A Generalized Class of Kumaraswamy Lindley Distribution with Applications to Lifetime Data

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Abstract

In this paper, a new class of generalized distribution called the Exponentiated Kumaraswamy Lindley (EKL) distribution as well as related sub-distributions are proposed. This class of distributions contains the Kumaraswamy Lindley (KL), generalized Lindley (GL), Lindley (L) distributions as special cases. Series expansion of the density is obtained. Statistical properties of this class of distributions, including hazard function, reverse hazard function, monotonicity property, shapes, moments, reliability, quantile function, mean deviations, Bonferroni and Lorenz curves, entropy and Fisher information are derived. Method of maximum likelihood is used to estimate the parameters of this new class of distributions. Finally, real data examples are discussed to illustrate the applicability of this class of distributions.

Mathematics Subject Classification : 60E05; Secondary 62E15
Keywords: Generalized Distribution; Lindley Distribution; Kumaraswamy Distribution; Exponentiated Distribution.

1 Introduction

Jones [7] explored the background and genesis of the Kumaraswamy (Kum) distribution (Kumaraswamy [8]) and, more importantly, made clear some similarities and differences between the beta and Kum distributions. Among the advantages are: simple normalizing constant; the distribution and quantile functions have simple explicit formula which do not involve special functions; explicit formula for moments of order statistics and L-moments. However, compared to Kum distribution, the beta distribution has the following advantages: simpler formula for moments and moment generating function (mgf); a one-parameter sub-family of symmetric distributions; simpler moment estimation

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and more ways of generating the distribution via physical processes. Gupta and Kundu [6] provided a review and recent developments on the exponentiated exponential distribution. Cordeiro et al. [3] studied the Kumaraswamy Weibull (KW) distribution and applied it to failure time data.

Lindley [9] used a mixture of exponential and length-biased exponential distributions to illustrate the difference between fiducial and posterior distributions. This mixture is called the Lindley (L) distribution. Ghitany et al. [4] studied the statistical properties of the Lindley distribution. Sankaran [16] obtained and studied the Poisson-Lindley distribution.

Motivated by the advantages of the generalized or exponentiated Lindley distribution with respect to having a hazard function that exhibits increasing, decreasing and bathtub shapes, as well as the versatility and flexibility of the Kum distribution in modeling lifetime data, we propose and study a new class of distributions that inherit these very important and desirable properties, and also contains several sub-models with quite a number of shapes.

In this article, we propose a new distribution, called Exponentiated Kumaraswamy Lindley (EKL) distribution which generalizes the exponentiated Lindley or generalized Lindley distribution. We discuss some structural properties of this distribution, derive the Fisher information matrix and estimate the parameters via the method of maximum likelihood. In section 2, some generalized Lindley distributions, Kum distribution, Kum-G distribution, and the corresponding probability density functions (pdf) are presented. Section 3 contains results on the generalized and EKL distributions, including the hazard and reverse hazard functions, monotonicity property, and various sub-distributions. In section 4, we present the moment of the KGL distribution. Reliability and quantile function are given in sections 5 and 6, respectively. Mean deviations are presented in section 7. Section 8 contains results on Bonferroni and Lorenz curves. Measures of uncertainty, Fisher information and distribution of order statistics are presented in section 9. Maximum likelihood estimates of the model parameters and asymptotic confidence intervals are given in section 10. Section 11 contains applications of the proposed model to real data, followed by concluding remarks in section 12.

2 Some Basic Utility Notions

In this section, some generalized Lindley distributions and the Kumaraswamy generalized distribution are presented. The following series expansion is useful in subsequent sections: For $|\omega| < 1$ and $b > 0$ a real non-integer, we have

$$(1-\omega)^{b-1} = \sum_{j=0}^{\infty}(-1)^j \binom{b-1}{j} \omega^j.$$

2.1 Some Generalized Lindley Distributions

Nadarajah et al. [12] studied the mathematical and statistical properties of the generalized Lindley (GL) distribution. The cdf and pdf of the GL distribution are given by

$$G_{GL}(x;\alpha,\lambda) = \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\alpha}, \tag{1}$$

and

$$g_{GL}(x;\alpha,\lambda) = \frac{\alpha\lambda^2}{1+\lambda}(1+x)\left[1 - \frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\alpha-1}\exp(-\lambda x), \tag{2}$$

respectively, for $x > 0$, $\lambda > 0$, $\alpha > 0$. This distribution is essentially the exponentiated Lindley distribution. Zakerzadeh and Dolati [18] presented and studied another generalization of the Lindley distribution. These generalizations of the Lindley distribution are considered to be useful life distributions and are suitable for modeling data with different types of hazard rate functions: increasing, decreasing, bathtub and unimodal. These models constitute flexible family of distributions in terms of the varieties of shapes and hazard functions.

The one parameter cdf of the Lindley distribution [9] is given by

$$G_L(x;\lambda) = 1 - \frac{1+\lambda+\lambda x}{1+\lambda}e^{-\lambda x}, \quad \text{for } x > 0, \text{ and } \lambda > 0. \tag{3}$$

The corresponding Lindley pdf is given by

$$g_L(x;\lambda) = \frac{\lambda^2(1+x)}{1+\lambda}e^{-\lambda x}, \quad \text{for } x > 0, \text{ and } \lambda > 0. \tag{4}$$

