ESTIMATION OF A PARTIALLY LINEAR VARYING-COEFFICIENT EV MODEL UNDER RESTRICTED CONDITION

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Abstract: In this paper,we study a partially linear varying-coefficient errors-in-variables (EV) model under additional restricted condition.Both of the parametric and nonparametric components are measured with additive errors. The restricted estimators of parametric and nonparametric components are established based on modified profile least-squares method and local correction method,and their asymptotic properties are also studied under some regularity conditions.Some simulation studies are conducted to illustrate our approaches.

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1 Introduction

The varying-coefficient partially linear model takes the following form:

$$Y = X^\tau \beta + Z^\tau \alpha(T) + \varepsilon, \tag{1}$$

where $\alpha(\cdot) = (\alpha_1(\cdot), \cdots, \alpha_q(\cdot))^\tau$ is a q-dimensional vector of unknown coefficient functions, $\beta = (\beta_1, \cdots, \beta_p)^\tau$ is a p-dimensional vector of unknown regression coefficients and ε is an independent random error with $E(\varepsilon) = 0, Var(\varepsilon) = \sigma^2$ almost certain. Model(1.1) has been studied in a great deal of literature. Examples can be found in the studies of Zhang et al.[9], Zhou and You[10],Xia and Zhang[11],Fan and Huang[12], among others. However,the covariates X,Z are often measured with errors in many practical applications.Some authors consider the case where the covariate X is measured with additive errors,and Z and T are errors free. For example, You and Chen[1] have proposed a modified profile least squares approach to estimate the parametric component. Hu et al.[2]and Wang et al. [3] have obtained confidence region of the parametric component by the empirical likelihood method. Some authors such as Feng[4] consider the case where the covariate Z is measured with additive errors,and X and T are errors free.

In this paper, we discuss the following model in which both of the parametric and nonparametric components are measured with additive errors.

$$\begin{cases} Y = X^\tau \beta + Z^\tau \alpha(T) + \varepsilon, \\ V = X + \eta, \\ W = Z + u, \\ A\beta = b, \end{cases} \tag{2}$$

where η,u are the measurement errors,η is independent of $(X^\tau, Z^\tau, T, \varepsilon, u)$,$u$ is independent of $(X^\tau, Z^\tau, T, \varepsilon, \eta)$. We also assume that $Cov(\eta) = \Sigma_\eta, Cov(u) = \Sigma_u$, where Σ_η,Σ_u is known.If Σ_η,Σ_u is unknown,we also can estimate them by repeatedly measuring V, W. A is a $k \times p$ matrix of known constants and b is a k-vector of known constants. We shall also assume that rank$(A) = k$.

2 Estimation

Suppose that $\{V_i, W_i, T_i, Y_i), i = 1, \cdots, n\}$ is an independent identically distributed(iid) random sample which comes from model (1.2).That is,they satisfy

$$\begin{cases} Y_i = X_i{}^\tau \beta + Z_i{}^\tau \alpha(T_i) + \varepsilon_i, \\ V_i = X_i + \eta_i, \\ W_i = Z_i + u_i, \end{cases} \tag{3}$$

where the explanatory variable X_i is measured with additive errors, $V_i = (V_{i1}, \cdots, V_{ip})^\tau$ is the surrogate variable of X_i, the explanatory variable Z_i is also measured with additive errors, $W_i = (W_{i1}, \cdots, W_{iq})^\tau$ is the surrogate variable of Z_i, $\alpha(T_i) = (\alpha_1 T(_i), \cdots, \alpha_q(T_i))^\tau$, and $\{\varepsilon_i\}_{i=1}^n$ are independent and identically distributed(iid) random errors with $E(\varepsilon_i) = 0$,$Var(\varepsilon_i) = \sigma^2 < \infty$.

We first assume that β is known,then the first equation of model (2.1) can be rewritten as

$$Y_i - X_i{}^\tau \beta = Z_i{}^\tau \alpha(T_i) + \varepsilon_i, \quad i = 1, \cdots, n \tag{4}$$

Clearly, model (2.2) can be treated as a varying coefficient model.Then, we apply a local linear regression technique to estimate the varying coefficient functions $\alpha(T)$. For T_i in a small neighborhood of T, one can approximate $\alpha_j(T_i)$ locally by a linear function

$$\alpha_j(T_i) \approx \alpha_j(T) + \alpha_j'(T)(T_i - T) \equiv a_j + b_j(T_i - T),\ j = 1, \cdots, q, \tag{5}$$

This leads to the following weighted local least-squares problem: find $\{(a_j, b_j), j = 1, \cdots, q\}$ to minimize

$$\sum_{i=1}^{n}\{(Y_i - X_i^\tau \beta) - \sum_{i=1}^{q}[a_j + b_j(T_i - T)]Z_{ij}\}^2 K_h(T_i - T), \tag{6}$$

where K is a kernel function, h is a bandwidth and $K_h(\cdot) = K(\cdot/h)/h$.

