ENDOMORPHISM RINGS OF MODULES

HAMZA HAKMI

ABSTRACT. Let M be a module over a ring R and E_M the endomorphism ring of M. The concern is study some fundamental properties of E_M when E_M is regular or semipotent. New results obtained include necessary and sufficient conditions for E_M to be regular or semipotent. New substructures of E_M are studied and its relationship with the Tot of E_M.

1. Introduction.

In this paper rings R, are associative with identity unless otherwise indicated. All modules over a ring R are unitary right modules. We write $J(R)$ and $U(R)$ for the Jacobson radical and the group of units of a ring R. A submodule N of a module M is said to be *small* in M, if $N+K \neq M$ for any proper submodule K of M [1]. Also, a submodule Q of a module M is said to be *large* (*essential*) in M if $Q \cap K \neq 0$ for every nonzero submodule K of M [1]. For a submodule N of a module M, we use $N \subseteq^{\oplus} M$ to mean that N is a direct summand of M, and write $N \leq_e M$ and $N \ll M$ to indicate that N is an large, respectively small, submodule of M. We use the notation: $E_M = \mathrm{End}_R(M)$, $\nabla(E_M) = \{\alpha : \alpha \in E_M; \mathrm{Im}(\alpha) \ll M\}$ and $\triangle(E_M) = \{\alpha : \alpha \in E_M; \mathrm{Ker}(\alpha) \leq_e M\}$. It is known that $\nabla(E_M)$ and $\triangle(E_M)$ are ideals in E_M [1].

2. Regular Endomorphism Rings.

We start with the following fundamental lemma which gives information about relationship between any two elements of E_M.

Lemma 2.1. Let M_R be a module and $\alpha, \beta \in E_M$. The following hold:
(1) $\mathrm{Im}(\alpha) + \mathrm{Im}(1-\alpha\beta) = M$.
(2) $\mathrm{Im}(\alpha - \alpha\beta\alpha) = \mathrm{Im}(\alpha) \cap \mathrm{Im}(1-\alpha\beta)$.
(3) $\mathrm{Im}(\beta) + \mathrm{Im}(1-\beta\alpha) = M$.
(4) $\mathrm{Im}(\beta - \beta\alpha\beta) = \mathrm{Im}(\beta) \cap \mathrm{Im}(1-\beta\alpha)$.
(5) $\mathrm{Ker}(\alpha) \cap \mathrm{Ker}(1-\beta\alpha) = 0$.
(6) $\mathrm{Ker}(\alpha - \alpha\beta\alpha) = \mathrm{Ker}(\alpha) + \mathrm{Ker}(1-\beta\alpha)$.
(7) $\mathrm{Ker}(\beta) \cap \mathrm{Ker}(1-\alpha\beta) = 0$.
(8) $\mathrm{Ker}(\beta - \beta\alpha\beta) = \mathrm{Ker}(\beta) + \mathrm{Ker}(1-\alpha\beta)$.

Proof. (1) It is clear, hence $M = \mathrm{Im}(\alpha\beta) + \mathrm{Im}(1-\alpha\beta) \subseteq \mathrm{Im}(\alpha) + \mathrm{Im}(1-\alpha\beta) \subseteq M$. Similarly, (3) holds.

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(2) It is clear that $\mathrm{Im}(\alpha-\alpha\beta\alpha) \subseteq \mathrm{Im}(\alpha) \cap \mathrm{Im}(1-\alpha\beta)$, hence $\mathrm{Im}(\alpha-\alpha\beta\alpha) = \alpha(\mathrm{Im}(1-\beta\alpha)) \subseteq \mathrm{Im}(\alpha)$ and $\mathrm{Im}(\alpha-\alpha\beta\alpha) = (1-\alpha\beta)(\mathrm{Im}(\alpha)) \subseteq \mathrm{Im}(1-\alpha\beta)$.
Let $x \in \mathrm{Im}(\alpha) \cap \mathrm{Im}(1-\alpha\beta)$. Then $x = \alpha(y) = (1-\alpha\beta)(z)$ where $y, z \in M$ and $z = x + \alpha\beta(z) = \alpha(y + \beta(z)) \in \mathrm{Im}(\alpha)$. For $y_0 = y + \beta(z)$; $z = \alpha(y_0)$. Thus, $x = (1-\alpha\beta)(z) = (1-\alpha\beta)\alpha(y_0) = (\alpha-\alpha\beta\alpha)(y_0) \in \mathrm{Im}(\alpha-\alpha\beta\alpha)$. Similarly, (4) holds. (5) and (7) are clear.
(6) It is clear that $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\alpha-\alpha\beta\alpha)$ and $\mathrm{Ker}(1-\beta\alpha) \subseteq \mathrm{Ker}(\alpha-\alpha\beta\alpha)$, so $\mathrm{Ker}(\alpha)+\mathrm{Ker}(1-\beta\alpha) \subseteq \mathrm{Ker}(\alpha-\alpha\beta\alpha)$. Let $y \in \mathrm{Ker}(\alpha-\alpha\beta\alpha)$. Then $\alpha(y) = \alpha\beta\alpha(y)$. Since $y = \beta\alpha(y) + (1-\beta\alpha)(y)$; $\beta\alpha(y) \in \mathrm{Ker}(1-\beta\alpha)$ and $(1-\beta\alpha)(y) \in \mathrm{Ker}(\beta\alpha)$ which implies $y \in \mathrm{Ker}(\alpha) + \mathrm{Ker}(1-\beta\alpha)$. Similarly, (8) holds. □