Lindley distribution is a mixture of exponential and gamma distributions, that is $f(x;\lambda) = (1-p)f_G(x;\lambda) + pf_E(x;\lambda)$ with $p = \frac{\lambda}{1+\lambda}$, where $f_G(x;\lambda) \equiv GAM(2,\lambda)$, and $f_E(x;\lambda) \equiv EXP(\lambda)$. Now, let Y_1 and Y_2 be two independently gamma distributed random variables with parameters (α,λ) and $(\alpha+1,\lambda)$, respectively. For $\gamma > 0$, let $X = Y_1$ with probability $\frac{\lambda}{\lambda+\gamma}$ and $X = Y_2$ with probability $\frac{\gamma}{\lambda+\gamma}$, then the pdf of X (see Zakerzadeh and Dolati [18]) is given by

$$f_{GL}(x;\alpha,\lambda,\gamma) = \frac{\lambda^2(\lambda x)^{\alpha-1}(\alpha+\gamma x)e^{-\lambda x}}{(\lambda+\gamma)\Gamma(\alpha+1)},$$

for $x > 0$, $\lambda > 0$, $\alpha > 0$, $\gamma > 0$. Note that when $\alpha = \gamma = 1$, we obtain the Lindley pdf given in equation (4). When $\gamma = 0$ we have the gamma pdf with parameters α and λ. If $\alpha = 1$ and $\gamma = 0$ the resulting pdf is the exponential pdf with parameter λ.

2.2 Kum-Generalized Distribution

Kumaraswamy [8] in his paper proposed a two-parameter distribution (Kum distribution) defined in (0, 1). Its cdf and pdf are given by:

$$F(x;a;b) = 1 - (1 - x^a)^b, \quad \text{and} \quad f(x;a,b) = abx^{a-1}(1 - x^a)^{b-1},$$

respectively, for $x \in (0,1)$, $a > 0$, $b > 0$. The parameters a and b are the shape parameters. Let $G(x)$, be an arbitrary baseline cdf in the interval $(0,1)$. Kum-G distribution has cdf $F(x;a,b)$ and pdf $f(x;a,b)$ defined by

$$F(x;a,b) = 1 - (1 - [G(x)]^a)^b, \tag{5}$$

and

$$f(x;a,b) = abg(x)[G(x)]^{a-1}(1 - [G(x)]^a)^{b-1} \quad \text{for } a > 0, b > 0, \tag{6}$$

where $g(x) = \frac{dG(x)}{dx}$ is the pdf corresponding to the baseline cdf $G(x)$.

3 Exponentiated Kumaraswamy Lindley Distribution

Exponentiated distributions are very important in statistics as indicated by Mudholkar and Srivastava [10] who proposed and studied the exponentiated Weibull distribution to analyze bathtub failure data. Gupta et al. [5] introduced and developed the general class of exponentiated distributions. The authors defined and studied the exponentiated exponential distribution. Nadarajah and Kotz [13] introduced the exponentiated Fréchet distribution, and Nadarajah [11] proposed and developed the exponentiated Gumbel distribution. For a baseline cdf $G(x)$, in general the exponentiated version $F(x) = [G(x)]^\delta$ is quite different and flexible to accommodate both monotone and non-monotone hazard rate functions. Exponentiated distributions are indeed quite different from the baseline cdf $G(x)$. In particular, $F(x) = [1 - \exp(-\lambda x)]^\delta$ has a constant hazard rate λ when $\delta = 1$, increasing hazard rate if $\delta > 1$ and decreasing hazard rate if $\delta < 1$.

Now, with the choice of $G(x)$ in the Kum-generalized distribution as the GL distribution, we obtain the KGL distribution. The four-parameter KGL cdf and pdf are given by

$$F_{KGL}(x;\alpha,\lambda,a,b) = 1 - \left\{1 - \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{a\alpha}\right\}^b, \tag{7}$$

and

$$\begin{aligned} f_{KGL}(x;\alpha,\lambda,a,b) &= \frac{ab\alpha\lambda^2}{1+\lambda}(1+x)\exp(-\lambda x) \\ &\times \left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{a\alpha-1} \\ &\times \left\{1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{a\alpha}\right\}^{b-1}, \qquad (8) \end{aligned}$$

for $x>0$, $\alpha>0$, $\lambda>0$, $a>0$, $b>0$, respectively. We can set the dependent parameter $a\alpha=\theta$, so the KGL cdf and pdf reduce to Kumaraswamy Lindley (KL) distribution with cdf and pdf given by:

$$F_{KL}(x;\lambda,\theta,b) = 1-\left\{1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta}\right\}^{b}, \qquad (9)$$

and

$$\begin{aligned} f_{KL}(x;\lambda,\theta,b) &= \frac{b\theta\lambda^2}{1+\lambda}(1+x)\exp(-\lambda x)\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta-1} \\ &\times \left\{1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta}\right\}^{b-1}, \qquad (10) \end{aligned}$$

for $x>0$, $\lambda>0$, $\theta>0$, $b>0$, respectively.

Consider the exponentiated Kumaraswamy Lindley (EKL) distribution with cdf and pdf given by:

$$F_{EKL}(x;\lambda,\theta,b,\delta) = \left\{1-\left\{1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta}\right\}^{b}\right\}^{\delta}, \qquad (11)$$

and

$$\begin{aligned} f_{EKL}(x;\lambda,\theta,b,\delta) &= \delta[F_{KL}(x)]^{\delta-1}f_{KL}(x) \\ &= \frac{\delta b\theta\lambda^2}{1+\lambda}(1+x)\exp(-\lambda x)\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta-1} \\ &\times \left\{1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta}\right\}^{b-1} \\ &\times \left\{1-\left\{1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta}\right\}^{b}\right\}^{\delta-1}, \qquad (12) \end{aligned}$$

for $x > 0$, $\lambda > 0$, $\theta > 0$, $b > 0$, $\delta > 0$, respectively.

If $\delta = b = 1$, the EKL distribution is exactly identical to the beta generalized Lindley (BGL) distribution (Oluyede and Yang [14]) with pdf given by

$$f_{BGL}(x;\lambda,\theta) \quad = \quad \frac{\theta\lambda^2}{1+\lambda}(1+x)\exp(-\lambda x)\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta-1},$$

for $x > 0$, $\lambda > 0$, $\theta > 0$. Figure 1 illustrates some possible shapes of the pdf of the EKL distribution.