The solution to problem (2.4) is given by

$$(\hat{a}_1, \cdots, \hat{a}_q, \cdots, h\hat{b}_1, \cdots, h\hat{b}_q) = \{(D_T^Z)^\tau \omega_T D_T^Z\}^{-1}(D_T^Z)^\tau \omega_T (Y - X\beta), \tag{7}$$

where

$$D_T^Z = \begin{pmatrix} Z_1^\tau & \frac{T_1 - T}{h} Z_1^\tau \\ \vdots & \vdots \\ Z_n^\tau & \frac{T_n - T}{h} Z_n^\tau \end{pmatrix}; \quad M = \begin{pmatrix} Z_1^\tau \alpha(T_1) \\ \vdots \\ Z_n^\tau \alpha(T_n) \end{pmatrix}; \quad Z = (Z_1, Z_2, \cdots, Z_n)^\tau;$$

$Y = (Y_1, Y_2, \cdots, Y_n)^\tau$;$X = (X_1, X_2, \cdots, X_n)^\tau$;$\omega_T = diag(K_h(T_1 - T), \cdots, K_h(T_n - T))$. If one ignores the measurement error and replaces Z_i by W_i in (2.5), one can show that the resulting estimator is inconsistent.To eliminate the estimation error caused by the measurement error,Using the method in literature[5], we modify (2.5) by local correction as follow:

$$(\hat{a}_1, \cdots, \hat{a}_q, \cdots, h\hat{b}_1, \cdots, h\hat{b}_q) = \{(D_T^W)^\tau \omega_T D_T^W - \Omega\}^{-1}(D_T^W)^\tau \omega_T (Y - X\beta), \tag{8}$$

then we obtain the following corrected local linear estimator for $\{\alpha(\cdot), j = 1, \cdots, q\}$ as

$$\hat{\alpha}(T) = (\hat{\alpha}_1(T), \cdots, \hat{\alpha}_q(T))^\tau = (I_q\ 0_q)\{(D_T^W)^\tau \omega_T D_T^W - \Omega\}^{-1}(D_T^W)^\tau \omega_T (Y - X\beta), \tag{9}$$

where,$\Omega = \sum_{i=1}^{n} \Sigma_u \otimes \begin{pmatrix} 1 & (T_i - T)/h \\ (T_i - T)/h & ((T_i - T)/h)^2 \end{pmatrix} K_h(T_i - T)$.

For the sake of descriptive convenience,we denote $R_i = \{(D_{T_i}^W)^\tau \omega_{T_i} D_{T_i}^W - \Omega\}^{-1}(D_{T_i}^W)^\tau \omega_{T_i}$, $S_i = (W_i^\tau\ 0_q^\tau)R_i$, $Q_i = (I_q\ 0_q)R_i$, $S = (S_1^\tau, \cdots, S_n^\tau)^\tau$, $\tilde{Y}_i = Y_i - S_i Y$, $\tilde{V}_i = V_i - V^\tau S_i^\tau$, then, minimize

$$\sum_{i=1}^{n}\{Y_i - V_i^\tau \beta - W_i^\tau \hat{\alpha}(Ti)\}^2 - \sum_{i=1}^{n} \hat{\alpha}^\tau(T_i)\Sigma_u \hat{\alpha}(T_i) - \sum_{i=1}^{n} \beta^\tau \Sigma_\eta \beta, \tag{10}$$

we obtain the modified profile least squares estimator of β

$$\hat{\beta} = \{\sum_{i=1}^{n}(\tilde{V}_i \tilde{V}_i^\tau - V^\tau Q_i^\tau \Sigma_u Q_i V - \Sigma_\eta)\}^{-1}\{\sum_{i=1}^{n}(\tilde{V}_i \tilde{Y}_i - V^\tau Q_i^\tau \Sigma_u Q_i Y)\}, \tag{11}$$

Moreover, the estimator of $\alpha(\cdot)$ is obtained as

$$\tilde{\alpha}(T) = (\tilde{\alpha}_1(T), \cdots, \tilde{\alpha}_q(T))^\tau = (I_q\ 0_q)\{(D_T^W)^\tau \omega_T D_T^W - \Omega\}^{-1}(D_T^W)^\tau \omega_T (Y - V\hat{\beta}). \quad (12)$$

