Exact analytical solution for the heat transfer of nanofluids via two transformations

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Abstract

The boundary layer flow of nanofluids are usually described by a system of nonlinear differential equations with infinity boundary conditions. These boundary conditions at infinity are transformed into classical boundary conditions via two different transformations. Accordingly, the original heat transfer equation is changed into a new one which is expressed in terms of the new variable. The exact solutions have been obtained in terms of the exponential function for the stream function and in terms of the incomplete Gamma function for the temperature distribution. Furthermore, it is found in this paper that a certain transformation reduces the computational work that required to obtain the exact solution of the heat transfer equation. Hence, such transformation is recommended for future analysis of similar physical problems. Besides, the other published exact solution was expressed in terms of the WhittakerM function which is more complicated than the generalized incomplete Gamma function of the current analysis. It is important to refer to that the analytical procedure followed in our paper is easier and more direct than the one considered in a previous published work.

Keywords: Nanofluid; boundary layer; stretching sheet; exact solution.

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1 Introduction

Nanofluids is a relatively new area of research which attracted attention in recent years because of their applications in engineering and applied sciences. The flow and heat transfer of such nanofluids are usually described by a system of nonlinear differential equations. Such system can be solved using numerical methods [1] or series methods such as Adomian decomposition method (ADM) [2-7], differential transformation/Taylor method (DTM) [8-9], and homotopy

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perturbation method (HPM) [10-13]. The solutions using any of the just mentioned methods are usually called approximate numerical or analytical solutions. These approximate solutions have been extensively used to investigate various physical models because the non availability of the exact solutions of these models. However, the exact solution of any system is the best if it can be obtained. Such exact solution gives us a better understanding about the phenomena involved in the physical model than the approximate solutions. Due to the difficulty of obtaining the exact solution many authors implement either approximate numerical methods or approximate analytical methods. However, we are interested in this paper in the exact solution of an example from nanofluid mechanics. Hence, an analytical approach shall be suggested with the help of two transformations to facilitate the task of the current study. The example considered in this paper is governed by the following system of nonlinear differential equations ([14], [15]):

$$f'''(\eta) + (1-\phi)^{2.5}\left[1-\phi+\phi(\rho_s/\rho_f)\right]\left[f(\eta)f''(\eta) - (f'(\eta))^2\right] - M(1-\phi)^{2.5}f'(\eta) = 0, \quad (1)$$

$$\frac{1}{Pr}\left(\frac{k_{nf}}{k_f}\right)\frac{1}{[1-\phi+\phi(\rho C_p)_s/(\rho C_p)_f]}\theta''(\eta) + f(\eta)\theta'(\eta) = 0, \quad (2)$$

which represent the steady laminar two-dimensional flow of an incompressible viscous nanofluid past a linearly semi-infinite stretching sheet under the influence of a constant magnetic. The primes denote the differentiation with respect to a similarity variable η. f and θ are the dimensionless stream function and temperature, respectively, ϕ is the solid volume fraction, ρ_f and ρ_s are the densities, $(\rho C_p)_f$ and $(\rho C_p)_s$ are the heat capacitances, M is the magnetic parameter, Pr is the Prandtl number, and k_{nf} is the thermal conductivity defined as follows ([16], [17])

$$\frac{k_{nf}}{k_f} = \frac{(k_s+2k_f) - 2\phi(k_f - k_s)}{(k_s+2k_f) + \phi(k_f - k_s)}, \quad (3)$$

where k_f and k_s are the thermal conductivities, where $()_f$ and $()_s$ denote the basic fluid and solid fractions, respectively. The flow is subject to the boundary conditions:

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0, \quad (4)$$

$$\theta(0) = 1, \quad \theta(\infty) = 0. \quad (5)$$

The exact solution of the stream function $f(\eta)$ shall be discussed in the next section. In the subsequent section, we discuss the effectiveness of two transformations in obtaining the exact solution of the heat transfer equation to get $\theta(\eta)$ for the present physical model.