From Lemma 2.1 we derive the following

Corollary 2.2. Let M_R be a module and $\alpha \in E_M$, $n \in \mathbb{N}^\star$. The following hold:
(1) $M = \mathrm{Im}(\alpha) + \mathrm{Im}(1-\alpha^n)$.
(2) $\mathrm{Im}(\alpha-\alpha^{n+1}) = \mathrm{Im}(\alpha) \cap \mathrm{Im}(1-\alpha^n)$.
(3) $\mathrm{Ker}(\alpha) \cap \mathrm{Ker}(1-\alpha^n) = 0$.
(4) $\mathrm{Ker}(\alpha-\alpha^{n+1}) = \mathrm{Ker}(\alpha) + \mathrm{Ker}(1-\alpha^n)$.

The following Lemma is continuation of Lemma 2.1 [2].

Lemma 2.3. Let M_R be a module and $\alpha, \beta \in E_M$. The following hold:
(1) $1-\alpha\beta$ is onto if and only if $1-\beta\alpha$ is onto.
(2) $1-\alpha\beta$ is one-to-one if and only if $1-\beta\alpha$ is one-to-one.
(3) $1-\alpha\beta \in U(E_M)$ if and only if $1-\beta\alpha \in U(E_M)$.

Proof. (1)($\Rightarrow$). Suppose that $\mathrm{Im}(1-\alpha\beta) = M$. Then $\mathrm{Im}(\beta-\beta\alpha\beta) = \mathrm{Im}(\beta) \cap \mathrm{Im}(1-\beta\alpha) = \mathrm{Im}(\beta(1-\alpha\beta)) = \beta(\mathrm{Im}(1-\alpha\beta)) = \mathrm{Im}(\beta)$ by Lemma 2.1(4). So $\mathrm{Im}(\beta) \subseteq \mathrm{Im}(1-\beta\alpha)$. Thus $M = \mathrm{Im}(\beta) + \mathrm{Im}(1-\beta\alpha) = \mathrm{Im}(1-\beta\alpha)$. Similarly, ($\Leftarrow$) holds.
(2)($\Rightarrow$). Suppose that $\mathrm{Ker}(1-\alpha\beta) = 0$. Let $x \in \mathrm{Ker}(1-\beta\alpha)$. Then $(1-\beta\alpha)(x) = 0$ and that $(\alpha-\alpha\beta\alpha)(x) = (1-\alpha\beta)(\alpha(x)) = 0$, so $\alpha(x) \in \mathrm{Ker}(1-\alpha\beta) = 0$ and $x \in \mathrm{Ker}(\alpha)$. Thus $\mathrm{Ker}(1-\beta\alpha) \subseteq \mathrm{Ker}(\alpha)$. By Lemma 2.1(5), $\mathrm{Ker}(1-\beta\alpha) = \mathrm{Ker}(\alpha) \cap \mathrm{Ker}(1-\beta\alpha) = 0$. Similarly, ($\Leftarrow$) holds.
(3) By (1) and (2). □

An element a of a ring R is called *regular* if $a = aba$ for some $b \in R$. A ring R is called *regular ring* if each $a \in R$ is regular. The next Proposition gives information about $\alpha \in E_M$, when α is a regular element.

Proposition 2.4. Let M_R be a module and $\alpha \in E_M$. The following are equivalent:
(1) There exists $\beta \in E_M$ such that $\alpha = \alpha\beta\alpha$.
(2) $\mathrm{Im}(\alpha)$ and $\mathrm{Ker}(\alpha)$ are direct summands of M.
(3) There exists $\beta \in E_M$ such that $\mathrm{Im}(\alpha) \cap \mathrm{Im}(1-\alpha\beta) = 0$.
(4) There exists $\beta \in E_M$ such that $\mathrm{Ker}(\alpha) + \mathrm{Ker}(1-\beta\alpha) = M$.

Proof. (1) $\Leftrightarrow$ (2). By [5 , Lemma 3.1].