As for the estimator $\hat{\beta}$ is consistent and asymptotically normal. However, restriction conditions $A\beta = b$ were not satisfied. In order to solve this problem, we will construct a restricted estimator, which is not only consistent but also satisfies the linear restrictions. To apply the Lagrange multiplier technique, we define the following Lagrange function corresponding to the restrictions $A\beta = b$ as

$$F(\beta, \lambda) = \sum_{i=1}^n \{Y_i - V_i^\tau \beta - W_i^\tau \hat{\alpha}(Ti)\}^2 - \sum_{i=1}^n \hat{\alpha}^\tau(T_i)\Sigma_u \hat{\alpha}(T_i) - \sum_{i=1}^n \beta^\tau \Sigma_\eta \beta + 2\lambda^\tau (A\beta - b), \quad (13)$$

where λ is a $k \times 1$ vector that contains the Lagrange multipliers. By differentiating $F(\beta, \lambda)$ with respect to β and λ, we obtain the following equations:

$$\frac{\partial F(\beta, \lambda)}{\partial \beta} = \big[\sum_{i=1}^n (\tilde{V}_i \tilde{Y}_i - V^\tau Q_i^\tau \Sigma_u Q_i Y) - A^\tau \lambda\big] - \sum_{i=1}^n (\tilde{V}_i \tilde{V}_i^\tau - V^\tau Q_i^\tau \Sigma_u Q_i V - \Sigma_\eta)\beta = 0, \quad (14)$$

$$\frac{\partial F(\beta, \lambda)}{\partial \lambda} = 2(A\beta - b) = 0, \quad (15)$$

Solving the Eq. (2.12) with respect to β , we get

$$\beta = \hat{\beta} - \big\{\sum_{i=1}^n (\tilde{V}_i \tilde{V}_i^\tau - V^\tau Q_i^\tau \Sigma_u Q_i V - \Sigma_\eta)\big\}^{-1} A^\tau \lambda.$$

We substitute β into the Eq. (2.13) and we have

$$b = A\hat{\beta} - A\big\{\sum_{i=1}^n (\tilde{V}_i \tilde{V}_i^\tau - V^\tau Q_i^\tau \Sigma_u Q_i V - \Sigma_\eta)\big\}^{-1} A^\tau \lambda.$$

As the inverse matrix of $A\big\{\sum_{i=1}^n (\tilde{V}_i \tilde{V}_i^\tau - V^\tau Q_i^\tau \Sigma_u Q_i V - \Sigma_\eta)\big\}^{-1} A^\tau$ exists, then we can write the estimator of λ as

$$\hat{\lambda} = \big\{A\big[\sum_{i=1}^n (\tilde{V}_i \tilde{V}_i^\tau - V^\tau Q_i^\tau \Sigma_u Q_i V - \Sigma_\eta)\big]^{-1} A^\tau\big\}^{-1}(A\hat{\beta} - b). \quad (16)$$

Then, the restricted estimator of β is obtained as

$$\hat{\beta}_r = \hat{\beta} - \big\{\sum_{i=1}^n (\tilde{V}_i \tilde{V}_i^\tau - V^\tau Q_i^\tau \Sigma_u Q_i V - \Sigma_\eta)\big\}^{-1} A^\tau \big\{A\big[\sum_{i=1}^n (\tilde{V}_i \tilde{V}_i^\tau - V^\tau Q_i^\tau \Sigma_u Q_i V - \Sigma_\eta)\big]^{-1} A^\tau\big\}^{-1}(A\hat{\beta} - b), \quad (17)$$

Moreover, the restricted estimator of $\alpha(\cdot)$ is obtained as

$$\tilde{\alpha}_r(T) = (I_q\ 0_q)\{(D_T^W)^\tau \omega_T D_T^W - \Omega\}^{-1}(D_T^W)^\tau \omega_T (Y - V\hat{\beta}_r). \quad (18)$$

3 Asymptotic normality

The following assumption will be used.

A1. The random variable T has a bounded support $\Im$. Its density function $f(\cdot)$ is Lipschitz continuous and $f(\cdot) > 0$.

A2. There is an $s > 2$, such that $E\|\varepsilon_1\|^{2s} < \infty, E\|u_1\|^{2s} < \infty, E\|\eta_1\|^{2s} < \infty, E\|X_1\|^{2s} < \infty, E\|Z_1\|^{2s} < \infty$, and for some $\delta < 2 - s^{-1}$, there is $n^{2\delta-1}h \to \infty$ as $n \to \infty$.

A3. $\{\alpha_j(\cdot), j = 1, \cdots, q\}$ have continuous second derivatives in $T \in \Im$.

A4. The function $K(\cdot)$ is a symmetric density function with compact support. and the bandwidth h satisfies $nh^2/(\log n)^2 \to \infty, nh^8 \to \infty$ as $n \to \infty$.

