrix as follows:

$$X = \begin{bmatrix} 0 & \cdots & x_{1i} & \cdots & x_{1N} \\ \vdots & \ddots & \vdots & \iddots & \vdots \\ x_{i1} & \cdots & 0 & \cdots & x_{iN} \\ \vdots & \iddots & \vdots & \ddots & \vdots \\ x_{N1} & \cdots & x_{Nj} & \cdots & 0 \end{bmatrix}$$

where x_{ij} represents the loan of bank i to bank j. The sum of rows $l_i = \sum_j x_{ij}$ represents bank i's total loans to other banks and $b_j = \sum_i x_{ij}$ is bank j's total liabilities to other banks.

The construction of matrix X is straightforward if the data on interbank loans are available. Otherwise, we need to make some assumptions on banks' balance sheets to fill in this matrix. Banks' balance sheets are usually accessible, though they show only total interbank borrowing and lending. A simple assumption in this case is that interbank lending and borrowing is equally distributed among other banks (Upper 2007). Another assumption is that interbank lending and borrowing are proportional to other banks' assets (for example, Gauthier et al. 2010). We have used the second approach in constructing interbank loans in this study. For this purpose, we selected the six largest Canadian banks and estimated the interbank borrowing and lending among them as well as their deposits, foreign assets and loans based on their public balanced sheets in 2009. These six banks account for 90.3% of all banking assets in Canada. When the matrix of interbank loans has been constructed, we need to specify the shock to the system and its impacts. Based on these shocks and their impacts, the probability of contagion is estimated. We assume that the portfolio of bank holdings do not change during this process.

This study extends previous literature in two directions. First, instead of assuming a random/exogenous idiosyncratic shock to the banking system, we define shocks to the banking system as extremes of the stresses to the Canadian banking system suggested by Illing and Liu

(2003) either using the bootstrap or by applying the extreme value theory to their measures. Then, the impacts of these shocks on each bank's asset will be calculated. Banks that default in the first round of shocks are fundamentally insolvent. We argue that using the extreme value theory gives us a better estimate of the shocks to the system than the empirical distributions used in Elsinger et al. (2006) and Gauthier et al. (2010) since we can also include out-of-sample shocks in the estimation. Second, we will combine importance sampling with the extreme value theory to improve the estimate of the probability of failure. The next section introduces time series and EVT modeling.

3. GARCH/ARMA modeling and EVT

This section explains the theoretical background and techniques we will use later in this study. The section starts with a short introduction to times series modeling and how these techniques can be applied to exchange rates to obtain i.i.d. residuals. The objective is to use these residuals in EVT modeling in Section 4 and the Monte Carlo simulation in Section 5. Then, the extreme value theory and its applications will be discussed. Finally, there will be a short introduction to importance sampling and its application in simulation of rare events.

3.1.Time series modeling

An autoregressive-moving average (ARMA) model is a type of random process which is used to model time series processes. An ARMA(p,q) representation of a time series is as follows:

$$y_t = c + \sum_{i=1}^{p} \varphi_i y_{t-i} + \sum_{j=1}^{q} \psi_j \varepsilon_{t-j} + \varepsilon_t. \qquad (1)$$

where $\varepsilon_t \sim \text{iid}(0, \sigma^2)$.

A property many financial time series possess is that their variance conditional on their past history may change over time. In other words, they may show time varying conditional heteroskedasticity. The "generalized autoregressive conditional heteroskedasticity" (GARCH) modeling is a method to capture this volatile behaviour of the maximum of financial time series. A formulation of a GARCH(p,q) model is as follows:

$$y_t = c + \varepsilon_t, \qquad (2)$$

$$\varepsilon_t = z_t \sigma_t,$$

$$\sigma_t^2 = a_0 + \sum_{i=1}^{p} a_i \epsilon_{t-i}^2 + \sum_{j=1}^{q} b_j \sigma_{t-j}^2,$$

where c is the mean of y_t, ε_t is the residual of the series, and $z_t \sim iid\ (0,1)$ is the standardized residual of the series. The coefficients a_i, $i = 0, \cdots, p$, and b_j, $j = 1, \cdots, q$, are all assumed to be positive to ensure that the conditional variance σ^2_t is always positive. A GARCH(p,q) model may be combined with an ARMA(r,s) model as follows:

$$y_t = c + \sum_{i=1}^{r} \varphi_i y_{t-i} + \sum_{j=1}^{s} \psi_j \varepsilon_{t-j} + \varepsilon_t, \qquad (3)$$

$$\varepsilon_t = z_t \sigma_t,$$

$$\sigma_t^2 = a_0 + \sum_{i=1}^{p} a_i \epsilon_{t-i}^2 + \sum_{j=1}^{q} b_j \sigma_{t-j}^2.$$

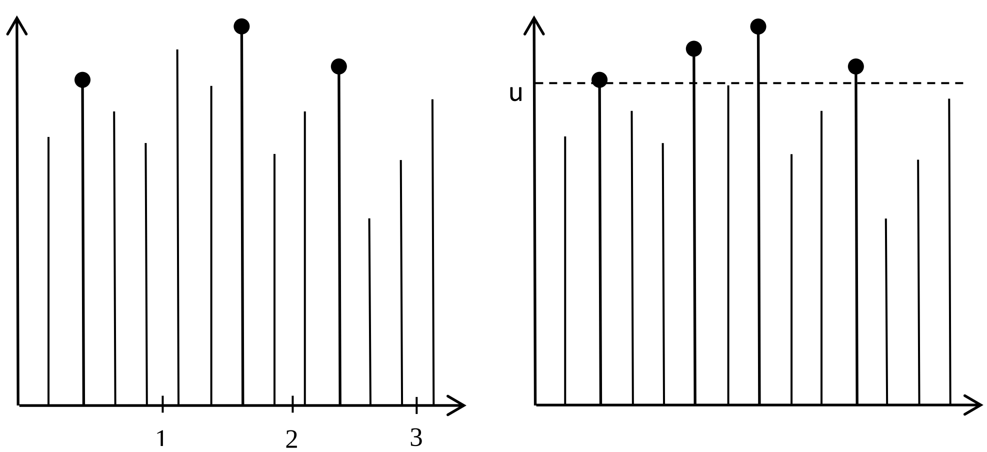
In Section 4 we filter the US exchange rate with a GARCH(1,1)/ARMA(1,1) to obtain standardized i.i.d. residuals.

3.2. Extreme value theory

Extreme value theory (EVT) is a framework to analyze the tail behaviour of a distribution. The majority of parametric methods use a normal distribution approximation to model time series data. One of the drawbacks of using a normal approximation to model the extreme values of financial data is that the probability of high quantiles are underestimated since financial data are usually fat-tailed. Using a fat-tailed distribution such as Student-t or log-normal improves the approximation, but may not still fully capture the tail behaviour of the series. On the other hand, non-parametric methods to model extreme values show another disadvantage by being unable to estimate out-of-sample quantiles. EVT attempts to overcome these problems by parameterizing the tail part of the data series. EVT is analogous to the central limit theorem (CLT) in this sense but for the extremes of a distribution.

This section includes a short introduction to EVT based on Embrechts et al. (1997), Coles (2001), and Smith (2002). Schafgans et al. (1990), Hols and de Vries (1991), Danielsson and de Vries (1997), Gilli and Kellezi (2006), and Gencay and Selcuk (2006) are examples of studies that used EVT to model the extreme values of exchange rates and financial data.