2 Exact solution for the stream function $f(\eta)$

Equations (1) and (2) can be rewritten as

$$f'''(\eta) + \alpha\left[f(\eta)f''(\eta) - (f'(\eta))^2\right] - \gamma f'(\eta) = 0, \tag{6}$$

$$\tau\theta''(\eta) + f(\eta)\theta'(\eta) = 0, \tag{7}$$

where

$$\alpha = (1-\phi)^{2.5}\left[1-\phi+\phi(\rho_s/\rho_f)\right], \quad \gamma = M(1-\phi)^{2.5}, \tag{8}$$

$$\tau = \frac{1}{Pr}\left(\frac{k_{nf}}{k_f}\right)\frac{1}{[1-\phi+\phi(\rho C_p)_s/(\rho C_p)_f]}. \tag{9}$$

In order to solve the system (6-7) with the boundary conditions (4-5) we begin with solving the f-differential equation (6). To do that, the following assumption is assumed

$$f(\eta) = a + b\ \mathrm{e}^{-\beta\eta}. \tag{10}$$

Here, it should be noted that the infinity boundary condition in (4) is already satisfied provided that $\beta > 0$. On using Eq. (10) into Eq. (6), we have

$$\beta b\left(\beta^2 - \alpha a\beta - \gamma\right)\mathrm{e}^{-\beta\eta} = 0, \tag{11}$$

which leads to

$$\beta^2 - \alpha a\beta - \gamma = 0. \tag{12}$$

Applying the first two boundary conditions given in (4), we obtain

$$a + b = 0, \quad b\beta + 1 = 0. \tag{13}$$

In view of Eqs. (11-13), we have

$$a = \frac{1}{\beta}, \quad b = -\frac{1}{\beta}, \quad \beta = \sqrt{\alpha+\gamma}. \tag{14}$$

Therefore,

$$f(\eta) = \frac{1}{\beta}\left(1 - \mathrm{e}^{-\beta\eta}\right). \tag{15}$$

Inserting Eq. (15) into Eq. (7), yields

$$\tau\theta''(\eta) + \frac{1}{\beta}\left(1 - \mathrm{e}^{-\beta\eta}\right)\theta'(\eta) = 0. \tag{16}$$

Regarding the heat transfer equation, i.e. Eq. (16), we shall discuss in the next section the use of two transformations for obtaining the exact solution in terms of the generalized incomplete gamma function.

3 Exact solution of the heat transfer $\theta(\eta)$

3.1 Transformation 1

Suppose the following transformation: $t = 1 - e^{-\beta\eta}$. The unbounded domain of the independent variable $\eta \in [0, \infty)$ can be changed into a bounded one by using a new independent variable t (say)$\in [0, 1)$ using the transformation [18-20]:

$$t = 1 - e^{-\beta\eta}. \tag{17}$$

Accordingly, the governing system should be expressed in terms of the new variable t. In order to do that, we introduce the following relations between the derivatives w.r.t η and the derivatives w.r.t t:

$$\frac{d}{d\eta}(\square) = \beta(1-t)\frac{d}{dt}(\square), \tag{18}$$

$$\frac{d^2}{d\eta^2}(\square) = \beta^2\left[(1-t)^2\frac{d^2}{dt^2}(\square) - (1-t)\frac{d}{dt}(\square)\right]. \tag{19}$$

The relations given by Eqs. (18-19) are obtained by using the chain rule in the differential calculus. Therefore, the system Eq. (16) becomes

$$\beta^2\tau(1-t)^2\theta''(t) + (1-t)(t-\beta^2\tau)\theta'(t) = 0, \tag{20}$$

which can be simplified to

$$\beta^2\tau(1-t)\theta''(t) + (t-\beta^2\tau)\theta'(t) = 0, \tag{21}$$

subject to the following set of boundary conditions

$$\theta(0) = 1, \quad \theta(1) = 0. \tag{22}$$

From Eq. (21) and using the separation of variables, we get

$$\frac{\theta''(t)}{\theta'(t)} = \frac{1}{\beta^2\tau}\left(\frac{\beta^2\tau - t}{1-t}\right), \tag{23}$$

or

$$\frac{\theta''(t)}{\theta'(t)} = \frac{1}{\beta^2\tau}\left(1 + \frac{\beta^2\tau - 1}{1-t}\right). \tag{24}$$

Integrating Eq. (24) once w.r.t. t from 0 to t, we have

$$\ln\left[\frac{\theta'(t)}{\theta'(0)}\right] = \frac{1}{\beta^2\tau}\left[t - (\beta^2\tau - 1)\ln(1-t)\right]. \tag{25}$$

Therefore

$$\theta'(t) = \theta'(0)\,(1-t)^{\frac{1-\beta^2\tau}{\beta^2\tau}}\;\mathrm{e}^{\frac{t}{\beta^2\tau}}. \tag{26}$$

Integrating Eq. (26) once again w.r.t. t from 0 to t, we obtain

$$\theta(t) = \theta(0) + \theta'(0)\int_0^t (1-\sigma)^{\frac{1-\beta^2\tau}{\beta^2\tau}}\;\mathrm{e}^{\frac{\sigma}{\beta^2\tau}}\,d\sigma. \tag{27}$$