A5. The matrix $\Gamma(T) = E(Z_1 Z_1^\tau|T)$ is nonsingular,$E(X_1 X_1^\tau|T)$ and $\Phi(T) = E(Z_1 X_1^\tau|T)$ are all Lipschitz continuous.

The following notations will be used.

Let $c_n = \{(nh)^{-1}\log n\}^{1/2}, \tilde{X}_i = X_i - X^\tau S_i^\tau, \tilde{\eta}_i = \eta_i - \eta^\tau S_i^\tau, \tilde{\varepsilon}_i = \varepsilon_i - \varepsilon^\tau S_i^\tau, \mu_k = \int_{-\infty}^{+\infty} t^k K(t)dt, \nu_k = \int_{-\infty}^{+\infty} t^k K^2(t)dt, k = 0, 1, 2, 3.$

Theorem 1. Assume that the conditions A1-A5 hold, Then the estimator $\hat{\beta}_r$ of β is asymptotically normal,namely,

$$\sqrt{n}(\hat{\beta}_r - \beta) \to_L N(0, \Sigma),$$

where $\to_L$ denotes the convergence in distribution, and

$$\Sigma = \Sigma_1^{-1}\Lambda\Sigma_1^{-1} - \Sigma_1^{-1}\Sigma_2\Sigma_1^{-1}\Lambda\Sigma_1^{-1} - \Sigma_1^{-1}\Lambda\Sigma_1^{-1}\Sigma_2\Sigma_1^{-1} + \Sigma_1^{-1}\Sigma_2\Sigma_1^{-1}\Lambda\Sigma_1^{-1}\Sigma_2\Sigma_1^{-1},$$

$$\Sigma_1 = E(X_1 X_1^\tau) - E(\Phi^\tau(T_1)\Gamma^{-1}(T_1)\Phi(T_1)),$$

$$\Sigma_2 = A^\tau(A\Sigma_1^{-1}A^\tau)^{-1}A,$$

$$\begin{aligned}\Lambda = {} & E(\varepsilon_1 - u_1^\tau\alpha(T_1) - \eta_1^\tau\beta)^2\Sigma_1 + E(\varepsilon_1 - \eta_1^\tau\beta)^2 E\{\Phi^\tau(T_1)\Gamma^{-1}(T_1)\Sigma_u\Gamma^{-1}(T_1)\Phi(T_1)\} \\ & + E\{\Phi^\tau(T_1)\Gamma^{-1}(T_1)(u_1u_1^\tau - \Sigma_u)\alpha(T_1)\}^{\otimes 2} + E(\varepsilon_1 - u_1^\tau\alpha(T_1))^2\Sigma_\eta \\ & + E\{(\eta_1\eta_1^\tau - \Sigma_\eta)\beta\beta^\tau(\eta_1\eta_1^\tau - \Sigma_\eta)\},\end{aligned}$$

$A^{\otimes 2}$ means AA^τ.

Theorem 2. Assume that the conditions A1-A5 hold, Then

$$\sqrt{nh}(\tilde{\alpha}_r(T) - \alpha(T) - \frac{1}{2}h^2\frac{\mu_2^2 - \mu_1\mu_3}{\mu_2 - \mu_1^2}\alpha''(T)) \to_L N(0, \Delta),$$

where $\Delta = \frac{\mu_2^2 v_0 - 2\mu_1\mu_2 v_1 + \mu_1^2 v_2}{(\mu_2 - \mu_1^2)^2} f(T)^{-1}\Sigma^*$; $\Sigma^* = \Gamma^{-1}(T)\Big[E(\varepsilon_1 - \eta_1^\tau\beta)^2\Gamma(T) + E(\varepsilon_1 - \eta_1^\tau\beta)^2\Sigma_u + E\{\xi_1\alpha(T)\alpha^\tau(T)\xi_1^\tau\}\Big]\Gamma^{-1}(T)$; $\xi_1 = \Sigma_u - u_1u_1^\tau - Z_1u_1^\tau$.

4 Simulation

We illustrate the proposed method through a simulated example. The data are generated from the following model

$$Y = sin(32t)X_1 + 2Z_1 + 3Z_2 + \varepsilon, V_1 = X_1 + \eta_1, W_1 = Z_1 + u_1, W_2 = Z_2 + u_2, \quad (19)$$

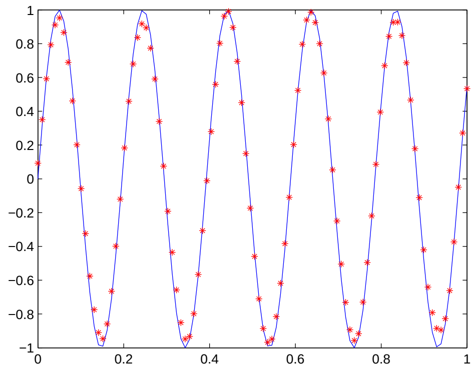
where $X_1 \sim N(5,1), Z_1 \sim N(1,1), Z_2 \sim N(1,1),\ \eta_1 \sim N(0,0.16), u_1 \sim N(0,0.25), u_2 \sim N(0,0.25)$.To gain an idea of the effect of the distribution of the error on our results, we take the following two different types of the error distribution,(1)$\varepsilon \sim N(0,0.16)$,(2)$\varepsilon \sim U(-1,1)$.The kernel function $K(x) = \frac{3}{4}(1-x^2)I_{|x|\leq 1}$ and bandwidth $h = \frac{1}{40}$ are used in our simulation studies,respectively.