(1) $\Rightarrow$ (3). Suppose that $\alpha = \alpha\beta\alpha$ for some $\beta \in E_M$. Then $\alpha - \alpha\beta\alpha = 0$, by Lemma 2.1, $0 = \mathrm{Im}(\alpha - \alpha\beta\alpha) = \mathrm{Im}(\alpha) \cap \mathrm{Im}(1 - \alpha\beta)$.
(3) $\Rightarrow$ (1). Since $\mathrm{Im}(\alpha - \alpha\beta\alpha) = \mathrm{Im}(\alpha) \cap \mathrm{Im}(1 - \alpha\beta) = 0$ by Lemma 2.1 and our hypothesis, implies that $\alpha - \alpha\beta\alpha = 0$.
(1) $\Rightarrow$ (4). Suppose that $\alpha = \alpha\beta\alpha$ for some $\beta \in E_M$. Then $\alpha - \alpha\beta\alpha = 0$ and $\mathrm{Ker}(\alpha - \alpha\beta\alpha) = M = \mathrm{Ker}(\alpha) + \mathrm{Ker}(1 - \beta\alpha)$ by Lemma 2.1.
(4) $\Rightarrow$ (1). If $M = \mathrm{Ker}(\alpha) + \mathrm{Ker}(1 - \beta\alpha)$; by Lemma 2.1 $\mathrm{Ker}(\alpha - \alpha\beta\alpha) = M$, so $\alpha - \alpha\beta\alpha = 0$. $\square$

The following Theorem describe the principal left and right ideals of E_M when E_M is regular.

Theorem 2.5. Let M be a module with E_M is a regular ring and $\alpha, \beta \in E_M$. The following hold:
(1) $\mathrm{Im}(\alpha) \subseteq \mathrm{Im}(\beta)$ if and only if $\alpha E_M \subseteq \beta E_M$.
(2) $\mathrm{Im}(\alpha) = \mathrm{Im}(\beta)$ if and only if $\alpha E_M = \beta E_M$.
(3) $\alpha E_M = \{\beta : \beta \in E_M;\ \mathrm{Im}(\beta) \subseteq \mathrm{Im}(\alpha)\}$.
(4) $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\beta)$ if and only if $(E_M)\beta \subseteq (E_M)\alpha$.
(5) $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\beta)$ if and only if $(E_M)\alpha = (E_M)\beta$.
(6) $(E_M)\alpha = \{\beta : \beta \in E_M;\ \mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\beta)\}$.

Proof. (1)($\Rightarrow$). Suppose that $\mathrm{Im}(\alpha) \subseteq \mathrm{Im}(\beta)$. Since E_M is regular, there exists $\mu \in E_M$ such that $\beta = \beta\mu\beta$. For $e = \beta\mu$; $e^2 = e \in E_M$ and $\mathrm{Im}(e) = \mathrm{Im}(\beta)$, so $\mathrm{Im}(\alpha) \subseteq \mathrm{Im}(e)$. Since $e = I_{\mathrm{Im}(e)}$; $e\alpha(x) = \alpha(x)$ for all $x \in M$. Thus, $\alpha = e\alpha = \beta\mu\alpha \in \beta E_M$. ($\Leftarrow$) It is clear. (2) and (3) by (1).
(4)($\Rightarrow$). Suppose that $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\beta)$. Since E_M is regular, there exists $\mu \in E_M$ such that $\alpha = \alpha\mu\alpha$. For $e = \mu\alpha$; $e^2 = e \in E_M$. It is clear that $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(e)$. Let $y \in \mathrm{Ker}(e)$. Then $\alpha(y) = \alpha\mu\alpha(y) = \alpha e(y) = 0$, so $y \in \mathrm{Ker}(\alpha)$. Thus, $\mathrm{Ker}(e) \subseteq \mathrm{Ker}(\alpha)$ and that $\beta(\mathrm{Ker}(e)) = \beta(\mathrm{Im}(1 - e)) = \mathrm{Im}(\beta(1 - e)) = 0$, so $\beta(1 - e) = 0$. Since $1 = e + (1 - e)$; $\beta = \beta e = (\beta\mu)\alpha \in (E_M)\alpha$. Thus, $(E_M)\beta \subseteq (E_M)\alpha$. ($\Leftarrow$) It is clear. (5) and (6) by (4). $\square$

3. Semipotent Endomorphism Rings.

An element a of a ring R is called *partially invertible* or *pi* for short, if a is a divisor of an idempotent [2]. The next Proposition gives information about $\alpha \in E_M$, when α is a divisor of an idempotent.

Proposition 3.1. Let M_R be a module and $\alpha \in E_M$. The following are equivalent:
(1) There exists $\beta \in E_M$ such that $\beta = \beta\alpha\beta$.
(2) There exists $\beta \in E_M$ such that $\mathrm{Im}(\alpha\beta)$, $\mathrm{Ker}(\alpha\beta)$ are direct summands of M.
(3) There exists $\beta \in E_M$ such that $\mathrm{Im}(\beta\alpha)$, $\mathrm{Ker}(\beta\alpha)$ are direct summands of M.
(4) There exists $\beta \in E_M$ such that $\mathrm{Im}(\beta) \cap \mathrm{Im}(1 - \beta\alpha) = 0$.
(5) There exists $\beta \in E_M$ such that $\mathrm{Ker}(\beta) + \mathrm{Ker}(1 - \alpha\beta) = M$.

