**Functional continuity of unital $B_0$-algebras with orthogonal bases**

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**Abstract.** Let $A$ be a unital $B_0$-algebra with an orthogonal basis, then every multiplicative linear functional on $A$ is continuous. This gives an answer to a problem posed by Z. Sawon and Z. Wronski.

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## I. Preliminaries

A topological algebra is a complex associative algebra which is also a Hausdorff topological vector space such that the multiplication is separately continuous. A locally convex algebra is a topological algebra whose topology is determined by a family of seminorms. A complete metrizable locally convex algebra is called a $B_0$-algebra. The topology of a $B_0$-algebra $A$ may be given by a countable family $(\|.\|_i)_{i\geq 1}$ of seminorms such that $\|x\|_i \leq \|x\|_{i+1}$ and $\|xy\|_i \leq \|x\|_{i+1}\|y\|_{i+1}$ for all $i \geq 1$ and $x, y \in A$. A multiplicative linear functional on a complex algebra $A$ is an algebra homomorphism from $A$ to the complex field. Let $A$ be a topological algebra. $M^*(A)$ denotes the set of all nonzero multiplicative linear functionals on $A$. $M(A)$ denotes the set of all nonzero continuous multiplicative linear functionals on $A$. A seminorm $p$ on $A$ is lower semicontinuous if the set $\{x \in A : p(x) \leq 1\}$ is closed in $A$.
Let $A$ be a topological algebra. A sequence $(e_n)_{n\geq 1}$ in $A$ is a basis if for each $x \in A$ there is a unique sequence $(\alpha_n)_{n\geq 1}$ of complex numbers such that $x = \Sigma_{n=1}^{\infty}\alpha_n e_n$. Each linear functional $e_n^* : A \to C$, $e_n^*(x) = \alpha_n$, is called a coefficient functional. If each $e_n^*$ is continuous, the basis $(e_n)_{n\geq 1}$ is called a Schauder basis. A basis $(e_n)_{n\geq 1}$ is orthogonal if $e_i e_j = \delta_{ij} e_i$ where $\delta_{ij}$ is the Kronecker symbol. If $(e_n)_{n\geq 1}$ is an orthogonal basis, then each $e_n^*$ is a multiplicative linear functional on $A$. Let $A$ be a topological algebra with an orthogonal basis $(e_n)_{n\geq 1}$. If $A$ has a unity $e$, then $e = \Sigma_{n=1}^{\infty} e_n$. Let $(x_k)_k$ be a net in $A$ converging to 0, since the multiplication is separately continuous, $e_n x_k = e_n^*(x_k)e_n \to_k 0$ and so $e_n^*(x_k) \to_k 0$. Then each orthogonal basis in a topological algebra is a Schauder basis. Let $f$ be a multiplicative linear functional on $A$. If $f(e_{n_0}) \neq 0$ for some $n_0 \geq 1$, then $f(x)f(e_{n_0}) = f(xe_{n_0}) = f(e_{n_0}^*(x)e_{n_0}) = e_{n_0}^*(x)f(e_{n_0})$ for all $x \in A$ and therefore $f = e_{n_0}^* \in M(A)$. This shows that $M(A) = \{e_n^* : n \geq 1\}$.
Here we consider unital $B_0$-algebras with orthogonal bases. These algebras were investigated in [3] where we can find examples of such algebras.

## II. Results

**Proposition II.1.** Let $(A, (\|.\|_i)_{i\geq 1})$ be a unital $B_0$-algebra with an orthogonal basis $(e_n)_{n\geq 1}$. Then there exists $x \in A$ such that the sequence $(e_n^*(x))_{n\geq 1}$ is not bounded.

**Proof.** Suppose that $\sup_{n\geq 1} |e_n^*(x)| < \infty$ for all $x \in A$. For each $x \in A$, let $\|x\| = \sup_{n\geq 1} |e_n^*(x)|$, $\|.\|$ is a lower semicontinuous norm on $A$ since $(e_n)_{n\geq 1}$ is a Schauder basis. Let $\tau_A$ be the topology on $A$ determined by the family $(\|.\|_i)_{i\geq 1}$ of seminorms. We define a new topology $\tau$ on $A$ described by the norm $\|.\|$ and the family $(\|.\|_i)_{i\geq 1}$ of seminorms. The topology $\tau$ is stronger than the topology $\tau_A$. By Garling's completeness theorem [1, Theorem 1], $(A, \tau)$ is complete. The topologies $\tau_A$ and $\tau$ are homeomorphic by the open mapping theorem. Then there exist $i_0 \geq 1$ and $M > 0$ such that $\|x\| \leq M\|x\|_{i_0}$ for all $x \in A$, hence $1 = |e_n^*(e_n)| \leq \|e_n\| \leq M\|e_n\|_{i_0}$ for all $n \geq 1$. This contradicts the fact that $e_n \to_n 0$.