For model (4.1) with restriction condition $\beta_1 + \beta_2 = 5$, We compare the performance of the unrestricted estimator with that of the restricted estimator in terms of sample mean (Mean), sample standard deviation (SD) and sample mean squared error (MSE). Simulations with sample size $n = 100, 200$. The simulation results are presented in Table 1.We can find that all the estimators of parameters are close to the true value. As the sample size increases, the biases, standard deviation and sample mean squared error of all the estimators decrease. It is noted that in all the scenarios we studied, the restricted corrected profile least-squares estimator of the parametric component outperforms the corresponding unrestricted estimator. The results are robust to the choice of error distributions.

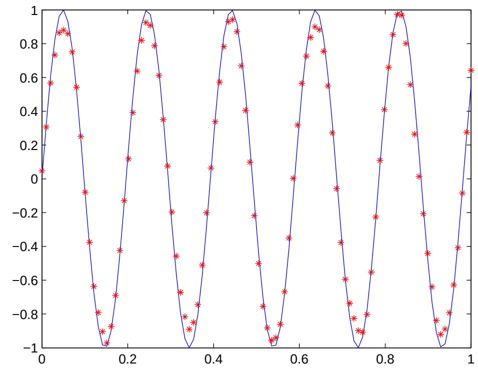
In addition, when the sample size is 200, we plot the estimated curve of the nonparametric component in Figure 1,2. $*$ indicate estimated value ,and use solid-line curve indicate actual value.then ,we found estimated results is fine.

Table 1: Finite sample performance of the restricted and unrestricted estimators

β	Error	n	Unrestricted			Restricted		
			Mean	SD	MSE	Mean	SD	MSE
$\beta_1 = 2$	$N(0,0.4^2)$	100	2.0441	0.0805	0.9984	2.0284	0.0547	0.0725
		200	1.9666	0.0637	0.0307	2.0095	0.0376	0.0246
	$U(-1,1)$	100	2.0514	0.0742	0.0528	2.0468	0.0541	0.0237
		200	1.9876	0.0652	0.0161	2.0109	0.0388	0.0129
$\beta_2 = 3$	$N(0,0.4^2)$	100	2.9262	0.0793	0.0865	2.9716	0.0547	0.0725
		200	2.9459	0.0669	0.0377	2.9905	0.0376	0.0246
	$U(-1,1)$	100	2.9497	0.0824	0.0318	2.9532	0.0541	0.0237
		200	2.9626	0.0679	0.0211	2.9891	0.0388	0.0129



$sin(32t)(\varepsilon N(0,0.4^2)$



$sin(32t)(\varepsilon U(-1,1))$

5 Proof

Lemma 1. Suppose that the conditions (A1)-(A5) hold, as $n \to \infty$, then

$$\sup_{T\in\Im}\left|\frac{1}{nh}\sum_{i=1}^{n}K(\frac{T_i-T}{h})(\frac{T_i-T}{h})^k Z_{ij_1}Z_{ij_2} - f(T)\Gamma_{j_1j_2}(T)\mu_k\right| = O(h^2+c_n)\ a.s.\ ,$$

$$\sup_{T\in\Im}\left|\frac{1}{nh}\sum_{i=1}^{n}K(\frac{T_i-T}{h})(\frac{T_i-T}{h})^k Z_{ij}\varepsilon_i\right| = O(c_n)\ a.s.\ ,$$

$$\sup_{T\in\Im}\left|\frac{1}{nh}\sum_{i=1}^{n}K(\frac{T_i-T}{h})(\frac{T_i-T}{h})^k Z_{ij}u_{ij}\right| = O(c_n)\ a.s.\ ,$$

where $j, j_1, j_2 = 1,\cdots,q; k = 0,1,2,3$. The proof of Lemma 1 can be found in Xia[6].