Proof. (1) $\Rightarrow$ (2). Suppose that $\beta = \beta\alpha\beta$ for some $\beta \in E_M$. Then $(\alpha\beta)^2 = \alpha\beta$ and so $\mathrm{Im}(\alpha\beta)$, $\mathrm{Ker}(\alpha\beta) = \mathrm{Im}(1 - \alpha\beta)$ are direct summands of M.
(2) $\Rightarrow$ (1). Suppose that (2) holds. By Lemma 2.1, there exists $g \in E_M$ such that

$(\alpha\beta)g(\alpha\beta) = \alpha\beta$. For $\mu = \beta g\alpha\beta g$; $\mu\alpha\mu = \mu$, gives (1).
Similarly, the equivalence (1) $\Leftrightarrow$ (3) holds.
(1) $\Rightarrow$ (4). Suppose (1) holds. Then $\beta - \beta\alpha\beta = 0$, by Lemma 2.1; $0 = \mathrm{Im}(\beta - \beta\alpha\beta) = \mathrm{Im}(\beta) \cap \mathrm{Im}(1 - \beta\alpha)$, gives (4).
(4) $\Rightarrow$ (1). Since $\mathrm{Im}(\beta - \beta\alpha\beta) = \mathrm{Im}(\beta) \cap \mathrm{Im}(1 - \beta\alpha) = 0$; by Lemma 2.1 and our hypothesis, implies that $\beta - \beta\alpha\beta = 0$.
(1) $\Rightarrow$ (5). If $\beta = \beta\alpha\beta$ for some $\beta \in E_M$; $\beta - \beta\alpha\beta = 0$, so $\mathrm{Ker}(\beta - \beta\alpha\beta) = M = \mathrm{Ker}(\beta) + \mathrm{Ker}(1 - \alpha\beta)$ by Lemma 2.1.
(5) $\Rightarrow$ (1). By Lemma 2.1, $\mathrm{Ker}(\beta - \beta\alpha\beta) = \mathrm{Ker}(\beta) + \mathrm{Ker}(1 - \alpha\beta) = M$, so $\beta - \beta\alpha\beta = 0$. □

Recall that a ring R is a *semipotent* ring by Zhou [6], also called an I_0-ring by Nicholson [4], if every principal left (resp. right) ideal not contained in $J(R)$ contains a nonzero idempotent.

Corollary 3.2. Let M_R be a module. The following are equivalent:
(1) E_M is a semipotent ring.
(2) For every $\alpha \in E_M \setminus J(E_M)$ there exists $\beta \in E_M$ such that $\beta = \beta\alpha\beta$.
(3) For every $\alpha \in E_M \setminus J(E_M)$ there exists $\beta \in E_M$ such that $\mathrm{Im}(\alpha\beta)$, $\mathrm{Ker}(\alpha\beta)$ are direct summands of M.
(4) For every $\alpha \in E_M \setminus J(E_M)$ there exists $\beta \in E_M$ such that $\mathrm{Im}(\beta\alpha)$, $\mathrm{Ker}(\beta\alpha)$ are direct summands of M.
(5) For every $\alpha \in E_M \setminus J(E_M)$ there exists $\beta \in E_M$ such that $\mathrm{Im}(\beta) \cap \mathrm{Im}(1-\beta\alpha) = 0$.
(6) For every $\alpha \in E_M \setminus J(E_M)$ there exists $\beta \in E_M$ such that $\mathrm{Ker}(\beta) + \mathrm{Ker}(1-\alpha\beta) = M$.

Proof. By Proposition 3.1. □

Let M be a module. Write:

$$\nabla_1(E_M) = \{\alpha : \alpha \in E_M, \ \ \mathrm{Im}(1 - \alpha\beta) = M \ \text{ for all } \ \beta \in E_M\}.$$

$$\nabla_2(E_M) = \{\alpha : \alpha \in E_M, \ \ \mathrm{Im}(1 - \beta\alpha) = M \ \text{ for all } \ \beta \in E_M\}.$$

It is clear that $\nabla_1(E_M)$ and $\nabla_2(E_M)$ are non-empty subsets in E_M, ($0 \in \nabla_1(E_M)$, $0 \in \nabla_2(E_M)$). In using Lemma 2.3(1), it is easy to see that $\nabla_1(E_M) = \nabla_2(E_M)$. Therefore, we use the notation:

$$\begin{aligned}\widehat{\nabla}(E_M) &= \{\alpha : \alpha \in E_M, \ \ \mathrm{Im}(1 - \alpha\beta) = M \ \text{ for all } \ \beta \in E_M\}.\\ &= \{\alpha : \alpha \in E_M, \ \ \mathrm{Im}(1 - \beta\alpha) = M \ \text{ for all } \ \beta \in E_M\}.\end{aligned}$$