Let $A$ be a unital $B_0$-algebra with an orthogonal basis $(e_n)_{n\geq 1}$. Sawon and Wronski [3, Theorem 2.1] proved that if there exists $x = \Sigma_{n=1}^{\infty} t_n e_n \in A$ such that $t_n \in R$, $t_n \leq t_{n+1}$ for $n \geq 1$ and $t_n \to \infty$, then every multiplicative linear functional on $A$ is continuous. In the proof, the authors used without justification that $f(x) \in R$ for every $f \in M^*(A) \setminus M(A)$. This fact is not evident as shown in the following result.

**Proposition II.2.** Let $A$ be a unital $B_0$-algebra with an orthogonal basis $(e_n)_{n\geq 1}$. If $x = \Sigma_{n=1}^{\infty} t_n e_n \in A$ such that $t_n \in R$, $t_n \leq t_{n+1}$ for $n \geq 1$ and $t_n \to \infty$, then $f(x) \in R$ for all $f \in M^*(A)$.

**Proof.** If $f \in M(A)$, $f = e_n^*$ for some $n \geq 1$, then $f(x) = t_n \in R$. If $f \in M^*(A) \setminus M(A)$, then $f(e_n) = 0$ for all $n \geq 1$. Suppose that $f(x) \notin R$, $f(x) = \alpha + i\beta$ with $\beta \neq 0$. Since $t_n \to \infty$, there exists $n_0 \geq 1$ such that $t_n \geq \alpha + |\beta|$ for all $n \geq n_0$. We define the sequence $(s_n)_{n\geq 1}$ by $s_n = t_{n_0}$ for $1 \leq n \leq n_0$ and $s_n = t_n$ for $n \geq n_0 + 1$. It is clear that $y = \Sigma_{n=1}^{\infty} s_n e_n \in A$ such that $s_n \geq \alpha + |\beta|$, $s_n \leq s_{n+1}$ for $n \geq 1$ and $s_n \to \infty$. Since $f(e_n) = 0$ for all $n \geq 1$, $f(y) = f(x) = \alpha + i\beta$. We have $f(|\beta|^{-1}y) = |\beta|^{-1}\alpha + i|\beta|^{-1}\beta$, then $f(|\beta|^{-1}y - |\beta|^{-1}\alpha e) = i|\beta|^{-1}\beta$. Set $z = |\beta|^{-1}y - |\beta|^{-1}\alpha e = \Sigma_{n=1}^{\infty} \frac{s_n - \alpha}{|\beta|} e_n$. The real sequence $(\frac{s_n-\alpha}{|\beta|})_{n\geq 1}$ is positive increasing and $\frac{s_n-\alpha}{|\beta|} \to \infty$, then $z^{-1} = \Sigma_{n=1}^{\infty} \frac{|\beta|}{s_n - \alpha} e_n \in A$ by [3, Theorem 0.1] and $f(z^{-1}) = -i|\beta|\beta^{-1}$, so $f(z+z^{-1}) = 0$. Set $v = z + z^{-1} = \Sigma_{n=1}^{\infty} (\frac{s_n-\alpha}{|\beta|} + \frac{|\beta|}{s_n-\alpha}) e_n$ and $v_n = \frac{s_n-\alpha}{|\beta|} + \frac{|\beta|}{s_n-\alpha}$ for all $n \geq 1$. Since the map $g : [1, \infty) \to R$, $g(x) = x + \frac{1}{x}$, is increasing and the sequence $(\frac{s_n-\alpha}{|\beta|})_{n\geq 1} \subset [1, \infty)$ is increasing, it follows that $(v_n)_{n\geq 1}$ is a positive increasing sequence and $v_n \to \infty$. By [3, Theorem 0.1], $v^{-1} = \Sigma_{n=1}^{\infty} \frac{1}{v_n} e_n \in A$ and therefore $f(e) = f(v)f(v^{-1}) = 0$. This contradicts the fact that $f$ is nonzero.