Lemma 2. Suppose that the conditions (A1)-(A5) hold, then

$$(D_T^W)^\tau \omega_T D_T^W - \Omega = nf(T)\Gamma(T)\otimes\begin{pmatrix}1 & \mu_1\\ \mu_1 & \mu_2\end{pmatrix}\{1+O_p(c_n)\},$$

$$(D_T^W)\omega_T V = nf(T)\Phi(T)\otimes(1,\mu_1)^\tau\{1+O_p(c_n)\},$$

$$(D_T^W)\omega_T W = nf(T)\Gamma(T)\otimes(1,\mu_1)^\tau\{1+O_p(c_n)\}.$$

The proof of Lemma 2 is similar to that of Lemma A.2 in Wang[3]. We here omit the detail.

Lemma 3. Let $G_1,\cdots,G_n$ be independent and identically distributed random variables.If $E|G_i|^s$ is bounded for $s>1$,then $\max_{1\le i\le n}|G_i|^s = o(n^{1/s})\ a.s.$.

The proof of Lemma 3 can be found in Shi[13].We here omit the detail.

Lemma 4. Suppose that the conditions (A1)-(A5) hold, then

$$\frac{1}{n}\sum_{i=1}^{n}\{\tilde{V}_i\tilde{V}_i^\tau - V^\tau Q_i^\tau\Sigma_u Q_i V - \Sigma_\eta\} \to E(X_1X_1^\tau) - E(\Phi^\tau(T_1)\Gamma^{-1}(T_1)\Phi(T_1))\ a.s.\ .$$

The proof of Lemma 4 is similar to that of Lemma 7.2 in Fan[12]. We here omit the detail.

Lemma 5. Assume that the conditions A1-A5 hold, Then the estimator $\hat{\beta}$ of β is asymptotically normal,namely,

$$\sqrt{n}(\hat{\beta}-\beta) \to_L N(0, \Sigma_1^{-1}\Lambda\Sigma_1^{-1}).$$

where Σ_1 and Λ are defined in Theorem 1.

Proof. By (2.9),we have

$$\sqrt{n}(\hat{\beta}-\beta) = \sqrt{n}\Big\{\sum_{i=1}^{n}(\tilde{V}_i\tilde{V}_i^\tau - V^\tau Q_i^\tau\Sigma_u Q_i V - \Sigma_\eta)\Big\}^{-1}\Big\{\sum_{i=1}^{n}[\tilde{V}_i(\tilde{Y}_i - \tilde{V}_i^\tau\beta) - V^\tau Q_i^\tau\Sigma_u Q_i(Y - V\beta) + \Sigma_\eta\beta]\Big\},$$

By Lemma 1 , Lemma 2 and Lemma 3 we have

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\Big\{\sum_{i=1}^{n}[\tilde{V}_i(\tilde{Y}_i - \tilde{V}_i^\tau\beta) - V^\tau Q_i^\tau\Sigma_u Q_i(Y - V\beta) + \Sigma_\eta\beta]\Big\}\\
&= \frac{1}{\sqrt{n}}\Big\{\sum_{i=1}^{n}[X_i - \Phi^\tau(T_i)\Gamma^{-1}(T_i)Z_i][\varepsilon_i - u_i^\tau\alpha(T_i) - \eta_i^\tau\beta] - \Phi^\tau(T_i)\Gamma^{-1}(T_i)u_i(\varepsilon_i - \eta_i^\tau\beta)\\
&\quad + \Phi^\tau(T_i)\Gamma^{-1}(T_i)(u_iu_i^\tau - \Sigma_u)\alpha(T_i) + \eta_i(\varepsilon_i - u_i^\tau\alpha(T_i)) - (\eta_i\eta_i^\tau - \Sigma_\eta)\beta + o_p(1)\\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}J_{in} + o_p(1).
\end{aligned}$$

then

$$\begin{aligned}
Cov(J_{in}) = {} & E\{[\varepsilon_i - u_i^\tau\alpha(T_i) - \eta_i^\tau\beta][X_i - \Phi^\tau(T_i)\Gamma^{-1}(T_i)Z_i]\}^{\otimes 2} + E\{\Phi^\tau(T_i)\Gamma^{-1}(T_i)u_i\\
&(\varepsilon_i - \eta_i^\tau\beta)\}^{\otimes 2} + E\{\Phi^\tau(T_i)\Gamma^{-1}(T_i)(u_iu_i^\tau - \Sigma_u)\alpha(T_i)\}^{\otimes 2} + E\{\eta_i(\varepsilon_i - u_i^\tau\alpha(T_i))\}^{\otimes 2}\\
&+E\{(\eta_i\eta_i^\tau - \Sigma_\eta)\beta\}^{\otimes 2}
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}Cov(J_{in}) = {} & E(\varepsilon_1 - u_1^\tau\alpha(T_1) - \eta_1^\tau\beta)^2\Sigma_1 + E(\varepsilon_1 - \eta_1^\tau\beta)^2 E\{\Phi^\tau(T_1)\Gamma^{-1}(T_1)\\
&\Sigma_u\Gamma^{-1}(T_1)\Phi(T_1)\} + E\{\Phi^\tau(T_1)\Gamma^{-1}(T_1)(u_1u_1^\tau - \Sigma_u)\alpha(T_1)\}^{\otimes 2}\\
&+ E(\varepsilon_1 - u_1^\tau\alpha(T_1))^2\Sigma_\eta + E\{(\eta_1\eta_1^\tau - \Sigma_\eta)\beta\beta^\tau(\eta_1\eta_1^\tau - \Sigma_\eta)\}.
\end{aligned}$$