Graph 3 – Block maxima (left panel) and exceedances over a threshold (right panel) approaches



There are two approaches to model extreme values of a data series. In the first approach, the data is partitioned into successive blocks, where the block maxima represent extreme values. Theorem 1 shows the limiting distribution of these maxima. In the second approach, the exeedances over a selected threshold are considered to be extreme values. Theorem 2 models the limiting distribution of these exeedances. These two approaches are presented in Graph 3. The second approach is more efficient in modeling exceedances in time series and financial data.

Theorem 1[4] (Fisher and Tippett, 1928; Gnednko, 1943) - Let $M_n = Max\,\{X_1, \ldots , X_n\}$, where X_1, … , X_n are i.i.d. from an arbitrary distribution F. If there exist sequences of constants $\{a_n > 0\}$ and $\{b_n\}$ such that $p\{(M_n - b_n)/a_n \leq z\} \rightarrow G(z)$ as $n \rightarrow \infty$ for a non-degenerate distribution function G, then G is a member of the "generalized extreme value" (GEV) family,

$$G(z) = \exp\left\{-\left[1 + \xi\left(\frac{z-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\right\}, \qquad (4)$$

defined on $\{z: 1 + \xi\,(z - \mu)/\sigma > 0\}$, where $-\infty < \mu < \infty$, $\sigma > 0$, and $-\infty < \xi < \infty$.

In the above equation, μ and σ are the location and scale parameters, and ξ is called the shape parameter. G belongs to one of the three standard extreme value distributions of Frechet, Weibull, or Gumbel as $\xi > 0$, $\xi < 0$, or $\xi = 0$ after some change of parameters:

Frechet: $$G(x) = \begin{cases} 0, & x \leq 0, \\ \exp(-x^{-\alpha}), & x > 0, \alpha > 0; \end{cases} \qquad (5)$$

Weibull: $$G(x) = \begin{cases} \exp(-(-x)^{\alpha}), & x < 0, \alpha > 0, \\ 1, & x \geq 0; \end{cases} \qquad (6)$$

Gumbell: $$G(x) = \exp(-\exp(-x)),\ -\infty < x < \infty; \qquad (7)$$

We should note that Theorem 1 states that if the (normalized) sequence of maxima has a limit, that limit will be a member of the GEV family. However, it does not guarantee the existence of a limit.

The Frechet distribution is heavy-tailed and the one that is usually used in financial modeling as exchange rates and other financial data have usually shown to be fat-tailed. Roughly speaking, the tail distribution function of a light-tailed random variable decays at an exponential

[4] For proofs and further explanations readers may refer to Leadbetter et al. (1983), Coles (2001), and Resnick (2007).

rate or faster, while it decays at a slower rate for heavy-tailed random variables. Rare events occur differently in light-tailed and heavy-tailed random variables. It can be shown that the most likely way for the sum of heavy-tailed random variables to become large is by one of the random variables to become large. In contrast in the light-tailed case, all of the random variables in the sum contribute to the sum becoming large (Juneja and Shahabuddin 2006). That means a large deviation in a light-tailed setting happens most likely due to small occurrences in a specific path instead of having a big event. Studies show that bank failures usually happen due to a single bad event and not a series of consecutive smaller negative returns that may add up to the same highly negative result (Danfelsson and de Vries 1997). The reason is that during a gradual decline of the market or negative returns, banks and other financial institutes can react and adjust themselves rather than letting the losses accumulate until a failure happens.

In a different approach, we may model the behaviour of extreme events over a threshold given by the following conditional probability:

$$F_u(y) = p\left(X\text{-}u \leq y \middle| X > u\right) = \frac{F(y+u)\text{-}F(u)}{1\text{-}F(u)}, \;\; 0 \leq y < x_0. \qquad (8)$$

where x_0 is the (finite or infinite) right endpoint of *F*. Graph 4 presents the relationship between the tail part of a distribution and the original distribution.

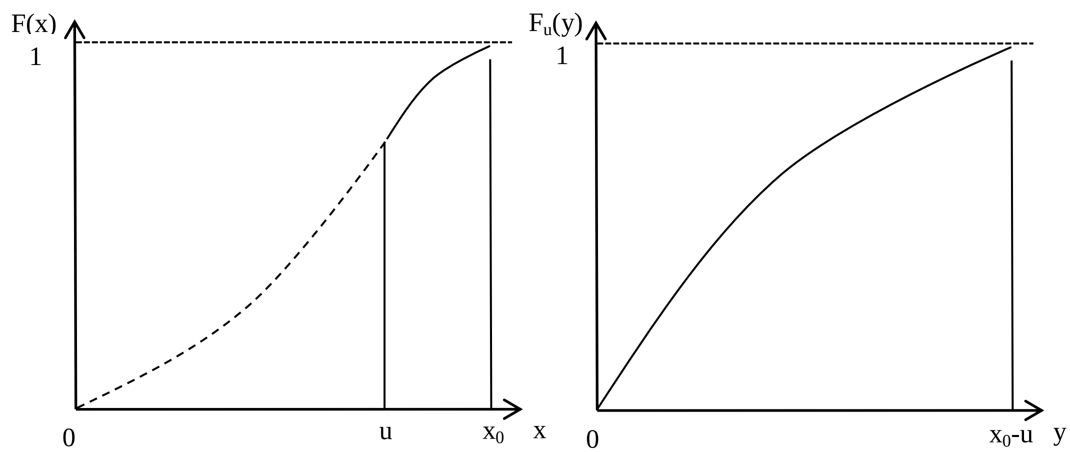
Theorem 2[5] (Pickands 1975; Balkema and de Haan 1974)- Let X_1, X_2, X_3, … be a sequence of independent random variables from *F*. Let $M_n = \text{Max}\,\{X_1, \ldots, X_n\}$. Suppose *F* satisfies the EVT, meaning that for large *n*, $p(M_n \leq z) \approx G(z)$, where *G* is defined as in Theorem 1. Then, for large enough *u*, the distribution function of (X-u), conditional on $X > u$, can be approximated by the "generalized Pareto distribution" (GPD),

[5] For proofs and further explanations readers may refer to Leadbetter et al. (1983) and Coles (2001).

$$H(y) = \begin{cases} 1 - \left(1 + \frac{\xi y}{\tilde{\sigma}}\right)^{-\frac{1}{\xi}}, & \xi \neq 0 \\ 1 - exp\left(-\frac{y}{\tilde{\sigma}}\right), & \xi = 0 \end{cases} \tag{9}$$

defined on $\{y: y > 0 \; and \; (1 + \xi y/\tilde{\sigma}) > 0\}$, where $\tilde{\sigma} = \sigma + \xi(u - \mu)$. Note that the shape parameter ξ is the same in *G* and *H*.

Graph 4- Original distribution function and estimation of the tail part



It is worth mentioning that analyzing data commonly shows that extreme conditions persist over several consecutive periods. This phenomenon is also observed frequently in financial time series data. It is worth recalling that the EVT theories assume the observations are i.i.d. and stationary. With respect to the high volatility and dependency of exchange rate time series data, some authors suggest using a GARCH model to remove the volatility and long-term dependency of the data before using an EVT model (McNeil and Frey 2000; Smith 2003, and Andersen et al. 2009). However, we may still need to deal with the short-term dependency in the data. As mentioned before, we will use a combination of GARCH and ARMA models in Section 4 to remove the long- and short-term dependencies in the data to obtain i.i.d. residuals to which EVT may be applied.