The following result is due to Sawon and Wronski [3, p.109], the proof is given for completeness.

**Proposition II.3([3]).** Let $(A, (\|.\|_i)_{i\geq 1})$ be a unital $B_0$-algebra with an orthogonal basis $(e_n)_{n\geq 1}$. Then the set $N$ of all positive integers can be split into two disjoint subsets $N_1$ and $N_2$ such that by putting $A_1 = \overline{span}(e_n)_{n\in N_1}$ and $A_2 = \overline{span}(e_n)_{n\in N_2}$, we have
(1) $A = A_1 \oplus A_2$;
(2) if $f$ is a multiplicative linear functional on $A$ such that $f \notin M(A)$, then $f_{/A_1} = 0$.

**Proof.** By Proposition II.1, there is $x = \Sigma_{n=1}^{\infty} t_n e_n \in A$ such that the sequence $(t_n)_{n\geq 1}$ is not bounded. Then there exists a subsequence $(t_{k_n})_{n\geq 1}$ of $(t_n)_{n\geq 1}$ such that $|t_{k_n}| \geq n^2$ for all $n \geq 1$. For each $i \geq 1$, there is $M_i > 0$ such that $\|t_n e_n\|_i \leq M_i$ for all $n \geq 1$. Let $i \geq 1$ and $n \geq 1$, $n^2 \|e_{k_n}\|_i \leq |t_{k_n}| \|e_{k_n}\|_i = \|t_{k_n} e_{k_n}\|_i \leq M_i$, then $\|e_{k_n}\|_i \leq n^{-2} M_i$. This implies that $\Sigma_{n=1}^{\infty} e_{k_n}$ is absolutely convergent. Let $A_1 = \overline{span}\{e_{k_n} : n \geq 1\}$, $A_1$ is a unital $B_0$-algebra with an orthogonal basis $(e_{k_n})_{n\geq 1}$ and $\Sigma_{n=1}^{\infty} n^{\frac{1}{2}} e_{k_n} \in A_1$ since $n^{\frac{1}{2}} \|e_{k_n}\|_i \leq n^{-\frac{3}{2}} M_i$ for all $i \geq 1$ and $n \geq 1$. Set $N_1 = \{k_n : n \geq 1\}$, $N_2 = N \setminus N_1$ and $A_2 = \overline{span}\{e_n : n \in N_2\}$. $A_2$ is a $B_0$-algebra with an orthogonal basis $(e_n)_{n\in N_2}$ and the unity $u_2 = e - u_1$ where $u_1 = \Sigma_{n=1}^{\infty} e_{k_n}$ is the unity of $A_1$. Let $x \in A$, $x = xe = x(u_1 + u_2) = xu_1 + xu_2 \in A_1 + A_2$, then $A = A_1 \oplus A_2$. If $f$ is a multiplicative linear functional on $A$ such that $f \notin M(A)$, $f_{/A_1}$ is a multiplicative linear functional on $A_1$ such that $f_{/A_1}(e_{k_n}) = 0$ for all $n \geq 1$. Since $\Sigma_{n=1}^{\infty} n^{\frac{1}{2}} e_{k_n} \in A_1$, $f_{/A_1}$ is continuous on $A_1$ by [3, Theorem 2.1] and therefore $f_{/A_1} = 0$.

Sawon and Wronski [3, p.109] posed the following problem:

**Problem.** Let $A$ be a unital $B_0$-algebra with an orthogonal basis $(e_n)_{n\geq 1}$. Does there exist a maximal subalgebra $A_1' = \overline{span}\{e_n : n \in N_1'\}(N_1' \subset N)$ of $A$ for which (1) and (2) hold?

**Proposition II.4.** Let $A$ be a unital $B_0$-algebra with an orthogonal basis $(e_n)_{n\geq 1}$. Then the following assertions are equivalent:
$(i)$ $A = \overline{span}\{e_n : n \in N\}$ is a maximal subalgebra of itself for which (1) and (2) hold;
$(ii)$ every multiplicative linear functional on $A$ is continuous.