In view of the first condition in (22), we have

$$\theta(t) = 1 + \theta'(0)I(t), \tag{28}$$

where

$$I(t) = \int_0^t (1-\sigma)^{\frac{1-\beta^2\tau}{\beta^2\tau}}\;\mathrm{e}^{\frac{\sigma}{\beta^2\tau}}\,d\sigma. \tag{29}$$

The integration in the right hand side can be analytically solved in terms of a well known special function as declared by the following procedure. We first suppose that

$$\mu = 1-\sigma. \tag{30}$$

Accordingly

$$I(t) = \mathrm{e}^{\frac{1}{\beta^2\tau}}\int_{1-t}^{1} \mu^{\frac{1-\beta^2\tau}{\beta^2\tau}}\;\mathrm{e}^{-\frac{\mu}{\beta^2\tau}}\,d\mu. \tag{31}$$

We then assume that

$$z = \frac{\mu}{\beta^2\tau}, \qquad d\mu = \beta^2\tau dz. \tag{32}$$

Accordingly

$$I(t) = \left(\beta^2\tau\right)^{\frac{1}{\beta^2\tau}}\mathrm{e}^{\frac{1}{\beta^2\tau}}\int_{\frac{1-t}{\beta^2\tau}}^{\frac{1}{\beta^2\tau}} z^{\frac{1}{\beta^2\tau}-1}\mathrm{e}^{-z}\,dz. \tag{33}$$

Using the definition of the generalized incomplete Gamma function, we have

$$I(t) = \left(\beta^2\tau\right)^{\frac{1}{\beta^2\tau}}\mathrm{e}^{\frac{1}{\beta^2\tau}}\;\Gamma\left(\frac{1}{\beta^2\tau}, \frac{1-t}{\beta^2\tau}, \frac{1}{\beta^2\tau}\right). \tag{34}$$

On inserting (34) into (28), we get

$$\theta(t) = 1 + \theta'(0)\left(\beta^2\tau\right)^{\frac{1}{\beta^2\tau}}\mathrm{e}^{\frac{1}{\beta^2\tau}}\;\Gamma\left(\frac{1}{\beta^2\tau}, \frac{1-t}{\beta^2\tau}, \frac{1}{\beta^2\tau}\right). \tag{35}$$

Applying the boundary condition $\theta(1) = 0$, we obtain

$$\theta'(0) = -\frac{\left(\beta^2\tau\right)^{\frac{-1}{\beta^2\tau}}\mathrm{e}^{\frac{-1}{\beta^2\tau}}}{\Gamma\left(\frac{1}{\beta^2\tau}, 0, \frac{1}{\beta^2\tau}\right)}. \tag{36}$$

From Eq. (35) and Eq. (36), we have

$$\theta(t) = 1 - \frac{\Gamma\left(\frac{1}{\beta^2\tau}, \frac{1-t}{\beta^2\tau}, \frac{1}{\beta^2\tau}\right)}{\Gamma\left(\frac{1}{\beta^2\tau}, 0, \frac{1}{\beta^2\tau}\right)}, \tag{37}$$

which can be simplified to

$$\theta(t) = \frac{\Gamma\left(\frac{1}{\beta^2\tau}, 0, \frac{1-t}{\beta^2\tau}\right)}{\Gamma\left(\frac{1}{\beta^2\tau}, 0, \frac{1}{\beta^2\tau}\right)}. \tag{38}$$

In terms of η, we have the following final form for the temperature distribution

$$\theta(\eta) = \frac{\Gamma\left(\frac{1}{\beta^2\tau}, 0, \frac{\mathrm{e}^{-\beta\eta}}{\beta^2\tau}\right)}{\Gamma\left(\frac{1}{\beta^2\tau}, 0, \frac{1}{\beta^2\tau}\right)}. \tag{39}$$

This exact solution can be verified by direct substitution into Eq. (16). It is also important here to mention that the exact solution of the heat transfer equation has been already obtained in [14] in terms of the WhittakerM function. However, the current exact solution is expressed in terms of the generalized incomplete Gamma function which is easier than the WhittakerM special function. It may be reasonable to refer here to that the analytical procedure we followed in this paper not only easier but also direct in comparison to the one considered in [14]. Furthermore, it agrees with the exact solution obtained very recently by Ebaid et. al [15] (Eq. (26)) in the absence of the slip parameter.