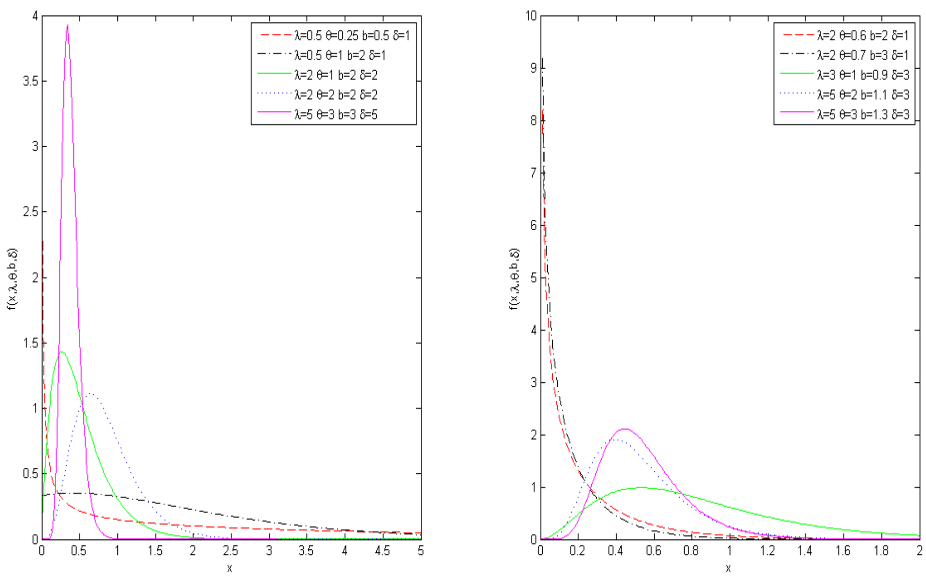

Figure 1: Plots of the pdf of EKL distribution for selected values of the parameters

3.1 Expansion of Density

In this section, the series expansion of the EKL pdf is presented. When $b > 0$ and $\delta > 0$ are real non-integer, we use the following series representations

$$[1-G_{GL}(x)]^{b-1} = \sum_{i=0}^{\infty}(-1)^i\binom{b-1}{i}[G_{GL}(x)]^i,$$

and

$$\{1-[1-G_{GL}(x)]^b\}^{\delta-1} = \sum_{j,r=0}^{\infty}(-1)^{j+r}\binom{\delta-1}{j}\binom{bj}{r}[G_{GL}(x)]^r.$$

From the above expansions and equation (12), we can write the EKL density as

$$\begin{aligned} f_{EKL}(x;\lambda,\theta,b,\delta) &= \frac{\delta b\theta\lambda^2}{1+\lambda}(1+x)\exp(-\lambda x)\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta-1} \\ &\times \sum_{i=0}^{\infty}(-1)^i\binom{b-1}{i}[G_{GL}(x)]^i \\ &\times \sum_{j,r=0}^{\infty}(-1)^{j+r}\binom{\delta-1}{j}\binom{bj}{r}[G_{GL}(x)]^r \\ &= \frac{\delta b\theta\lambda^2}{1+\lambda}(1+x)\exp(-\lambda x) \\ &\times \sum_{i,j,r=0}^{\infty} w_{i,j,r}\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta(i+r+1)-1}, \end{aligned} \quad (13)$$

where the coefficient $w_{i,j,r}$ is

$$w_{i,j,r} = w_{i,j,r}(b,\delta) = (-1)^{i+j+r}\binom{b-1}{i}\binom{\delta-1}{j}\binom{bj}{r},$$

and $\sum_{i=0}^{\infty} w_{i,j,r} = 1$, for $x > 0$, $\lambda > 0$, $\theta > 0$, $b > 0$, $\delta > 0$. Consequently, the EKL density is given by three infinite weighted power series sums of the baseline distribution function $G_{GL}(x)$.

Note here that we have considered the case when $b > 0$ and $\delta > 0$ are non-integer, however the other cases can be similarly derived.

3.2 Some Sub-models of EKL Distribution

In this section, we present the sub-models of EKL distribution for selected values of the parameters θ, b, and δ.

1. $\boldsymbol{\delta = 1}$

   If $\delta = 1$, this is the Kumaraswamy Lindley (KL) distribution with cdf and pdf given by

$$F_{KL}(x;\lambda,\theta,b) = 1-\left\{1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta}\right\}^{b},$$

   and

$$\begin{aligned} f_{KL}(x;\lambda,\theta,b) &= \frac{b\theta\lambda^2}{1+\lambda}(1+x)\exp(-\lambda x)\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta-1} \\ &\times \left\{1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta}\right\}^{b-1}, \end{aligned}$$

for $x > 0$, $\lambda > 0$, $\theta > 0$, $b > 0$, respectively.

2. $\boldsymbol{\theta = 1}$
If $\theta = 1$, the EKL cdf and pdf reduces to:

$$F_{EKL}(x;\lambda,b,\delta) = \left\{1 - \left[\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^b\right\}^\delta,$$

and

$$\begin{aligned} f_{EKL}(x;\lambda,b,\delta) &= \frac{\delta b\lambda^2}{1+\lambda}(1+x)\exp(-\lambda x) \\ &\times \left[\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{b-1} \\ &\times \left\{1 - \left[\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^b\right\}^{\delta-1}, \end{aligned}$$

for $x > 0$, $\lambda > 0$, $b > 0$, $\delta > 0$, respectively.