**Proof.** $(i) \Rightarrow (ii)$ : Let $f$ be a multiplicative linear functional on $A$ such that $f \notin M(A)$, $f$ is zero on $A$ by $(i)$. Then every multiplicative linear functional on $A$ is continuous.
$(ii) \Rightarrow (i)$ : It is clear that $A$ satisfies (1). Let $f$ be a multiplicative linear functional on $A$ such that $f \notin M(A)$, then $f$ is zero on $A$ since $f$ is continuous, hence $A$ satisfies (2).

**Proposition II.5.** Let $(t_n)_{n\geq n_0}$ be a complex sequence, the following assertions are equivalent:
$(i)$ $\Sigma_{n=n_0}^{\infty}|t_n - t_{n+1}| < \infty$;
$(ii)$ there exists $M > 0$ such that $|t_q| + \Sigma_{n=p}^{q-1}|t_n - t_{n+1}| \leq M$ for all $q > p \geq n_0$.

**Proof.** $(i) \Rightarrow (ii)$ : Let $\epsilon > 0$, there exists $n_1 \geq n_0$ such that $\Sigma_{k=n}^{\infty}|t_k - t_{k+1}| \leq \epsilon$ for every $n \geq n_1$. let $m > n \geq n_1$, $|t_n - t_m| \leq |t_n - t_{n+1}| + ... + |t_{m-1} - t_m| \leq \epsilon$. Then the sequence $(t_n)_{n\geq n_0}$ converges, so there is $M_0 > 0$ such that $|t_n| \leq M_0$ for all $n \geq n_0$. Let $q > p \geq n_0$, $|t_q| + \Sigma_{n=p}^{q-1}|t_n - t_{n+1}| \leq M_0 + \Sigma_{n=n_0}^{\infty}|t_n - t_{n+1}|$.
$(ii) \Rightarrow (i)$ : Let $p \geq n_0$ and $q = p+1$, $\Sigma_{n=n_0}^{p}|t_n - t_{n+1}| \leq |t_q| + \Sigma_{n=n_0}^{q-1}|t_n - t_{n+1}| \leq M$. Then the sequence $(\Sigma_{n=n_0}^{p}|t_n - t_{n+1}|)_{p\geq n_0}$ is positive increasing and bounded, so it is convergent i.e. $\Sigma_{n=n_0}^{\infty}|t_n - t_{n+1}| < \infty$.

**Proposition II.6.** Let $(A, (\|.\|_i)_{i\geq 1})$ be a unital $B_0$-algebra with an orthogonal basis $(e_n)_{n\geq 1}$. If $(t_n)_{n\geq n_0}$ is a complex sequence such that $\Sigma_{n=n_0}^{\infty}|t_n - t_{n+1}| < \infty$, then $\Sigma_{n=n_0}^{\infty}t_n e_n \in A$.

**Proof.** Let $q > p \geq n_0$, by using the equality $t_n = t_q + \Sigma_{k=n}^{q-1}(t_k - t_{k+1})$ for every $p \leq n < q$, we obtain that $\Sigma_{n=p}^{q-1}t_n e_n = t_q(e_p + ... + e_{q-1}) + \Sigma_{k=p}^{q-1}(t_k - t_{k+1})(e_p + ... + e_k)$. Let $i \geq 1$, $\|\Sigma_{n=p}^{q-1}t_n e_n\|_i \leq |t_q|\|e_p + ... + e_{q-1}\|_i + \Sigma_{k=p}^{q-1}|t_k - t_{k+1}|\|e_p + ... + e_k\|_i \leq (|t_q| + \Sigma_{k=p}^{q-1}|t_k - t_{k+1}|)\sup_{p\leq k\leq q}\|e_p + ... + e_k\|_i \leq M\sup_{p\leq k\leq q}\|e_p + ... + e_k\|_i$ by Proposition II.5. Let $\epsilon > 0$, since $e = \Sigma_{n=1}^{\infty}e_n \in A$, there is $n_1 \geq n_0$ such that $\|e_p + ... + e_k\|_i \leq \epsilon M^{-1}$ for $n_1 \leq p \leq k$, hence $\sup_{p\leq k\leq q}\|e_p + ... + e_k\|_i \leq \epsilon M^{-1}$ for $n_1 \leq p < q$. Then $\|\Sigma_{n=p}^{q-1}t_n e_n\|_i \leq \epsilon$ for $n_1 \leq p < q$. This shows that $\Sigma_{n=n_0}^{\infty}t_n e_n$ is convergent in $A$.