3.2 Transformation 2:

Suppose the following transformation: $t = \mathrm{e}^{-\beta\eta}$, hence we have

$$\frac{d}{d\eta}(\square) = -\beta t \frac{d}{dt}(\square), \tag{40}$$

$$\frac{d^2}{d\eta^2}(\square) = \beta^2 \left[t^2 \frac{d^2}{dt^2}(\square) + t\frac{d}{dt}(\square)\right]. \tag{41}$$

Therefore, the Eq. (16) becomes

$$\beta^2\tau\ t\theta''(t) + \left(\beta^2\tau - 1 + t\right)\theta'(t) = 0, \tag{42}$$

subject to the following set of boundary conditions

$$\theta(0) = 0, \quad \theta(1) = 1. \tag{43}$$

We rewrite Eq. (42) as

$$\frac{\theta''(t)}{\theta'(t)} = -\frac{1}{\beta^2\tau}\left(\frac{\beta^2\tau - 1}{t} + 1\right). \tag{44}$$

Integrating Eq. (44) once w.r.t. t from 0 to t, we obtain

$$\ln\left[\frac{\theta'(t)}{\theta'(1)}\right] = -\frac{1}{\beta^2\tau}\left[t + (\beta^2\tau - 1)\ln(t) - 1\right], \tag{45}$$

or

$$\theta'(t) = \theta'(1)\ \mathrm{e}^{\frac{1}{\beta^2\tau}}\ t^{\frac{1}{\beta^2\tau}-1}\mathrm{e}^{-\frac{t}{\beta^2\tau}}. \tag{46}$$

Integrating Eq. (46) once again w.r.t t from 0 to t and using the boundary condition $\theta(0) = 0$, we obtain

$$\theta(t) = \theta'(1)\ \mathrm{e}^{\frac{1}{\beta^2\tau}}(\beta^2\tau)^{\frac{1}{\beta^2\tau^2}}\int_0^{\frac{t}{\beta^2\tau}} z^{\frac{1}{\beta^2\tau}-1}\mathrm{e}^{-z}\ dz. \tag{47}$$

In terms of the generalized incomplete Gamma function, Eq. (47) can be rewritten as

$$\theta(t) = \theta'(1)\ \mathrm{e}^{\frac{1}{\beta^2\tau}}(\beta^2\tau)^{\frac{1}{\beta^2\tau^2}}\ \Gamma\left(\frac{1}{\beta^2\tau}, 0, \frac{t}{\beta^2\tau}\right). \tag{48}$$

On applying the boundary condition $\theta(1) = 1$, yields

$$\theta'(1) = \frac{\mathrm{e}^{-\frac{1}{\beta^2\tau}}(\beta^2\tau)^{-\frac{1}{\beta^2\tau^2}}}{\Gamma\left(\frac{1}{\beta^2\tau}, 0, \frac{1}{\beta^2\tau}\right)}. \tag{49}$$

Inserting this value of $\theta'(1)$ into Eq. (48), we have

$$\theta(t) = \frac{\Gamma\left(\frac{1}{\beta^2\tau}, 0, \frac{t}{\beta^2\tau}\right)}{\Gamma\left(\frac{1}{\beta^2\tau}, 0, \frac{1}{\beta^2\tau}\right)}, \tag{50}$$

which in terms of η gives the following exact solution:

$$\theta(\eta) = \frac{\Gamma\left(\frac{1}{\beta^2\tau}, 0, \frac{\mathrm{e}^{-\beta\eta}}{\beta^2\tau}\right)}{\Gamma\left(\frac{1}{\beta^2\tau}, 0, \frac{1}{\beta^2\tau}\right)}. \tag{51}$$

Here, we can easily observe that less computational work is needed to getting the exact solution of the heat transfer equation. Therefore, the transformation $t = \mathrm{e}^{-\beta\eta}$ may be recommended for any future analysis to analyze similar physical models.

4 Conclusion

The nonlinear differential equations governing the flow and heat transfer of nanofluids in the presence of a magnetic field have been solved exactly. Two transformations have been suggested to change the domain from unbounded domain into a bounded one. It was observed that one of the two transformations is easier than the other where few steps were required in getting the exact solution. Moreover, the obtained exact solution for the heat transfer equation agreed with the results in literature at a special case. Furthermore, the other previous exact solution published in [14] (Eq. 26) was expressed in terms of the WhittakerM function which is more complicated special function than the generalized incomplete Gamma function of the current analysis. It may be reasonable to refer here to that the analytical procedure followed in our paper is easier and more direct than the one considered in [14].

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