3.3. Importance sampling

Another technique we will use in simulation is importance sampling. Importance sampling is a change of measure to increase the probability of events of interest in a Monte Carlo simulation. The probability will then be adjusted to reflect this change of measure. Importance sampling can be explained as follows. Suppose we want to evaluate the following integral:

$$E_f[h(x)] = \int_{\chi} h(x)f(x)dx. \qquad (10)$$

Under the normal Monte Carlo approach, we generate a sample (X_1, …, X_m) from the density f to approximate (10) by $\bar{h}_m = \frac{1}{m}\sum_{j=1}^{m} h(x_j)$ since $\bar{h}_m$ converges almost surely to $E_f[h(x)]$ by the SLLN. However, this approach is not efficient if the events of interest happen rarely. To improve the result, we may make the events of interest occur more frequently than it would happen in the normal Monte Carlo method. This can happen by generating random variables from another distribution g that oversamples from the portion of the state space that receives lower probability under f. Later, importance weights correct for this bias. The new estimator will then be $\bar{h}_m^* = \frac{1}{m}\sum_{j=1}^{m} h(x_j)\,\omega^*(x_j)$, where $x_j \sim g$, and $\omega^*(x_j) = f(x_j)/g(x_j)$ are importance weights. This approach is called importance sampling and is based on an alternative representation of $E_f[h(x)]$:

$$E_f[h(x)] = \int h(x)f(x)dx = \int h(x)\frac{f(x)}{g(x)}g(x)dx. \qquad (11)$$

4. Application of GARCH/ARMA and EVT to exchange rates

This section is an analysis of the nominal US exchange rate versus Canadian dollars by using EVT. Since the US dollar constitutes 86% of the value of the "Canadian effective exchange rate", we concentrate on modeling this variable to analyze the impact of foreign exchange rate shocks on the banking system in this study. The link of a change in exchange rates to the change in bank assets is as follows. Suppose A is the bank's foreign assets in foreign currency. Let R_t be the exchange rate of the foreign currency at time *t*. Then, the value of the foreign assets in Canadian dollars will be $S_t = A.R_t$. Consequently, the change in the value of foreign assets due to a change in exchange rate will be $\Delta S_t = A. \Delta R_t$, where $\Delta R_t = R_t - R_{t-1}$. If the value of the foreign exchange rate depreciates, i.e. $\Delta R_t < 0$, the bank will lose part of its value, ΔS_t, which may lead to its bankruptcy. We assume all other variables are constant to simplify the analysis. Since we are interested in the maximum of losses, we model the negative changes in the exchange rate.

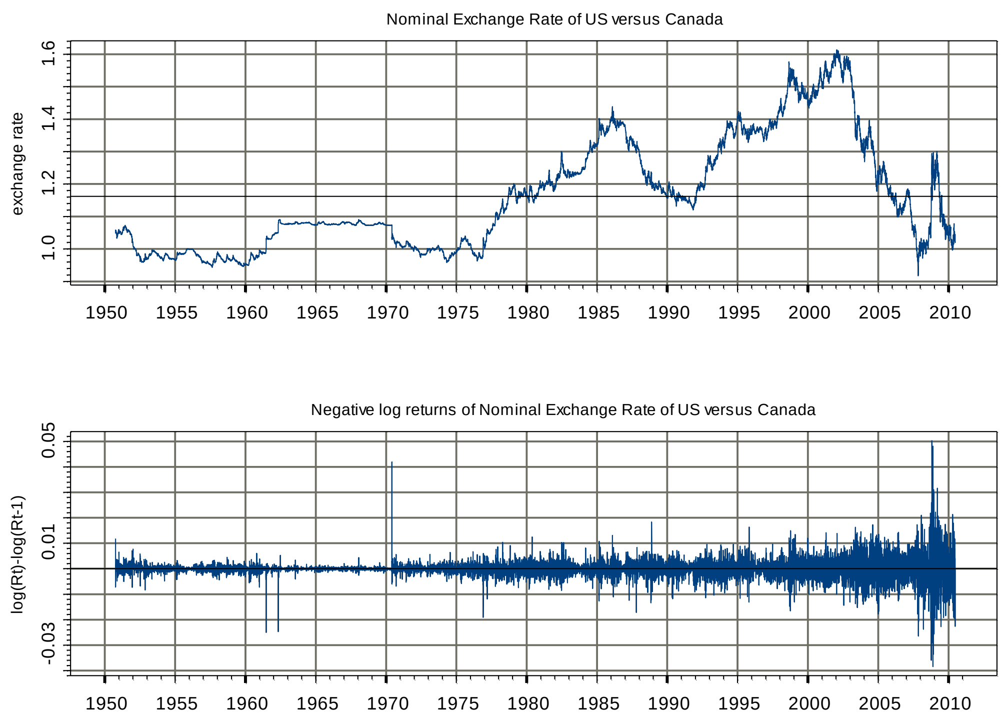
4.1. Application of EVT to exchange rates

The data covers the daily noon spot US exchange rate from November 2, 1950 to June 24, 2010. The data set contains 15031 observations. In what follows, we first re-express the exchange rate as a product of returns and then use GARCH/ARMA to obtain i.i.d. residuals. Let R_t be the nominal daily exchange rate of US versus Canada (denoted by USXch in the graphs). Consistent with the literature, we define the negative returns of exchange rate as $r_t = -\log(R_t/R_{t-1}) = - [\log(R_t) – \log(R_{t-1})]$.

Graph 5 presents the plots of the exchange rate R_t and the negative returns r_t. Unit root tests suggest that the exchange rate R_t is not mean stationary, but the negative return r_t is.

However, the ARCH test, the autocorrelation function (ACF) and partial autocorrelation functions (PACF) in Graph 6 suggest that there are long- and short-term dependencies in r_t. As explained in Section 3, the EVT modeling cannot be applied on this series. Graph 6 also presents the histogram and qq-plot of r_t against the normal distribution. The mean, standard deviation, and skewness of r_t are zero, 0.0032, and 0.303. The kurtosis of r_t is 26, which suggests the negative returns of the exchange rate are heavy-tailed.

Graph 5 – US exchange rate versus Canadian dollars



We use a GARCH(1,1) with ARMA(1,1) model to remove the long-and short-term dependencies in r_t. Table 1 presents the estimated results. As shown in Table 1, all coefficients are highly significant. The model can be represented as follows:

$$r_t = c + \alpha r_{t-1} + \beta \varepsilon_{t-1} + \varepsilon_t, \qquad (12)$$