3. $\boldsymbol{b = 1}$
If $b = 1$, this is the generalized Lindley (GL) distribution. The GL cdf and pdf are given by

$$F_{GL}(x;\alpha,\lambda) = \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^\alpha,$$

and

$$f_{GL}(x;\alpha,\lambda) = \frac{\alpha\lambda^2}{1+\lambda}(1+x)\left[1 - \frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\alpha-1}\exp(-\lambda x),$$

where $\alpha = \theta\delta$ in this case, for $x > 0$, $\lambda > 0$, $\alpha > 0$, respectively.

4. $\boldsymbol{\delta = \theta = 1}$
If $\delta = \theta = 1$, the EKL cdf and pdf reduces to:

$$F_{EKL}(x;\lambda,b) = 1 - \left[\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^b,$$

and

$$f_{EKL}(x;\lambda,b) = \frac{b\lambda^2}{1+\lambda}(1+x)\exp(-\lambda x)\left[\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{b-1},$$

for $x > 0$, $\lambda > 0$, $b > 0$, respectively.

5. $\boldsymbol{\delta = \theta = b = 1}$
If $\delta = \theta = b = 1$, ten we have the Lindley (L) distribution given by equation (3).

3.3 Hazard and Reverse Hazard Functions

In this section, the hazard and reverse hazard functions of the EKL distribution are presented. Graphs of these functions for selected values of the parameters λ, θ, b, and δ are also presented. The hazard and reverse hazard functions of EKL distribution are given by

$$\begin{aligned}
h_{_{EKL}}(x;\lambda,\theta,b,\delta) &= \frac{f_{EKL}(x;\lambda,\theta,b,\delta)}{\bar{F}_{EKL}(x;\lambda,\theta,b,\delta)} \\
&= \frac{\delta b\theta\lambda^2}{1+\lambda}(1+x)\exp(-\lambda x)\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta-1} \\
&\times \left\{1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta}\right\}^{b-1} \\
&\times \left\{1-\left\{1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta}\right\}^{b}\right\}^{\delta-1} \\
&\times \left\{1-\left\{1-\left\{1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta}\right\}^{b}\right\}^{\delta}\right\}^{-1},
\end{aligned}$$

and

$$\begin{aligned}
\tau_{_{EKL}}(x;\lambda,\theta,b,\delta) &= \frac{f_{EKL}(x;\lambda,\theta,b,\delta)}{F_{EKL}(x;\lambda,\theta,b,\delta)} \\
&= \frac{\delta b\theta\lambda^2}{1+\lambda}(1+x)\exp(-\lambda x)\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta-1} \\
&\times \left\{1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta}\right\}^{b-1} \\
&\times \left\{1-\left\{1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta}\right\}^{b}\right\}^{-1},
\end{aligned}$$

for $x > 0$, $\lambda > 0$, $\theta > 0$, $b > 0$, $\delta > 0$, respectively. The graphs of hazard function of EKL distribution are shown in Figure 2. These graphs show the variety of shapes for the EKL hazard function including bathtub, deceasing, and increasing hazard rate functions.

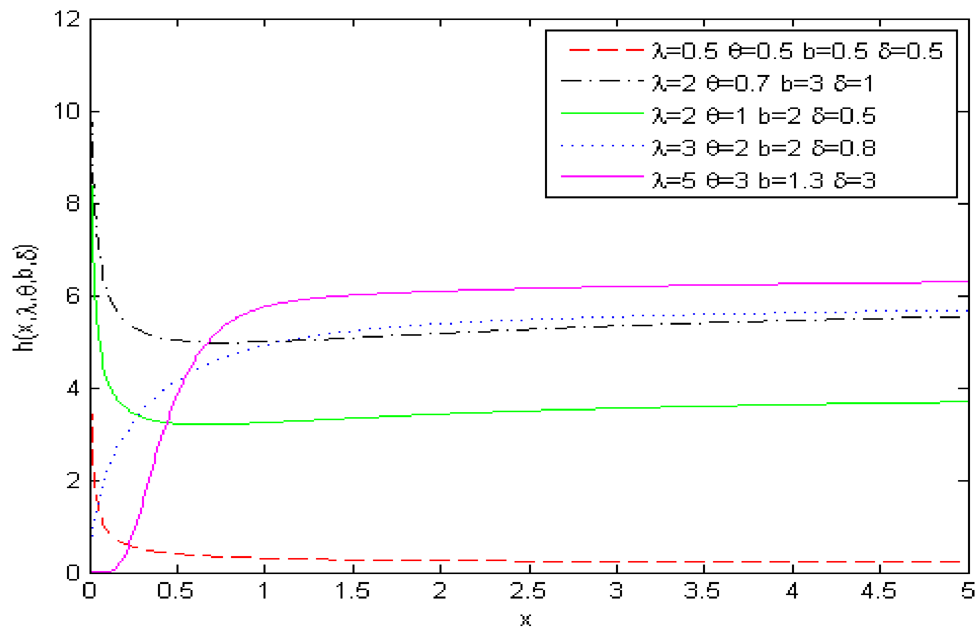

Figure 2: Plots of the hazard function of EKL distribution for selected values of the parameters