$\widehat{\nabla}(E_M)$ is a semi-ideal in E_M, which means that it is closed under arbitrary multiplication from either side, hence if $\alpha \in \widehat{\nabla}(E_M)$ and $\lambda \in E_M$; $\mathrm{Im}(1 - (\alpha\lambda)\beta) = \mathrm{Im}(1 - \alpha(\lambda\beta)) = M$ for all $\beta \in E_M$ and $\mathrm{Im}(1 - \beta(\lambda\alpha)) = \mathrm{Im}(1 - (\beta\lambda)\alpha) = M$ for all $\beta \in E_M$. Thus, $\alpha\lambda$, $\lambda\alpha \in \widehat{\nabla}E_M$.
It is clear that $\nabla(E_M) \subseteq \widehat{\nabla}(E_M)$.

Also, let M be a module. Write:

$$\triangle_1(E_M) = \{\alpha : \alpha \in E_M, \ \ \mathrm{Ker}(1 - \alpha\beta) = 0 \ \text{ for all } \ \beta \in E_M\}.$$

$$\triangle_2(E_M) = \{\alpha : \alpha \in E_M, \ \ \mathrm{Ker}(1 - \beta\alpha) = 0 \ \text{ for all } \ \beta \in E_M\}.$$

It is clear that $\triangle_1(E_M)$ and $\triangle_2(E_M)$ are non-empty subsets in E_M, ($0 \in \triangle_1(E_M)$, $0 \in \triangle_2(E_M)$). In using Lemma 2.3(2), it is easy to see that $\triangle_1(E_M) = \triangle_2(E_M)$. Therefore, we use the notation:

$$\widehat{\triangle}(E_M) = \{\alpha : \alpha \in E_M, \ \ \text{Ker}(1-\alpha\beta) = 0 \ \text{ for all } \ \beta \in E_M\}.$$

$$= \{\alpha : \alpha \in E_M, \ \ \text{Ker}(1-\beta\alpha) = 0 \ \text{ for all } \ \beta \in E_M\}.$$

$\widehat{\triangle}(E_M)$ is a semi-ideal in E_M, which means that it is closed under arbitrary multiplication from either side, hence if $\alpha \in \widehat{\triangle}(E_M)$ and $\lambda \in E_M$; $\text{Ker}(1-(\alpha\lambda)\beta) = \text{Ker}(1-\alpha(\lambda\beta)) = 0$ for all $\beta \in E_M$ and $\text{Ker}(1-\beta(\lambda\alpha)) = \text{Ker}(1-(\beta\lambda)\alpha) = 0$ for all $\beta \in E_M$. Thus, $\alpha\lambda,\ \lambda\alpha \in \widehat{\triangle}E_M$.

It is clear that $\triangle(E_M) \subseteq \widehat{\triangle}(E_M)$.

It is known that if $\alpha^2 = \alpha \in E_M$; $M = \text{Im}(\alpha) \oplus \text{Im}(1-\alpha)$ and $\text{Im}(\alpha) = \text{Ker}(1-\alpha)$, $\text{Ker}(\alpha) = \text{Im}(1-\alpha)$.

Following [2], $\text{Tot}(R) = \{a : a \in R;\ a \text{ is not pi }\}$.

Lemma 3.3. For any module M the following hold:
(1) $J(E_M) \subseteq \widehat{\nabla}(E_M) \cap \widehat{\triangle}(E_M)$.
(2) $\widehat{\nabla}(E_M) \cup \widehat{\triangle}(E_M) \subseteq \text{Tot}(E_M)$.
(3) $J(E_M) \subseteq \text{Tot}(E_M)$.

Proof. (1). Let $\alpha \in J(E_M)$. Then $\alpha\beta,\ \beta\alpha \in J(E_M)$ for every $\beta \in E_M$, so $g(1-\beta\alpha) = 1$, $(1-\alpha\beta)g_0 = 1$ for some $g, g_0 \in E_M$. Thus, $\text{Im}(1-\alpha\beta) = M$ and $\text{Ker}(1-\beta\alpha) = 0$. Therefore, $J(E_M) \subseteq \widehat{\nabla}(E_M) \cap \widehat{\triangle}(E_M)$.
(2). Let $\alpha \in \widehat{\nabla}(E_M)$. Then $\text{Im}(1-\alpha\lambda) = M$ for all $\lambda \in E_M$. Suppose that $\alpha \notin \text{Tot}(E_M)$, there exists $\beta \in E_M$ such that $0 \neq (\alpha\beta)^2 = \alpha\beta \in E_M$, so $\text{Ker}(\alpha\beta) = \text{Im}(1-\alpha\beta) = M$ and that $\alpha\beta = 0$ a contradiction. Thus $\alpha \in \text{Tot}(E_M)$. Let $\alpha \in \widehat{\triangle}(E_M)$. Then $\text{Ker}(1-\lambda\alpha) = 0$ for all $\lambda \in E_M$. If $\alpha \notin \text{Tot}(E_M)$, there exists $\beta \in E_M$ such that $0 \neq (\beta\alpha)^2 = \beta\alpha \in E_M$, so $\text{Im}(\beta\alpha) = \text{Ker}(1-\beta\alpha) = 0$ and that $\beta\alpha = 0$ a contradiction. Thus $\alpha \in \text{Tot}(E_M)$.
(3). By (1) and (2). □