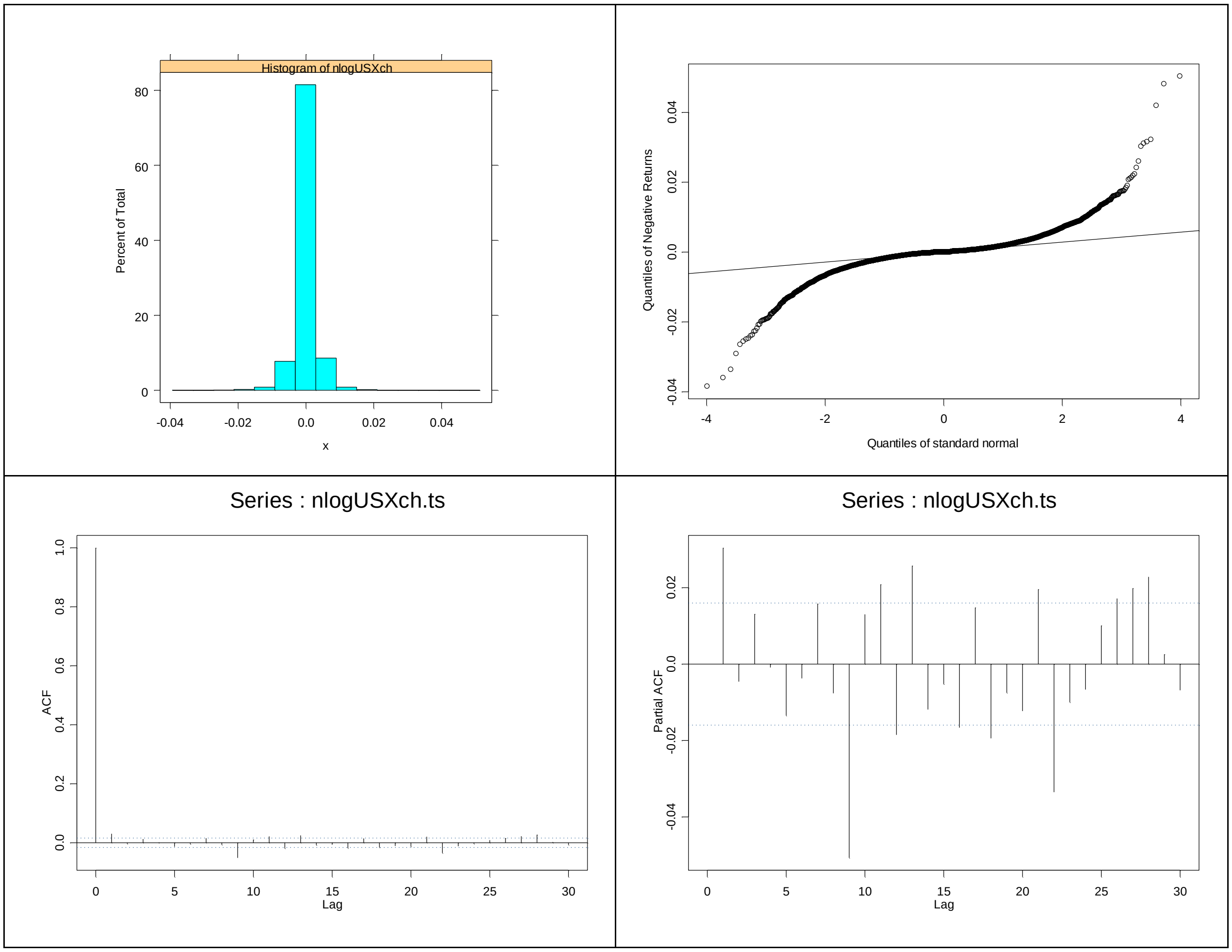
where

$$\varepsilon_t = \sigma_t z_t, \quad (13)$$

$$\sigma_t^2 = a + \lambda\varepsilon^2_{t-1} + \mu\sigma^2_{t-1}, \quad (14)$$

$$z_t \sim iid\,(0, 1). \quad (15)$$

Graph 6 – Histogram, qq-plot, ACF, and PACF of r_t (nlogUSXch.ts)



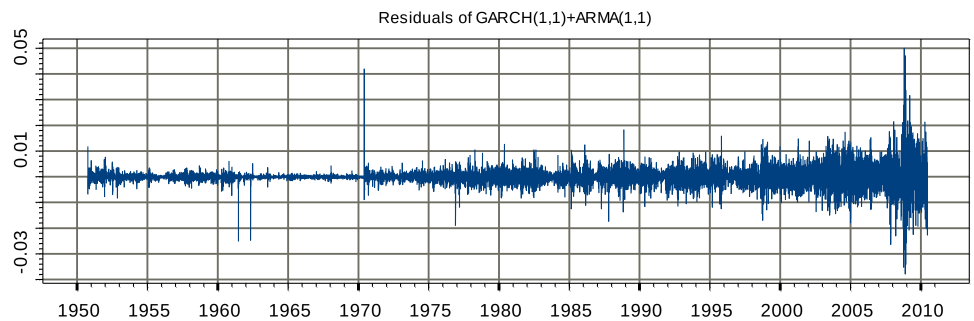
The estimated parameters are as follows: c = 0.0001, α = -0.3526, β = 0.4205, a = 0.0000002, λ = 0.1207, and μ = 0.8592. We denote the residuals and standardized residuals of the model by ε_t and z_t. Graph 7 compares these two series. Graph 8 presents the histogram, qq-plot against the normal distribution, ACF, and PACF of z_t. As the graphs suggest, there is no autocorrelation left in the data and the moving average effect has been significantly reduced. Also, the ARCH test suggests that no ARCH effect is left in z_t. Therefore, we can conclude that the GARCH(1,1)/ARMA(1,1) model filtered out all long- and short-term dependencies in the data. The mean, standard deviation, skewness, and kurtosis of z_t are -0.035, 0.95, 2.2, and 133. We will use these i.i.d. residuals in the following sections for the Monte Carlo and EVT analysis.

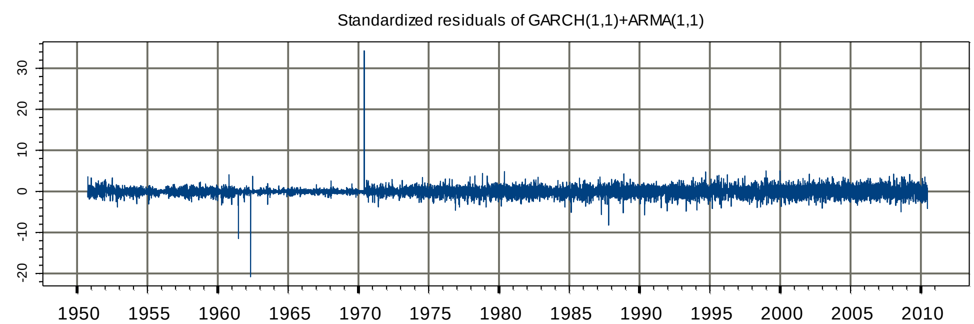
Table 1- GARCH(1,1) with ARMA(1,1) filtering of r_t

Estimated Coefficients:

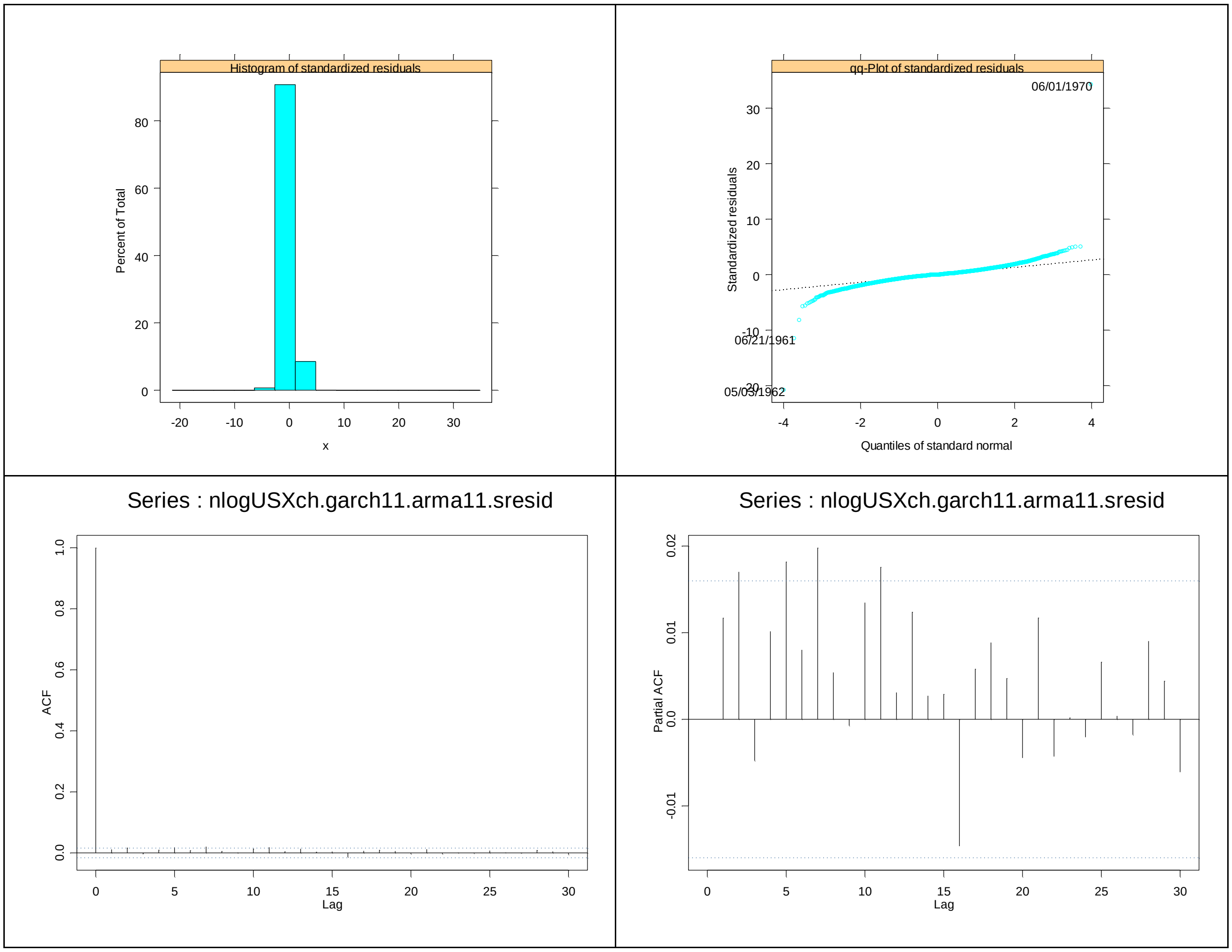
	Value	Std.Error	t value	Pr(>|t|)
c	9.091e-005	2.067e-005	4.399	5.487e-006
α	-3.526e-001	1.057e-001	-3.337	4.246e-004
β	4.205e-001	1.028e-001	4.089	2.178e-005
a	1.936e-007	5.132e-009	37.730	0.000e+000
λ	1.207e-001	3.421e-003	35.286	0.000e+000
μ	8.592e-001	3.616e-003	237.583	0.000e+000

Graph 7- Residuals (ε_t) and standardized residuals (z_t) of the model





Graph 8- Properties of the standardized residuals (z_t)

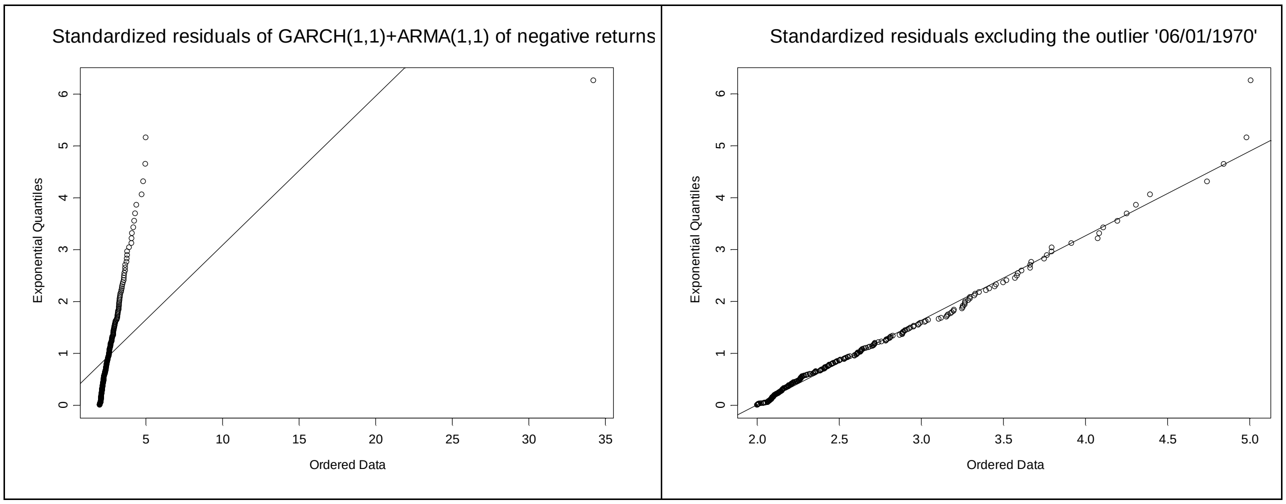


4.2. Application of EVT to exchange rates

In this section, we estimate the GPD parameters for the exeedances over a threshold on the residuals of the US-Canada exchange rates we obtained in Section 4.1. A method to understand the tail behaviour of the series is to create a qq-plot against the exponential distribution as a reference distribution. If the exeedances over thresholds are from an exponential distribution, then the shape parameter ξ is 0 and the qq-plot should be linear. Departures from linearity in the qq-plot then indicate either fat-tailed behaviour ($\xi > 0$) or bounded tails ($\xi < 0$).

Graph 9 presents the qq-plots of the exeedances of z_t under two cases where the outlier on 06/01/1970 is included (Graph 9-left) and excluded (Graph 9-right). We see a departure from the straight line when the outlier is included, but the qq-plot suggests an exponential distribution when we exclude this outlier (also, leaving this outlier out transform the heavy-tailed residuals to an exponential distribution).