Therefore, by Lemma 4,and central limit theorem, Slutsky theorem, we have

$$\sqrt{n}(\hat{\beta}-\beta) \to_L N(0, \Sigma_1^{-1}\Lambda\Sigma_1^{-1}).$$

Proof of Theorem 1. We first denote that

$$\begin{aligned} J_0 =: & I - \big\{\sum_{i=1}^{n}(\tilde{V}_i\tilde{V}_i^\tau - V^\tau Q_i^\tau \Sigma_u Q_i V - \Sigma_\eta)\big\}^{-1} A^\tau \big\{A\big[\sum_{i=1}^{n}(\tilde{V}_i\tilde{V}_i^\tau - V^\tau Q_i^\tau \Sigma_u Q_i V - \Sigma_\eta)\big]^{-1} A^\tau\big\}^{-1} A \\ = & I - \big\{\frac{1}{n}\sum_{i=1}^{n}(\tilde{V}_i\tilde{V}_i^\tau - V^\tau Q_i^\tau \Sigma_u Q_i V - \Sigma_\eta)\big\}^{-1} A^\tau \big\{A\big[\frac{1}{n}\sum_{i=1}^{n}(\tilde{V}_i\tilde{V}_i^\tau - V^\tau Q_i^\tau \Sigma_u Q_i V - \Sigma_\eta)\big]^{-1} A^\tau\big\}^{-1} A, \end{aligned}$$

By Lemma 4, we obtain

$$J_0 \xrightarrow{P} I - \Sigma_1^{-1}A^\tau[A\Sigma_1^{-1}A^\tau]^{-1}A =: J,$$

By (2.16) ,we have

$$\begin{aligned} \hat{\beta}_r - \beta = & \Big\{I - \big\{\sum_{i=1}^{n}(\tilde{V}_i\tilde{V}_i^\tau - V^\tau Q_i^\tau \Sigma_u Q_i V - \Sigma_\eta)\big\}^{-1} A^\tau \big\{A\big[\sum_{i=1}^{n}(\tilde{V}_i\tilde{V}_i^\tau \\ & - V^\tau Q_i^\tau \Sigma_u Q_i V - \Sigma_\eta)\big]^{-1} A^\tau\big\}^{-1} A\Big\}(\hat{\beta}-\beta) \\ = & J(\hat{\beta}-\beta) + (J_0 - J)(\hat{\beta}-\beta), \end{aligned}$$

Note that $J_0 - J = o_p(1)$ and $\hat{\beta}-\beta = O(n^{-1/2})$.It is easy to check that

$$(J_0 - J)(\hat{\beta}-\beta) = o_p(n^{-1/2}).$$

Invoking the Slutsky theorem and Lemma 5,we obtain the desired result.

Proof of Theorem 2. For T_i in a small neighborhood of T,and let $|T_i - T| < h$, we can approximate $\alpha(T_i)$ by the following Taylor expansion

$$\alpha(T_i) \approx \alpha(T) + \alpha'(T)(T_i - T) + \frac{1}{2}\alpha''(T_i - T)^2 + o_p(h^2),$$

Then, we have

$$M = \begin{pmatrix} Z_1^\tau\alpha(T_1) \\ \vdots \\ Z_n^\tau\alpha(T_n) \end{pmatrix} = D_T^Z \begin{pmatrix} \alpha(T) \\ h\alpha'(T) \end{pmatrix} + \begin{pmatrix} \frac{1}{2}Z_1^\tau\alpha''(T_1)(T_1 - T)^2 \\ \vdots \\ \frac{1}{2}Z_n^\tau\alpha''(T_n)(T_n - T)^2 \end{pmatrix},$$

By the expression of M,it is easy to see that

$$(D_T^W)^\tau\omega_T M = (D_T^W)^\tau\omega_T D_T^Z \begin{pmatrix} \alpha(T) \\ h\alpha'(T) \end{pmatrix} + \frac{h^2}{2}(D_T^W)^\tau\omega_T\Psi_T Z\alpha''(T) + o_p(h^2),$$

where $\Psi_T = diag\{((T_1 - T)/h)^2, \cdots, ((T_n - T)/h)^2\}$.