The following Proposition describe the Jacobson radical of E_M when E_M is a semipotent ring.

Proposition 3.4. Let M_R be a module with E_M is a semipotent ring. The following hold:
(1) $J(E_M) = \widehat{\nabla}(E_M)$.
(2) $J(E_M) = \widehat{\triangle}(E_M)$.
(3) $\widehat{\nabla}(E_M) = \widehat{\triangle}(E_M)$.

Proof. (1). By Lemma 3.3 we have $J(E_M) \subseteq \widehat{\nabla}(E_M)$. Let $\alpha \in \widehat{\nabla}(E_M)$. Then $\text{Im}(1-\alpha\lambda) = M$ for all $\lambda \in E_M$. If $\alpha \notin J(E_M)$ there exists $\beta \in E_M$ such that $0 \neq \beta = \beta\alpha\beta \in E_M$, therefore $0 \neq (\alpha\beta)^2 = \alpha\beta \in E_M$ and $\text{Ker}(\alpha\beta) = \text{Im}(1-\alpha\beta) = M$. So $\alpha\beta = 0$ a contradiction. Thus, $\alpha \in J(E_M)$.
(2). By Lemma 3.3 we have $J(E_M) \subseteq \widehat{\triangle}(E_M)$. Let $\alpha \in \widehat{\triangle}(E_M)$. Then $\text{Ker}(1 - \lambda\alpha) = 0$ for all $\lambda \in E_M$. If $\alpha \notin J(E_M)$ there exists $\beta \in E_M$ such that $0 \neq \beta =$

$\beta\alpha\beta \in E_M$, therefore $0 \neq (\beta\alpha)^2 = \beta\alpha \in E_M$ and $\mathrm{Im}(\beta\alpha) = \mathrm{Ker}(1-\beta\alpha) = 0$. So $\beta\alpha = 0$ a contradiction. Thus, $\alpha \in J(E_M)$.
(3). By (1) and (2). □

Corollary 3.5. Let M_R be a module with E_M is a semipotent ring and $\alpha \in E_M$. The following hold:
(1) $\alpha \in J(E_M)$ if and only if $\mathrm{Im}(1-\alpha\beta) = M$ for all $\beta \in E_M$, if and only if $\mathrm{Ker}(1-\beta\alpha) = 0$ for all $\beta \in E_M$.
(2) $\alpha \in J(E_M)$ if and only if $\mathrm{Im}(1-\beta\alpha) = M$ for all $\beta \in E_M$, if and only if $\mathrm{Ker}(1-\alpha\beta) = 0$ for all $\beta \in E_M$.

Proof. By Proposition 3.4. □

Theorem 3.6. (1). For every module M the following are equivalent:
(i) $\widehat{\nabla}(E_M) \subseteq J(E_M)$.
(ii) For every $\alpha \in E_M$ with $1-\alpha \in \widehat{\nabla}(E_M)$ is one-to-one.
(2). For every module M the following are equivalent:
(i) $\widehat{\triangle}(E_M) \subseteq J(E_M)$.
(ii) For every $\alpha \in E_M$ with $1-\alpha \in \widehat{\triangle}(E_M)$ is onto.

Proof. (1)$(i) \Rightarrow (ii)$. Let $\alpha \in E_M$ with $1-\alpha \in \widehat{\nabla}(E_M)$, by assumption $1-\alpha \in J(E_M)$, so $\alpha = 1-(1-\alpha) \in U(E_M)$. Thus, α is one-to-one.
$(ii) \Rightarrow (i)$. Let $\alpha \in \widehat{\nabla}(E_M)$, then $\mathrm{Im}(1-\alpha\beta) = M$ for all $\beta \in E_M$. Also, for all $\lambda \in E_M$; $\mathrm{Im}(1-(\alpha\beta)\lambda) = \mathrm{Im}(1-\alpha(\beta\lambda)) = M$, hence $\alpha \in \widehat{\nabla}(E_M)$. So $\alpha\beta = (1-(1-\alpha\beta)) \in \widehat{\nabla}(E_M)$, by assumption $1-\alpha\beta$ is one-to-one. Thus, $1-\alpha\beta \in U(E_M)$ and that $\alpha \in J(E_M)$.
(2)$(i) \Rightarrow (ii)$. Let $\alpha \in E_M$ with $1-\alpha \in \widehat{\triangle}(E_M)$, by assumption $1-\alpha \in J(E_M)$, so $\alpha = 1-(1-\alpha) \in U(E_M)$. Thus, α is onto.
$(ii) \Rightarrow (i)$. Let $\alpha \in \widehat{\triangle}(E_M)$, then $\mathrm{Ker}(1-\beta\alpha) = 0$ for all $\beta \in E_M$. Also, for all $\lambda \in E_M$; $\mathrm{Ker}(1-\lambda(\beta\alpha)) = \mathrm{Ker}(1-(\lambda\beta)\alpha) = 0$, hence $\alpha \in \widehat{\triangle}(E_M)$. So $\beta\alpha = (1-(1-\beta\alpha)) \in \widehat{\triangle}(E_M)$, is onto by assumption. Thus, $1-\beta\alpha \in U(E_M)$ and that $\alpha \in J(E_M)$. □