Graph 9- qq-Plot of z_t against the exponential distribution



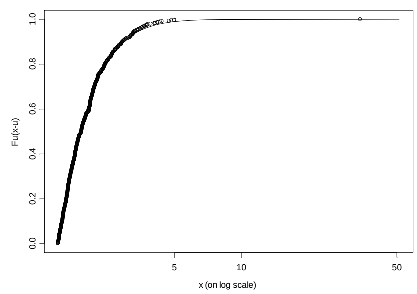
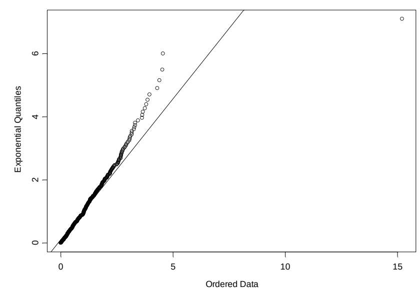
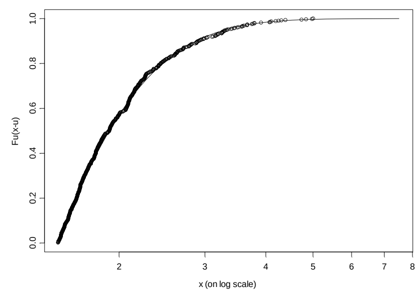
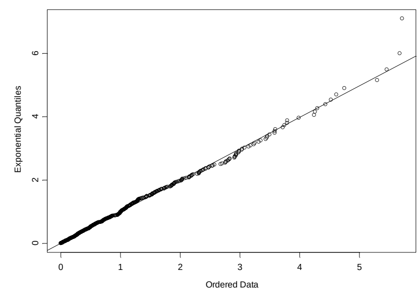
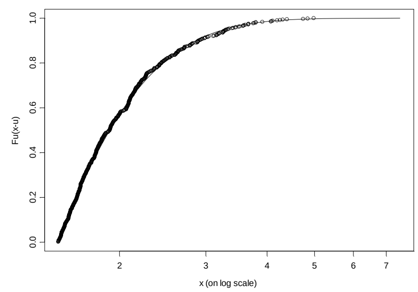
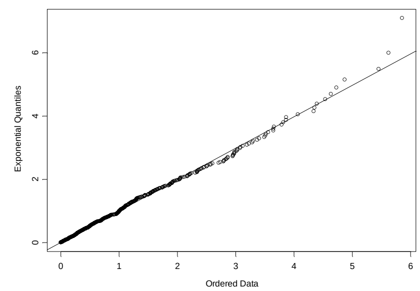
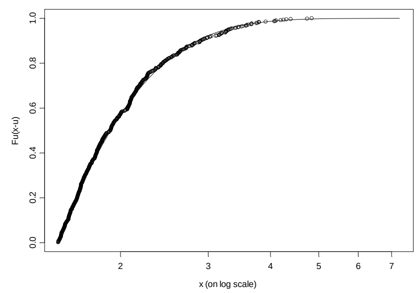
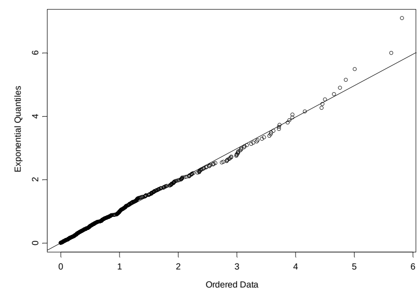
It is worth mentioning that on 06/01/1970, Canada went off fixed exchange rates with the US. After capturing all short and long run dependencies in the data by a GARCH/ARMA model, we observe that this is the largest shock received by the Canadian exchange rate since 1950. The magnitude of this shock is almost seven times larger than the next largest shock. We keep this outlier in the non-parametric analysis of Section 5. However, we do not include it in the EVT analysis since it does not fit models we estimate in this section.

We need to set a threshold to determine exceedances in the series and fit a model on them. A useful tool to find a threshold is the empirical mean excess function. To determine the threshold level, we need to find an interval where the mean excess function has a positive slope. Also, the estimated shape parameter should be relatively stable with respect to the threshold. As explained before, a right choice of the threshold is critical because if it is too low, we may violate the asymptotic basis of the model which leads to bias in estimation, and if it is too high, few observations will be left which will lead to a high variance. Examining different value for the threshold suggests that a threshold equal of 1.5 is appropriate for the estimation. Also, we remove the four right end points of the data one by one to examine the stability of the estimations. Graphs 10 present the estimated parameters and fitted models for u = 1.5. The first column presents the fitted models, and the second column compares the exeedances with the quantiles of an exponential distribution. As these graphs suggest, the exeedances fit the model quite well when the outlier of 06/01/1970 is excluded. The estimated parameters are also stable after removing this outlier.

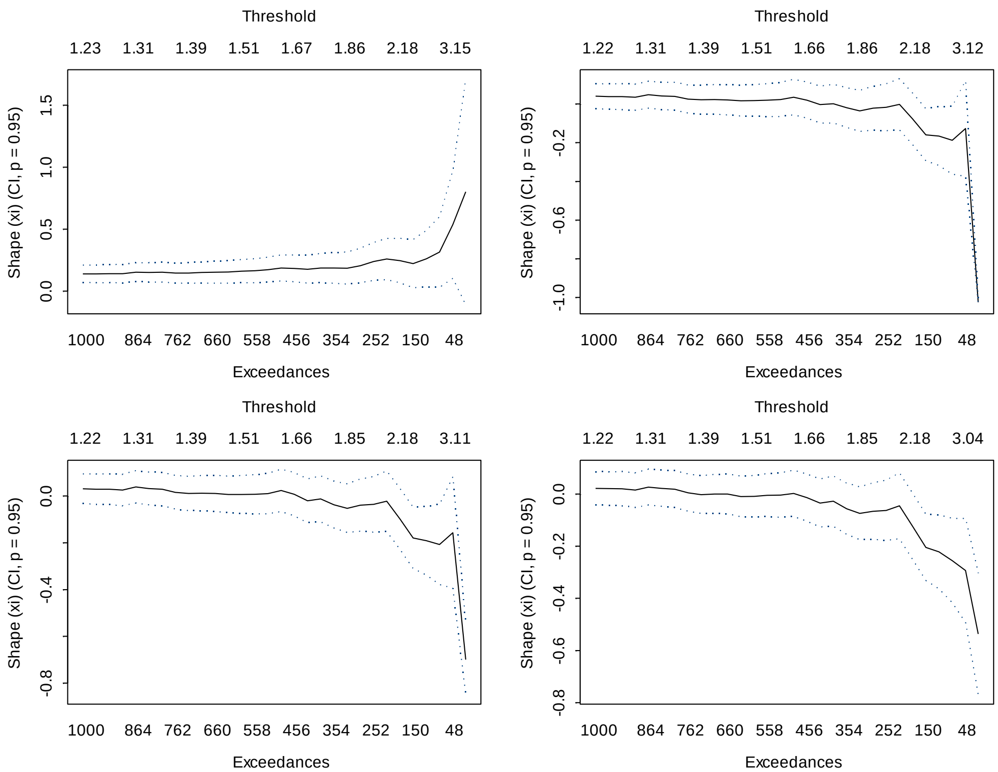
Graph 11 presents the stability of the shape parameter. This plot shows how the maximum likelihood estimator (MLE) of the shape parameter ξ varies with the selected threshold u. We should choose u from a region where the shape parameter ξ remains relatively stable. The upper-right graph belongs to the case where all exeedances are included, and the other three graphs present the cases when we remove the outliers one by one. According to Graph 11, the MLE of the shape parameter ξ is stable when we remove the outlier of 06/01/1970 and we keep over 250 exeedances.

These estimations suggest that when the outlier of 06/01/1970 is taken out, the estimated shape parameter is not statistically different from zero, and the scale parameter is around 0.59. The models fit the data fairly well. This suggests an exponential distribution with mean 0.59 for the tail part of z_t. A Kolmogorov-Smirnov test also confirms that the exeedances are from an exponential distribution with mean 0.59.

Graph 10- Exeedances over a threshold estimation; *u* = 1.5

Estimated parameters	**Excess Distribution**	**QQplot of Residuals**
Case 1: All data n = 604 ξ = 0.1552308 s.d. = 0.03734148 β(u) = 0.5291026 s.d. = 0.02902898		
Case 2: Excluding 06/01/1970 n = 603 ξ = 0.01414949 s.d. = 0.04459061 β(u) = 0.5889857 s.d. = 0.03556595		
Case 3: Excluding 06/01/1970, 12/31/1998 n = 602 ξ = 0.001761105 s.d. = 0.04445515 β(u) = 0.5915517 s.d. = 0.03567908		
Case 4: Excluding 06/01/1970, 12/31/1998, 12/29/1999 n = 601 ξ = -0.01185767 s.d. = 0.04468459 β(u) = 0.5947643 s.d. = 0.03598382		

Graph 11- The shape parameter for *u* = 1.5 when removing the outliers



5. Simulation results

This section presents the results of the simulation of the banking network. The section includes two parts. In the first part, we present the results of the bootstrap simulation. The second part shows how the results can be improved by using EVT and importance sampling techniques.