**Theorem II.7.** Let $(A, (\|.\|_i)_{i\geq 1})$ be a unital $B_0$-algebra with an orthogonal basis $(e_n)_{n\geq 1}$. Then every multiplicative linear functional on $A$ is continuous.

**Proof.** By Proposition II.6, $x = \Sigma_{n=1}^{\infty}\frac{1}{n}e_n \in A$. Let $f \in M^*(A)$ and $\alpha = f(x)$, then $f(\alpha e - x) = 0$. We have $\alpha e - x = \Sigma_{n=1}^{\infty}(\alpha - \frac{1}{n})e_n = \Sigma_{n=1}^{\infty}\frac{\alpha n-1}{n}e_n$. Let $I_\alpha = \{n \in N : n \neq \alpha^{-1}\}$, $I_\alpha = N$ if $\alpha^{-1} \notin N$ and $I_\alpha = N \setminus \{\alpha^{-1}\}$ if $\alpha^{-1} \in N$. Let $m_\alpha = \inf\{n \in N : |\alpha|n - 1 > 0\}$ and consider the complex sequence $(\frac{n}{\alpha n-1})_{n\geq m_\alpha}$. Let $n \geq m_\alpha$, $\frac{n}{\alpha n-1} - \frac{n+1}{\alpha(n+1)-1} = \frac{1}{(\alpha n-1)(\alpha(n+1)-1)}$. We have $|\alpha n - 1| \geq |\alpha|n - 1$ and $|\alpha(n+1) - 1| \geq |\alpha|(n+1) - 1 = |\alpha|n + |\alpha| - 1 \geq |\alpha|n - 1$. Let $n \geq m_\alpha$, $|\alpha|n - 1 > 0$, hence $\frac{1}{|\alpha n-1|} \leq \frac{1}{|\alpha|n-1}$ and $\frac{1}{|\alpha(n+1)-1|} \leq \frac{1}{|\alpha|n-1}$. Consequently $|\frac{n}{\alpha n-1} - \frac{n+1}{\alpha(n+1)-1}| = \frac{1}{|\alpha n-1||\alpha(n+1)-1|} \leq \frac{1}{(|\alpha|n-1)^2}$. Then $\Sigma_{n=m_\alpha}^{\infty}\frac{n}{\alpha n-1}e_n \in A$ by Proposition II.6 and therefore $\Sigma_{n\in I_\alpha}\frac{n}{\alpha n-1}e_n \in A$. If $\alpha^{-1} \notin N$ i.e. $I_\alpha = N$, $(\alpha e - x)\Sigma_{n=1}^{\infty}\frac{n}{\alpha n-1}e_n = e$, then $\alpha e - x$ is invertible, a contradiction. If $\alpha^{-1} \in N$, put $n_\alpha = \alpha^{-1}$, then $(\alpha e - x)\Sigma_{n\in I_\alpha}\frac{n}{\alpha n-1}e_n = \Sigma_{n\in I_\alpha}e_n$, hence $\Sigma_{n\in I_\alpha}e_n \in Ker(f)$ since $\alpha e - x \in Ker(f)$. If $e_{n_\alpha} \in Ker(f)$, then $e = e_{n_\alpha} + \Sigma_{n\in I_\alpha}e_n \in Ker(f)$, a contradiction. Finally $f(e_{n_\alpha}) \neq 0$ and therefore $f = e_{n_\alpha}^*$.

**Remark.** Proposition II.4 and Theorem II.7 give an answer to Sawon and Wronski's problem.

## References

[1] D. J. H. Garling, On topological sequence spaces, Proc. Cambridge Philos. Soc., 63 (1967), 997-1019.
[2] T. Husain, Orthogonal Schauder bases, Monographs and Textbooks in Pure and Applied Mathematics, 143, Marcel Dekker, New York, 1991.
[3] Z. Sawon and Z. Wronski, Fréchet algebras with orthogonal basis, Colloquium Mathematicum, 48 (1984), 103-110.
[4] W. Zelazko, Metric generalizations of Banach algebras, Rozprawy Matematyczne, 47, Warszawa, 1965.

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