Theorem 3.7. (1). For every module M the following are equivalent:
(i) $\mathrm{Tot}(E_M) = \nabla(E_M)$.
(ii) For every $\alpha \in E_M$ with $\mathrm{Im}(\alpha)$ is not small in M there exists $\beta \in E_M$ such that $\mathrm{Im}(\beta) \cap \mathrm{Im}(1-\beta\alpha) = 0$.
(iii) For every $\alpha \in E_M$ with $\mathrm{Im}(\alpha)$ is not small in M there exists $\beta \in E_M$ such that $\mathrm{Ker}(\beta) + \mathrm{Ker}(1-\alpha\beta) = M$.
(2). For every module M the following are equivalent:
(i) $\mathrm{Tot}(E_M) = \triangle(E_M)$.
(ii) For every $\alpha \in E_M$ with $\mathrm{Ker}(\alpha)$ is not large in M there exists $\beta \in E_M$ such that $\mathrm{Im}(\beta) \cap \mathrm{Im}(1-\beta\alpha) = 0$.
(iii) For every $\alpha \in E_M$ with $\mathrm{Ker}(\alpha)$ is not large in M there exists $\beta \in E_M$ such that $\mathrm{Ker}(\beta) + \mathrm{Ker}(1-\alpha\beta) = M$.

Proof.(1)$(i) \Rightarrow (ii)$. Let $\alpha \in E_M$ with $\mathrm{Im}(\alpha)$ is not small in M. Then $\alpha \notin \nabla(E_M)$,

by assumption there exists $\lambda \in E_M$ such that $0 \neq (\lambda\alpha)^2 = \lambda\alpha \in E_M$. For $\beta = \lambda\alpha\lambda$; $\beta\alpha\beta = \beta$. By Lemma 2.1, $0 = \mathrm{Im}(\beta - \beta\alpha\beta) = \mathrm{Im}(\beta) \cap \mathrm{Im}(1 - \beta\alpha)$.
$(ii) \Rightarrow (iii)$. Suppose (2) holds. Let $\alpha \in E_M$ with $\mathrm{Im}(\alpha)$ is not small in M. By assumption there exists $\beta \in E_M$ such that $\mathrm{Im}(\beta) \cap \mathrm{Im}(1-\beta\alpha) = 0$, by Lemma 2.1; $\beta - \beta\alpha\beta = 0$ and $M = \mathrm{Ker}(\beta - \beta\alpha\beta) = \mathrm{Ker}(\beta) + \mathrm{Ker}(1 - \alpha\beta)$.
$(iii) \Rightarrow (i)$. It is clear that $\nabla(E_M) \subseteq \mathrm{Tot}(E_M)$. Let $\alpha \in \mathrm{Tot}(E_M)$. Suppose that $\alpha \notin \nabla(E_M)$, then $\mathrm{Im}(\alpha)$ is not small in M, by assumption there exists $\beta \in E_M$ such that $M = \mathrm{Ker}(\beta) + \mathrm{Ker}(1 - \alpha\beta)$. By Proposition 3.1; $\beta = \beta\alpha\beta$, so $0 \neq (\beta\alpha)^2 = \beta\alpha \in E_M$ a contradiction, hence $\alpha \in \mathrm{Tot}(E_M)$. Thus $\alpha \in \nabla(E_M)$.
(2)$(i) \Rightarrow (ii)$. Let $\alpha \in E_M$ with $\mathrm{Ker}(\alpha)$ is not large in M. Then $\alpha \notin \triangle(E_M)$, by assumption there exists $\lambda \in E_M$ such that $0 \neq (\alpha\lambda)^2 = \alpha\lambda \in E_M$. For $\beta = \lambda\alpha\lambda$; $\beta\alpha\beta = \beta$. By Lemma 2.5, $\mathrm{Im}(\beta) \cap \mathrm{Im}(1 - \beta\alpha) = 0$.
$(ii) \Rightarrow (iii)$. Suppose (2) holds. Let $\alpha \in E_M$ with $\mathrm{Ker}(\alpha)$ is not large in M. By assumption there exists $\beta \in E_M$ such that $\mathrm{Im}(\beta) \cap \mathrm{Im}(1-\beta\alpha) = 0$, by Lemma 2.1; $\mathrm{Im}(\beta - \beta\alpha\beta) = 0$, so $\beta - \beta\alpha\beta = 0$ and $M = \mathrm{Ker}(\beta - \beta\alpha\beta) = \mathrm{Ker}(\beta) + \mathrm{Ker}(1-\alpha\beta)$.
$(iii) \Rightarrow (i)$. It is clear that $\triangle(E_M) \subseteq \mathrm{Tot}(E_M)$. Let $\alpha \in \mathrm{Tot}(E_M)$. Suppose that $\alpha \notin \triangle(E_M)$, then $\mathrm{Ker}(\alpha)$ is not large in M, by assumption there exists $\beta \in E_M$ such that $\mathrm{Ker}(\beta) + \mathrm{Ker}(1 - \alpha\beta) = M$. By Proposition 3.1; $\beta = \beta\alpha\beta$, so $0 \neq (\alpha\beta)^2 = \alpha\beta \in E_M$ a contradiction, hence $\alpha \in \mathrm{Tot}(E_M)$. Thus $\alpha \in \triangle(E_M)$.
□