5.1. Non-parametric simulation

The non-parametric simulation is based on reconstructing the exchange rate time series by resampling from the i.i.d. residuals we obtained in Section 4, and applying the reconstructed counter-factual series to the banking network to find out if any failure or contagion occurs in the network. By repeating this process *n* times and counting the number of failures and contagions in the system, we find the probability of failure and contagion. We assume the reserve value of banks varies from 1% to 10% of their total assets[6].

This process can be explained more precisely as follows. Consider the GARCH(1,1)/ARMA(1,1) model we estimated for the negative returns on US-Canada exchange rates in Section 4:

$$\hat{r}_t = c + \alpha\hat{r}_{t-1} + \beta\hat{\varepsilon}_{t-1} + \hat{\varepsilon}_t, \quad (16)$$

where

$$\hat{\varepsilon}_t = \hat{\sigma}_t\hat{z}_t, \quad (17)$$

$$\hat{\sigma}_t^{2} = a + \lambda\hat{\varepsilon}^{2}_{t-1} + \mu\hat{\sigma}^{2}_{t-1}, \quad (18)$$

$$\hat{z}_{\mathrm{t}} \sim iid\ (0, 1) \quad (19)$$

with c = 0.0001, α = -0.3526, β = 0.4205, a = 0.0000002, λ = 0.1207, and μ = 0.8592. We start from equation (19) and move backward to equation (16) to reconstruct the negative returns on exchange rate.

Suppose we are at time t. First, we sample with replacement from the i.i.d. standard residuals we obtained in Section 4 and will call it $\hat{z}_t$. At the same time, we construct $\hat{\sigma}_t^2$ using $\hat{\varepsilon}_{t-1}$ and $\hat{\sigma}_{t-1}$ (equation 18). Then, we rebuild the residual $\hat{\varepsilon}_t$ by using Equation 17. Finally, the negative returns on exchange rates will be constructed by Equation 16. We set the initial values of ε and σ ($\hat{\varepsilon}_0$ and $\hat{\sigma}_0$) as the means of these two variables in the original series evaluated in Section 4.

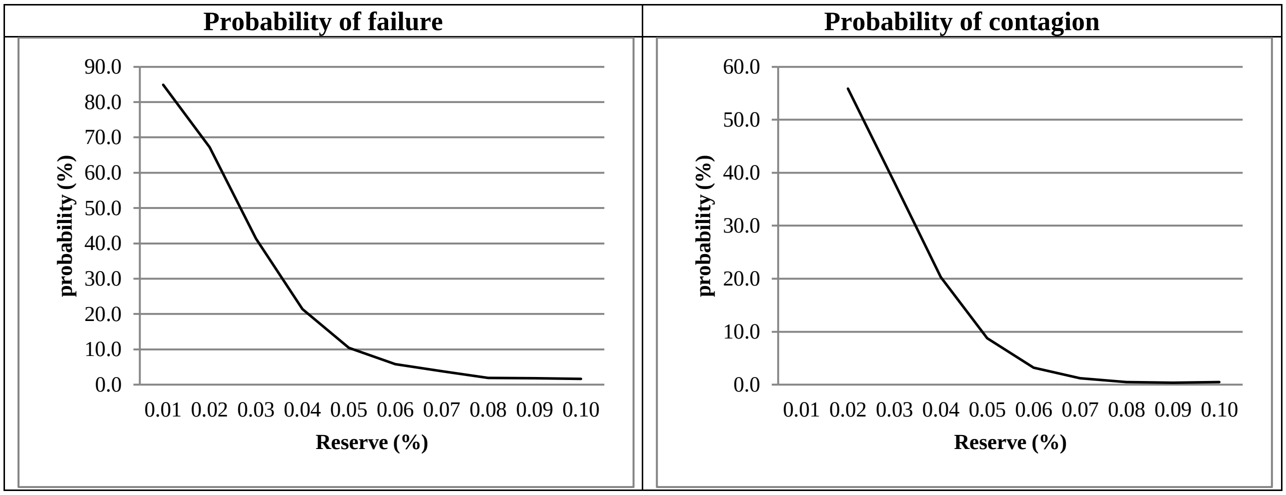
[6] As mentioned earlier, we are interested in showing the procedure of estimating the probability of failure and contagion, not the actual estimates. Therefore, the probabilities obtained in this section may not be true in reality. Estimating the correct probabilities needs more rigorous estimates of bank assets and considering all shocks to the banking system.

After constructing $\hat{r}_t$, we calculate the level change in the exchange rate $\Delta\hat{R}_t = \hat{R}_t - \hat{R}_{t-1}$, where $\hat{R}_t = \hat{R}_{t-1}exp(-\hat{r}_t)$. $\hat{R}_0$ is set to be the mean of R_t in the original series. At this point, we assess whether this change makes a failure and contagion in the banking system: if $-A_i.\Delta\hat{R}_t$ is less than the reserve value bank *i* keeps to cover the shocks, a failure happens. A_i is the value of foreign asset at bank *i*. If at least one of the banks fails, we test for the possibility of contagion: first, we update the assets of all banks. This includes two steps. In the first step, we calculate the remaining assets of the failed bank(s). After paying off the deposits, the remaining assets will be distributed among other banks according to their interbank loan ratios. In the next step, the net asset of all non-failed banks will be calculated with respect to the current exchange rate and interbank losses. If the net asset of one of the banks becomes negative at this time, contagion happens. Since the exchange rate has dropped and the value of banks' assets is lower, contagion due to default in repaying interbank loans is more probable now.

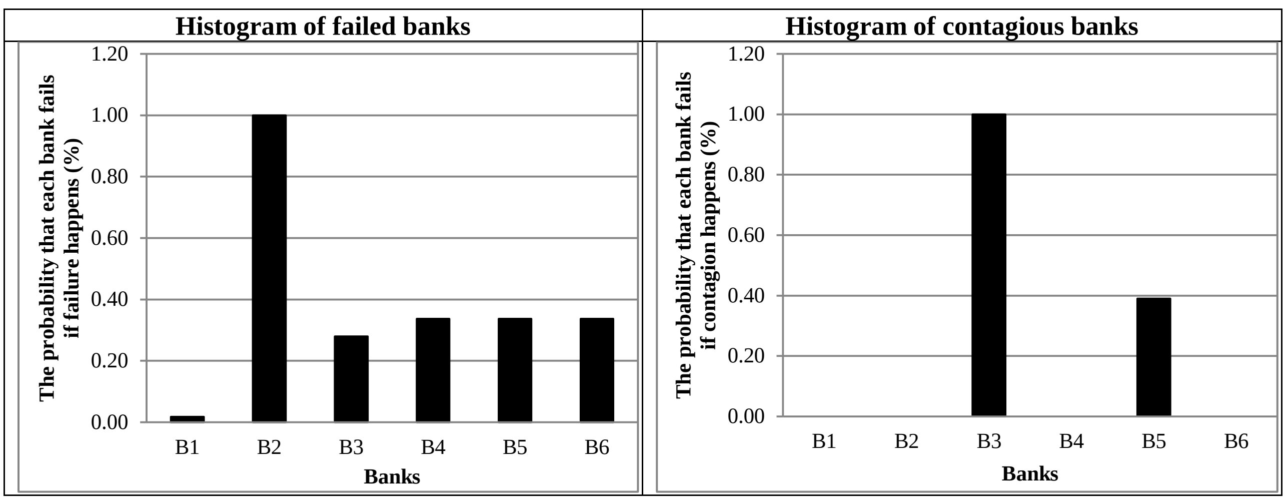
If no failure happens at time *t*, we assume the assets will return to the same level as before and the same process will be followed at time *t*+1. However, if a failure happens, we record it and will start a new series. This way, we can bootstrap new replications of counterfactual time series of exchange returns and assess whether there would a failure or not.