3.4 Monotonicity Property

In this section, we discuss the monotonicity properties of the EKL distribution. Let

$$V(x) = G_L(x;\lambda) = 1 - \frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x), \tag{14}$$

then from equation (12), we can rewrite EKL pdf as

$$\begin{aligned} f_{EKL}(x;\lambda,\theta,b,\delta) &= \frac{\delta b\theta\lambda^2}{1+\lambda}(1+x)\exp(-\lambda x)[V(x)]^{\theta-1} \\ &\times \; [1-[V(x)]^{\theta}]^{b-1}\{1-[1-[V(x)]^{\theta}]^{b}\}^{\delta-1}, \end{aligned} \tag{15}$$

for $x > 0$, $\lambda > 0$, $\theta > 0$, $b > 0$, $\delta > 0$. It follows that

$$\begin{aligned} \log\, f_{EKL}(x) &= \log\left(\frac{\delta b\theta\lambda^2}{1+\lambda}\right) + \log\,(1+x) - \lambda x + (\theta-1)\log\, V(x) \\ &+ \; (b-1)\log\,[1-[V(x)]^{\theta}] \\ &+ \; (\delta-1)\log\,\left(1-[1-[V(x)]^{\theta}]^{b}\right), \end{aligned} \tag{16}$$

and

$$\begin{aligned}\frac{\mathrm{d}\,\log\, f_{EKL}(x)}{\mathrm{d}x} &= \frac{1}{1+x} - \lambda + \frac{\theta - 1 + (1-b\theta)[V(x)]^{\theta}}{V(x)[1-[V(x)]^{\theta}]}V'(x) \\ &+ b\theta(\delta-1)\frac{[1-[V(x)]^{\theta}]^{b-1}[V(x)]^{\theta-1}V'(x)}{1-[1-[V(x)]^{\theta}]^{b}}, \qquad (17)\end{aligned}$$

where $V(x) = 1 - \frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)$, and $V'(x) = \frac{\mathrm{d}\,V(x)}{\mathrm{d}x} = \frac{\lambda^2}{1+\lambda}(1+x)\exp(-\lambda x)$.

Analysis: We know that $x > 0$, $\lambda > 0$, $\theta > 0$, $b > 0$, and $\delta > 0$, so that

$$V'(x) = \frac{\mathrm{d}\,V(x)}{\mathrm{d}x} = \frac{\lambda^2}{1+\lambda}(1+x)\exp(-\lambda x) > 0, \forall\, x > 0.$$

If $x \to 0$,

$$V(x) = 1 - \frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x) \to 0.$$

If $x \to \infty$,

$$V(x) = 1 - \frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x) \to 1,$$

since

$$\lim_{x\to\infty}(1+\lambda+\lambda x)\exp(-\lambda x) = \lim_{x\to\infty}\frac{1+\lambda+\lambda x}{\exp(\lambda x)} = \lim_{x\to\infty}\frac{\lambda}{\lambda\exp(\lambda x)} = 0.$$

Thus, $V(x)$ is monotonically increasing from 0 to 1. Now, as $0 < V(x) < 1$, we get $0 < [V(x)]^{\theta} < 1, \forall\,\theta > 0$, $[V(x)]^{\theta-1} > 0, \forall\,\theta > 0$, $0 < 1-[V(x)]^{\theta} < 1, \forall\,\theta > 0$, $0 < [1-[V(x)]^{\theta}]^{b} < 1, \forall\,\theta > 0$, $[1-[V(x)]^{\theta}]^{b-1} > 0, \forall\,\theta > 0$, and $0 < 1-[1-[V(x)]^{\theta}]^{b} < 1, \forall\,\theta > 0$. Then, we have

$$\frac{V'(x)}{V(x)[1-[V(x)]^{\theta}]} > 0, \quad \text{and} \quad \frac{[1-[V(x)]^{\theta}]^{b-1}[V(x)]^{\theta-1}V'(x)}{1-[1-[V(x)]^{\theta}]^{b}} > 0.$$

Also, from $x > 0$, we have $0 < \frac{1}{1+x} < 1$.

If $\lambda \geq 1$, $0 < \theta \leq 1$, $b\theta \geq 1$, and $0 < \delta \leq 1$, we get $\frac{\mathrm{d}\,\log\, f_{KGL}(x)}{\mathrm{d}x} < 0$, since $\frac{1}{1+x} - \lambda < 0$, $\theta - 1 + (1-b\theta)[V(x)]^{\theta} \leq 0$, and $b\theta(\delta-1) \leq 0$. In this case, $f_{EKL}(x;\lambda,\theta,b,\delta)$ is monotonically decreasing for all x.

If $\lambda < 1$, $f_{EKL}(x;\lambda,\theta,b,\delta)$ could attain a maximum, a minimum or a point of inflection according to whether

$$\frac{\mathrm{d}^2\,\log\, f_{EKL}(x)}{\mathrm{d}x^2} < 0, \frac{\mathrm{d}^2\,\log\, f_{EKL}(x)}{\mathrm{d}x^2} > 0, \text{or}\ \frac{\mathrm{d}^2\,\log\, f_{EKL}(x)}{\mathrm{d}x^2} = 0,$$

respectively.

3.5 Shape of Hazard Function

Note that if $x \to \infty$, then $\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x) \to 0$. Also,

$$\begin{aligned}\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta-1} &= \sum_{i=0}^{\infty}\binom{\theta-1}{i}\left[-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{i}\\ &\approx 1-(\theta-1)\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x).\end{aligned}$$

Consequently,

$$f_{EKL}(x;\lambda,\theta,b,\delta) \sim \frac{\delta b\theta^{b}\lambda^{b+1}}{(1+\lambda)^{b}}x^{b}\exp(-\lambda bx). \quad (18)$$

If $x \to 0$, then

$$f_{EKL}(x;\lambda,\theta,b,\delta) \sim \frac{\delta b^{\delta}\theta\lambda^{2\theta\delta}}{(1+\lambda)^{\theta\delta}}x^{\theta\delta}. \quad (19)$$

The cdf of EKL distribution is

$$F_{EKL}(x;\lambda,\theta,b,\delta) = \left\{1-\left\{1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta}\right\}^{b}\right\}^{\delta},$$

for $x>0$, $\lambda>0$, $\theta>0$, $b>0$, $\delta>0$.