A module Q is called *locally injective* [3] if, for every submodule $A \subseteq Q$, which is not large in Q, there exists an injective submodule $0 \neq B \subseteq Q$, with $A \cap B = 0$.

A module P is called *locally projective* [3] if, for every submodule $B \subseteq P$, which is not small in P, there exists a projective direct summand $0 \neq A \subseteq^{\oplus} P$, with $A \subseteq B$.

F. Kasch in [3] studied conditions on modules Q and P, which imply that $\mathrm{Tot}(E_Q) = \triangle(E_Q) = J(E_Q)$ and $\mathrm{Tot}(E_P) = \nabla(E_P) = J(E_P)$. He showed that these equalities hold if Q is injective, respectively P is semiperfect and projective. Also, it was proved in [3], that $\mathrm{Tot}(E_Q) = \triangle(E_Q)$ if Q is a locally injective module and $\mathrm{Tot}(E_P) = \nabla(E_P)$ if P is a locally projective module.

The following questions were raised by Kasch in [3].
(1) If Q is locally injective, then it is true that $\mathrm{Tot}(E_Q) = \triangle(E_Q) = J(E_Q)$?.
(2) If P is locally projective, then it is true that $\mathrm{Tot}(E_P) = \nabla(E_P) = J(E_P)$?.

Zhou in [6], proved that the answer to question (1) is "Yes" if a ring R is left Noetherian. But in general, the answer to the question is "No" by [6, Example 4.2]. During our study of answer to questions it is obtained the following results:

Corollary 3.8. (1). If Q is a locally injective module, then $\mathrm{Tot}(E_Q) = \triangle(E_Q) = \widehat{\triangle}(E_Q)$.
(2). If P is a locally projective module, then $\mathrm{Tot}(E_P) = \nabla(E_P) = \widehat{\nabla}(E_P)$.

Proof. (1). Since Q is locally injective, then $\mathrm{Tot}(E_Q) = \triangle(E_Q) \subseteq \widehat{\triangle}(E_Q) \subseteq \mathrm{Tot}(E_Q)$ by definition and Lemma 3.2, so $\mathrm{Tot}(E_Q) = \triangle(E_Q) = \widehat{\triangle}(E_Q)$.
(2). Since P is locally projective, then $\mathrm{Tot}(E_P) = \nabla(E_P) \subseteq \widehat{\nabla}(E_P) \subseteq \mathrm{Tot}(E_P)$ by

definition and Lemma 3.2, so $\mathrm{Tot}(E_P) = \nabla(E_P) = \widehat{\nabla}(E_P)$. □

Corollary 3.9. (1). If Q is a locally injective module and $\alpha \in E_Q$, then $\mathrm{Ker}(\alpha) \leq_e Q$ if and only if $\mathrm{Ker}(1 - \alpha\beta) = 0$ for all $\beta \in E_Q$ if and only if $\mathrm{Ker}(1 - \beta\alpha) = 0$ for all $\beta \in E_Q$.
(2). If P is a locally projective module and $\alpha \in E_P$, then $\mathrm{Im}(\alpha) \ll P$ if and only if $\mathrm{Im}(1 - \alpha\beta) = P$ for all $\beta \in E_P$ if and only if $\mathrm{Im}(1 - \beta\alpha) = P$ for all $\beta \in E_P$.

Proof by Corollary 3.8. □

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Department of Mathematics Faculty Of Science Damascus University
E-mail address: hhakmi-64@hotmail.com