Graph 12 presents the probability of failure and contagion for $n = 1000$ replications. B1 to B6 are the six largest Canadian banks considered for this study. As expected, the probability of failure and contagion decreases as banks increase their reserve ratio. When the reserve rate is as low is 1% of bank assets, we see failures in the network over 80% of time. The probability of failure and contagion reduces significantly as banks keep 10% of their assets in reserve.

Graph 12- Probability of failure and contagion by reserve value



Graph 13- Histogram of failure and contagion by bank



Graph 13 presents the distribution of the failed and contagious banks. B2 is always the first bank that fails. The reason is that B2 has the largest ratio of international assets to total assets. In fact, there is a negative relationship between a bank’s ratio of foreign assets to total assets and the probability of failure of the bank. On the other hand, B3 fails due to contagion whenever there is contagion in the system. B3 has the highest ratio of interbank loans to total assets.

One disadvantage of Monte Carlo simulation in estimating failure in the banking system is that failure and contagion are rare events by nature (reserve levels are kept high enough to cover the shocks). Therefore, estimating the probability of failure will be very computer intensive when the reserve ratio is high. In the next section, we show how the extreme value theory and importance sampling could improve the Monte Carlo simulation in this context.

5.2. Monte Carlo simulation using EVT

This section combines the Monte Carlo analysis of Section 5.1 with the EVT and importance sampling techniques introduced in Section 3. The idea is to increase the probability of failure in the network, and then adjust the probability with an appropriate change of measure. This task is done in two steps. First, the tail of the empirical distribution of the exchange rates will be substituted with the equivalent exponential distribution we estimated in Section 4 using the EVT. Then, we will use importance sampling to increase the probability of failure. In what follows, we explain this process in more details on how to use the EVT and importance sampling in the simulation.

As Section 5.1, the negative exchange rate changes are reconstructed by using equations (16) - (19). $\hat{z}_t$'s are sampled again with replacement from the i.i.d. standard residuals we obtained in Section 4.1. The only difference with Section 5.1 is that if $\hat{z}_t$ is larger than the threshold u = 1.5[7], we replace it with an *exponential*(λ) random variable. This replacement is justified by the EVT estimation we did in Section 4.2 where we showed the tail part of the residuals exhibits an exponential distribution with rate λ = 1/0.59. We also propose that with an appropriate change of measure, we could use another exponential distribution with a rate $\hat{\lambda} < \lambda$ to increase the probability of failure in the system. This is justified with the assumption that most failures

[7] In fact we do this replacement if $1.5 < \hat{z}_t < 34$, since we exclude the outlier from the model.

happen with larger shocks (i.e. in the tail of residual distribution). The rest of the process is the same as Section 5.1: we calculate the level changes in the exchange rate $\Delta\hat{R}_t = \hat{R}_t - \hat{R}_{t-1}$ and will assess whether a failure and contagion will happen in the system. We did two types of tests on this method as follows:

i) In the first step, we ran some simulations with $\lambda = 1/0.59$ to verify if this method works. The simulation results confirm that this change works quite well and the estimated results under this approach are very close to the non-parametric results obtained in Section 5.1 for all reserve rates. The only difference is that since we rely on a parametric model – versus the empirical distribution in Section 5.1 – we may also receive larger, out-of-sample shocks since the counterfactual shocks are not bound by the previous empirical shocks anymore. This improvement is one of the advantages of replacing the tail of the empirical distribution with a parametric distribution (EVT).

ii) In the second step, we want to increase the failure rate in the system to have an estimate of the probability of failure and contagion when the reserve ratio is high. The approach we took was to replace $\hat{z}_t > 1.5$ with an *exponential*($\lambda^{'} < 1/0.59$), and then use the likelihood ratio $\omega^*(z) = f(z|\lambda = 1/0.59)/f(z|\lambda')$ to adjust for the change of measure whenever this replacement happens in the reconstructed series. At the end, if a failure happens under this series, these likelihood ratios are multiplied to obtain the probability of this series to occur.

The first result we obtained was that the number of failures and contagions in the system increased as we increased the mean of the distribution $\lambda^{'}$. This result is consistent with theory and one of the advantages of adding importance sampling to this type of modeling. However, when the probability of failure was corrected for this change of measure, two problems happened. On one hand, the probability of failure was usually under-estimated. On the other hand, the estimated probabilities were too volatile. The conclusion of this section is that although combining EVT and importance sampling techniques with the Monte Carlo method can significantly increase the number of failures in the system and also include out-of-sample shocks, finding an appropriate change of measure would be difficult.

There can be different reasons why the probability of failures is underestimated under this approach. First, the shocks created under this method are sometimes unnecessarily too large. Even though these large shocks cause a failure in the system, since the probability of them are very low, the twisted measures will be very low as well. This could also significantly increase the variance of the shocks which could affect the change of the measure estimates when the simulation is based on low iterations. Second, consistent with other studies, we did not create a dynamic model of bank assets and the changes that occur in the system due to the changes in the exchange rates. This means if a failure does not happen in the system, the bank assets remain the same. Even though this set up simplifies the modeling of the banking system significantly, we may miss the cases where the failure may happen due to incremental shocks to the system. These modifications keep the door open for further research on this type of modeling.

6. Conclusion

This study simulated the shocks to a banking network and estimated the probability of failure and contagion in the system. We considered two sources of shocks: external shocks due to market and macroeconomic factors which impact the entire banking system, and idiosyncratic shocks due to failure of a single bank. We used bootstrap simulation and the extreme value theory to model the shocks. In doing so, we also used different techniques such as GARCH and importance sampling. GARCH modeling and other time series techniques have been used to convert the shocks to an i.i.d. sequence. Later, we showed that importance sampling could be used to accelerate the probability of failure in the system. We calibrated the model such that the banking network resembles the Canadian banking system.

One of the interesting findings of the empirical section of the paper is that the GARCH/ARMA model could identify large shocks to the US-Canada exchange rates, especially the one that happened on 06/01/1970 when Canada went off fixed exchange rates.

Moreover, the simulation results confirm that the probability of failure and contagion in the system decreases as the reserve ratio increases. The simulation results also show that the failure of banks due to exchange rate shocks has a positive relationship with the ratio of foreign assets to total assets. Similarly, having a high ratio of interbank loans increases the probability of contagion.

Another result is that the exeedances over a threshold can be replaced with the estimated EVT model and obtain the same simulated results. The advantage of using an EVT model is that importance sampling or other techniques could then be used to generate larger shocks and increase the number of failures in the system to obtain more accurate estimates or to reduce the number of iterations in the Monte Carlo simulation in the case of high reserve ratios (i.e. rare events). Also, out-of-sample shocks can be included in the model. Preliminary results show that combining importance sampling and EVT can significantly increase the number of failure and contagion in the system, though the standard deviation of the estimator can be large if the change of measure is naive. Finding a good change of measure could be one of the extensions of this study. The model can also be improved by better calibrating the banking system and the shocks.

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