If $x \to \infty$, then $\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x) \to 0$. Also,

$$\begin{aligned}\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta} &= \sum_{i=0}^{\infty}\binom{\theta}{i}\left[-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{i}\\ &\approx 1-\theta\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x),\end{aligned}$$

so that

$$F_{EKL}(x;\lambda,\theta,b,\delta) \approx 1-\delta\left[\theta\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{b}.$$

Also,

$$\begin{aligned}1-F_{EKL}(x;\lambda,\theta,b,\delta) &\approx \delta\left[\theta\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{b}\\ &= \frac{\delta\theta^{b}}{(1+\lambda)^{b}}(1+\lambda+\lambda x)^{b}[\exp(-\lambda x)]^{b}\\ &\sim \frac{\delta\theta^{b}}{(1+\lambda)^{b}}(\lambda x)^{b}[\exp(-\lambda x)]^{b}\\ &= \frac{\delta\theta^{b}\lambda^{b}}{(1+\lambda)^{b}}x^{b}\exp(-\lambda bx).\end{aligned}$$

That is,

$$1 - F_{EKL}(x; \lambda, \theta, b, \delta) \sim \frac{\delta \theta^b \lambda^b}{(1+\lambda)^b} x^b \exp(-\lambda b x). \tag{20}$$

If $x \to 0$, then $\frac{1+\lambda+\lambda x}{1+\lambda} \exp(-\lambda x) \to 1$, and $1 - \frac{1+\lambda+\lambda x}{1+\lambda} \exp(-\lambda x) \to 0$, so that

$$\left[1 - \frac{1+\lambda+\lambda x}{1+\lambda} \exp(-\lambda x)\right]^\theta \to 0,$$

and

$$\begin{aligned}\left\{1 - \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda} \exp(-\lambda x)\right]^\theta\right\}^b &= \sum_{i=0}^{\infty} \binom{b}{i} \\ &\times \left\{-\left[1 - \frac{1+\lambda+\lambda x}{1+\lambda} \exp(-\lambda x)\right]^\theta\right\}^i \\ &\approx 1 - b\left[1 - \frac{1+\lambda+\lambda x}{1+\lambda} \exp(-\lambda x)\right]^\theta.\end{aligned}$$

Thus,

$$F_{EKL}(x; \lambda, \theta, b, \delta) \sim \frac{b^\delta \lambda^{2\theta\delta}}{(1+\lambda)^{\theta\delta}} x^{\theta\delta}. \tag{21}$$

The hazard functions of EKL is given by

$$h_{EKL}(x; \lambda, \theta, b, \delta) = \frac{f_{EKL(x;\lambda,\theta,b,\delta)}}{\bar{F}_{EKL}(x; \lambda, \theta, b, \delta)} = \frac{f_{EKL}(x; \lambda, \theta, b, \delta)}{1 - F_{EKL}(x; \lambda, \theta, b, \delta)}, \tag{22}$$

for $x > 0$, $\lambda > 0$, $\theta > 0$, $b > 0$, $\delta > 0$.

If $x \to \infty$, with equations (18) and (20) in equation (22), we get

$$\begin{aligned}h_{EKL}(x; \lambda, \theta, b, \delta) &\sim \frac{\delta b \theta^b \lambda^{b+1} x^b \exp(-\lambda b x)/(1+\lambda)^b}{\delta \theta^b \lambda^b x^b \exp(-\lambda b x)/(1+\lambda)^b} \\ &= b\lambda.\end{aligned}$$

If $x \to 0$, with equations (19) and (21) in equation (22), we get

$$\begin{aligned}h_{EKL}(x; \lambda, \theta, b, \delta) &\sim \frac{\delta b^\delta \theta \lambda^{2\theta\delta} x^{\theta\delta}/(1+\lambda)^{\theta\delta}}{1 - b^\delta \lambda^{2\theta\delta} x^{\theta\delta}/(1+\lambda)^{\theta\delta}} \\ &\sim \frac{\delta b^\delta \theta \lambda^{2\theta\delta}}{(1+\lambda)^{\theta\delta}} x^{\theta\delta},\end{aligned}$$

since $\frac{b^\delta \lambda^{2\theta\delta}}{(1+\lambda)^{\theta\delta}} x^{\theta\delta} \to 0$, as $x \to 0$.

4 Moments of EKL Distribution

In this section, moments of the EKL distribution are presented. The following lemma is proved by using the result given by Nadarajah et al. [12].

Lemma 1

Let

$$K(m,n,p,q) = \int_0^\infty x^p(1+x)\left[1-\frac{1+n+nx}{1+n}\exp(-nx)\right]^{m-1}\exp(-qx)\,\mathrm{d}x.$$

1. If m is non-integer, we have

$$K(m,n,p,q) = \sum_{l=0}^{\infty}\sum_{k=0}^{l}\sum_{w=0}^{k+1}\binom{m-1}{l}\binom{l}{k}\binom{k+1}{w}\frac{(-1)^l n^k\Gamma(p+w+1)}{(1+n)^l(nl+q)^{p+w+1}}.$$

2. If m is an integer, we have

$$K(m,n,p,q) = \sum_{l=0}^{m-1}\sum_{k=0}^{l}\sum_{w=0}^{k+1}\binom{m-1}{l}\binom{l}{k}\binom{k+1}{w}\frac{(-1)^l n^k\Gamma(p+w+1)}{(1+n)^l(nl+q)^{p+w+1}}.$$

Proof. 1. If m is non-integer, then

$$\left[1-\frac{1+n+nx}{1+n}\exp(-nx)\right]^{m-1} = \sum_{l=0}^{\infty}\binom{m-1}{l}(-1)^l \times \left(\frac{1+n+nx}{1+n}\exp(-nx)\right)^l,$$

and

$$K(m,n,p,q) = \sum_{l=0}^{\infty}\binom{m-1}{l}\frac{(-1)^l}{(1+n)^l} \times \int_0^\infty x^p(1+x)(1+n+nx)^l\exp[-(nl+q)x]\,\mathrm{d}x.$$