$$(D_T^W)^\tau \omega_T D_T^Z \begin{pmatrix} \alpha(T) \\ h\alpha'(T) \end{pmatrix} = \{(D_T^W)^\tau \omega_T D_T^W - \Omega\} \begin{pmatrix} \alpha(T) \\ h\alpha'(T) \end{pmatrix} + \{-(D_T^W)^\tau \omega_T D_T^u + \Omega\} \begin{pmatrix} \alpha(T) \\ h\alpha'(T) \end{pmatrix}.$$

$$\{(D_T^W)^\tau \omega_T D_T^W - \Omega\}^{-1} (D_T^W)^\tau \omega_T \Psi_T Z \alpha''(T) = \frac{1}{\mu_2 - \mu_1^2} \begin{pmatrix} (\mu_2^2 - \mu_1\mu_3)\alpha''(T) \\ (\mu_3 - \mu_1\mu_2)\alpha''(T) \end{pmatrix} \{1 + o(1)\}\ a.s.\ ,$$

Recall the definition of $\tilde{\alpha}_r(T)$ in (2.16), we have

$$\begin{aligned}
\tilde{\alpha}_r(T) &= (I_q\ 0_q)\{(D_T^W)^\tau \omega_T D_T^W - \Omega\}^{-1}(D_T^W)^\tau \omega_T (Y - V\hat{\beta}_r) \\
&= (I_q\ 0_q)\{(D_T^W)^\tau \omega_T D_T^W - \Omega\}^{-1}(D_T^W)^\tau \omega_T M \\
&\quad + (I_q\ 0_q)\{(D_T^W)^\tau \omega_T D_T^W - \Omega\}^{-1}(D_T^W)^\tau \omega_T V(\beta - \hat{\beta}_r) \\
&\quad + (I_q\ 0_q)\{(D_T^W)^\tau \omega_T D_T^W - \Omega\}^{-1}(D_T^W)^\tau \omega_T (\varepsilon - \eta\beta) \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

As mentioned above

$$\begin{aligned}
I_1 &= \alpha(T) + \frac{1}{2}h^2 \frac{\mu_2^2 - \mu_1\mu_3}{\mu_2 - \mu_1^2}\alpha''(T) + (I_q\ 0_q)\{(D_T^W)^\tau \omega_T D_T^W - \Omega\}^{-1}\{-(D_T^W)^\tau \omega_T D_T^u + \Omega\} \begin{pmatrix} \alpha(T) \\ h\alpha'(T) \end{pmatrix} \\
&\quad + o_p(h^2),
\end{aligned}$$

By Lemma 1 and Lemma 2, we can obtain

$$(I_q\ 0_q)\{(D_T^W)^\tau \omega_T D_T^W - \Omega\}^{-1}(D_T^W)^\tau \omega_T (D_T^W)^\tau \omega_T V = \Gamma^{-1}(T)\Phi(T)\{1 + O_p(c_n)\},$$

Invoking Theorem 1, we yield that

$$\sqrt{nh} I_2 = \sqrt{nh}\Gamma^{-1}(T)\Phi(T)\{1 + O_p(c_n)\}O(n^{-1/2}) = o_p(1),$$

Similar to that of $A4 \sim A6$ in [5], we have

$$\sqrt{nh}\{(D_T^W)^\tau \omega_T D_T^W - \Omega\}^{-1}\Big\{(D_T^W)^\tau \omega_T(\varepsilon - \eta\beta) + \{-(D_T^W)^\tau \omega_T D_T^u + \Omega\} \begin{pmatrix} \alpha(T) \\ h\alpha'(T) \end{pmatrix} \Big\} \to_L N(0, \Xi)$$

where, Σ^* is defined in Theorem 2,and
$\Xi = f(T)^{-1}\Sigma^* \otimes \frac{1}{(\mu_2 - \mu_1)^2} \begin{pmatrix} \mu_2^2\nu_0 - 2\mu_1\mu_2\nu_1 + \mu_1^2\nu_2 & (\mu_1^2 + \mu_2)\nu_1 - \mu_1\mu_2\nu_0 - \mu_1\nu_2 \\ (\mu_1^2 + \mu_2)\nu_1 - \mu_1\mu_2\nu_0 - \mu_1\nu_2 & \nu_2 - \mu_1(2\nu_1 + \mu_1\nu_0) \end{pmatrix}.$
As mentioned above

$$\sqrt{nh}(\tilde{\alpha}_r(T) - \alpha(T) - \frac{1}{2}h^2 \frac{\mu_2^2 - \mu_1\mu_3}{\mu_2 - \mu_1^2}\alpha''(T)) \to_L N(0, \Delta).$$

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