Furthermore, l is an integer, so that

$$(1+n+nx)^l = \sum_{k=0}^{l}\binom{l}{k}(n+nx)^k = \sum_{k=0}^{l}\binom{l}{k}n^k(1+x)^k,$$

and

$$K(m,n,p,q) = \sum_{l=0}^{\infty}\binom{m-1}{l}\frac{(-1)^l}{(1+n)^l}\sum_{k=0}^{l}\binom{l}{k}n^k \times \int_0^\infty x^p(1+x)^{k+1}\exp[-(nl+q)x]\,\mathrm{d}x.$$

Now, k is an integer, so that

$$(1+x)^{k+1} = \sum_{w=0}^{k+1} \binom{k+1}{w} x^{w},$$

and

$$\begin{aligned} K(m,n,p,q) &= \sum_{l=0}^{\infty} \binom{m-1}{l} \frac{(-1)^{l}}{(1+n)^{l}} \sum_{k=0}^{l} \binom{l}{k} n^{k} \sum_{w=0}^{k+1} \binom{k+1}{w} \\ &\times \int_{0}^{\infty} x^{p+w} \exp[-(nl+q)x]\, \mathrm{d}x \\ &= \sum_{l=0}^{\infty} \sum_{k=0}^{l} \sum_{w=0}^{k+1} \binom{m-1}{l} \binom{l}{k} \binom{k+1}{w} \frac{(-1)^{l} n^{k} \Gamma(p+w+1)}{(1+n)^{l}(nl+q)^{p+w+1}}. \end{aligned} \tag{23}$$

2. If m is an integer, the index l in equation (23) stops at $m-1$, so that

$$K(m,n,p,q) = \sum_{l=0}^{m-1} \sum_{k=0}^{l} \sum_{w=0}^{k+1} \binom{m-1}{l} \binom{l}{k} \binom{k+1}{w} \frac{(-1)^{l} n^{k} \Gamma(p+w+1)}{(1+n)^{l}(nl+q)^{p+w+1}}.$$

□

The s^{th} moment of the EKL distribution, say μ_s', is given by

$$\mu_s' = \int_{0}^{\infty} x^{s} f_{EKL}(x;\lambda,\theta,b,\delta)\, \mathrm{d}x.$$

Let $b>0$ and $\delta>0$ be real non-integer, then from equation (13), we obtain

$$\begin{aligned} \mu_s' &= \frac{\delta b \theta \lambda^{2}}{1+\lambda} \sum_{i,j,r=0}^{\infty} (-1)^{i+j+r} \binom{b-1}{i} \binom{\delta-1}{j} \binom{bj}{r} \int_{0}^{\infty} x^{s}(1+x)\exp(-\lambda x) \\ &\times \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta(i+r+1)-1} \mathrm{d}x. \end{aligned}$$

Now, using Lemma 1 with $m=\theta(i+r+1)$, $n=\lambda$, $p=s$, $q=\lambda$, we have

$$\begin{aligned} \mu_s' &= \frac{\delta b \theta \lambda^{2}}{1+\lambda} \sum_{i,j,r=0}^{\infty} (-1)^{i+j+r} \binom{b-1}{i} \binom{\delta-1}{j} \binom{bj}{r} \\ &\times K(\theta(i+r+1),\lambda,s,\lambda). \end{aligned} \tag{24}$$

If $\theta > 0$ is non-integer, then the s^{th} moment of the EKL is given by

$$\begin{aligned}
\mu_s' &= \delta b\theta \sum_{i,j,r,l=0}^{\infty} \sum_{k=0}^{l} \sum_{w=0}^{k+1} \\
&\times \binom{b-1}{i}\binom{\delta-1}{j}\binom{bj}{r}\binom{\theta(i+r+1)-1}{l}\binom{l}{k}\binom{k+1}{w} \\
&\times \frac{(-1)^{i+j+r+l}\Gamma(s+w+1)}{(1+\lambda)^{l+1}\lambda^{s+w-k-1}(1+l)^{s+w+1}}. \qquad (25)
\end{aligned}$$

If $\theta > 0$ is an integer, then $\theta(i+r+1)$ is an integer, so that the index l in equation (25) stops at $\theta(i+r+1)-1$.

Note here that we have considered the case when $b > 0$ and $\delta > 0$ are non-integer, however the other cases can be similarly derived.

5 Reliability

In reliability and related areas, the stress-strength model describes the life of a component with random strength X, that is subjected to a random stress Y. The component will fail at the instant that the applied stress exceeds the strength, and the component will function satisfactorily whenever $X > Y$. We derive $R = P(X > Y)$, a measure of component reliability, when X and Y have independent $EKL(\lambda_1, \theta_1, b_1, \delta_1)$ and $EKL(\lambda_2, \theta_2, b_2, \delta_2)$ distributions, respectively. Note from equations (11) and (12) that

$$\begin{aligned}
R &= P(X > Y) \\
&= \int_0^\infty f_X(x;\lambda_1,\theta_1,b_1,\delta_1)F_Y(x;\lambda_2,\theta_2,b_2,\delta_2)\,\mathrm{d}x \\
&= \int_0^\infty \frac{\delta_1 b_1\theta_1\lambda_1^2}{1+\lambda_1}(1+x)\exp(-\lambda_1 x)\left[1-\frac{1+\lambda_1+\lambda_1 x}{1+\lambda_1}\exp(-\lambda_1 x)\right]^{\theta_1-1} \\
&\times \left\{1-\left[1-\frac{1+\lambda_1+\lambda_1 x}{1+\lambda_1}\exp(-\lambda_1 x)\right]^{\theta_1}\right\}^{b_1-1} \\
&\times \left\{1-\left\{1-\left[1-\frac{1+\lambda_1+\lambda_1 x}{1+\lambda_1}\exp(-\lambda_1 x)\right]^{\theta_1}\right\}^{b_1}\right\}^{\delta_1-1} \\
&\times \left\{1-\left\{1-\left[1-\frac{1+\lambda_2+\lambda_2 x}{1+\lambda_2}\exp(-\lambda_2 x)\right]^{\theta_2}\right\}^{b_2}\right\}^{\delta_2}\mathrm{d}x. \qquad (26)
\end{aligned}$$

Applying the series expansions

$$\left[1-\frac{1+\lambda_1+\lambda_1 x}{1+\lambda_1}\exp(-\lambda_1 x)\right]^{\theta_1-1} = \sum_{k=0}^{\infty}\sum_{m=0}^{k}\binom{\theta_1-1}{k}\binom{k}{m} \times \frac{(-1)^k\lambda_1^m x^m\exp(-\lambda_1 k x)}{(1+\lambda_1)^m}, \quad (27)$$

$$\left\{1-\left[1-\frac{1+\lambda_1+\lambda_1 x}{1+\lambda_1}\exp(-\lambda_1 x)\right]^{\theta_1}\right\}^{b_1-1} = \sum_{l,p=0}^{\infty}\sum_{n=0}^{p}\binom{b_1-1}{l}\binom{\theta_1 l}{p}\binom{p}{n} \times \frac{(-1)^{l+p}\lambda_1^n x^n\exp(-\lambda_1 p x)}{(1+\lambda_1)^n}, \quad (28)$$

$$\left\{\left\{1-\left[1-\frac{1+\lambda_1+\lambda_1 x}{1+\lambda_1}\exp(-\lambda_1 x)\right]^{\theta_1}\right\}^{b_1}\right\}^{\delta_1-1} = \sum_{q,e,f=0}^{\infty}\sum_{g=0}^{f}\binom{\delta_1-1}{q}\binom{b_1 q}{e}\binom{\theta_1 e}{f} \times \binom{f}{g}\frac{(-1)^{q+e+f}\lambda_1^g x^g e^{(-\lambda_1 f x)}}{(1+\lambda_1)^g}, (29)$$

$$\left\{\left\{1-\left[1-\frac{1+\lambda_2+\lambda_2 x}{1+\lambda_2}\exp(-\lambda_2 x)\right]^{\theta_2}\right\}^{b_2}\right\}^{\delta_2} = \sum_{t,h,i=0}^{\infty}\sum_{j=0}^{i}\binom{\delta_2}{t}\binom{b_2 t}{h}\binom{\theta_2 h}{i}\binom{i}{j} \times \frac{(-1)^{t+h+i}\lambda_2^j x^j\exp(-\lambda_2 i x)}{(1+\lambda_2)^j}, \quad (30)$$

and substituting equations (27), (28), (29), and (30) into equation (26), we get

$$\begin{aligned}
R &= \int_0^\infty \frac{\delta_1 b_1 \theta_1 \lambda_1^2}{1+\lambda_1}(1+x)\exp(-\lambda_1 x) \\
&\times \sum_{k=0}^{\infty}\sum_{m=0}^{k}\binom{\theta_1-1}{k}\binom{k}{m}\frac{(-1)^k \lambda_1^m x^m \exp(-\lambda_1 k x)}{(1+\lambda_1)^m} \\
&\times \sum_{l,p=0}^{\infty}\sum_{n=0}^{p}\binom{b_1-1}{l}\binom{\theta_1 l}{p}\binom{p}{n}\frac{(-1)^{l+p} \lambda_1^n x^n \exp(-\lambda_1 p x)}{(1+\lambda_1)^n} \\
&\times \sum_{q,e,f=0}^{\infty}\sum_{g=0}^{f}\binom{\delta_1-1}{q}\binom{b_1 q}{e}\binom{\theta_1 e}{f}\binom{f}{g}\frac{(-1)^{q+e+f} \lambda_1^g x^g \exp(-\lambda_1 f x)}{(1+\lambda_1)^g} \\
&\times \sum_{t,h,i=0}^{\infty}\sum_{j=0}^{i}\binom{\delta_2}{t}\binom{b_2 t}{h}\binom{\theta_2 h}{i}\binom{i}{j}\frac{(-1)^{t+h+i} \lambda_2^j x^j \exp(-\lambda_2 i x)}{(1+\lambda_2)^j}\,\mathrm{d}x \\
&= \delta_1 b_1 \theta_1 \sum_{k,l,p,q,e,f,t,h,i=0}^{\infty}\sum_{m=0}^{k}\sum_{n=0}^{p}\sum_{g=0}^{f}\sum_{j=0}^{i}\binom{\theta_1-1}{k}\binom{k}{m}\binom{b_1-1}{l}\binom{\theta_1 l}{p}\binom{p}{n} \\
&\times \binom{\delta_1-1}{q}\binom{b_1 q}{e}\binom{\theta_1 e}{f}\binom{f}{g}\binom{\delta_2}{t}\binom{b_2 t}{h}\binom{\theta_2 h}{i}\binom{i}{j} \\
&\times \frac{(-1)^{k+l+p+q+e+f+t+h+i}\lambda_1^{m+n+g+2}\lambda_2^j}{(1+\lambda_1)^{m+n+g+1}(1+\lambda_2)^j} \\
&\times \int_0^\infty (1+x)x^{m+n+g+j}\exp(-[\lambda_1(k+p+f+1)+\lambda_2 i]x)\,\mathrm{d}x. \qquad (31)
\end{aligned}$$

We use the following gamma functions in equation (31)

$$\begin{aligned}
&\int_0^\infty (1+x)x^{m+n+g+j}e^{-[\lambda_1(k+p+f+1)+\lambda_2 i]x}\,\mathrm{d}x \\
&= \int_0^\infty x^{m+n+g+j}e^{-[\lambda_1(k+p+f+1)+\lambda_2 i]x}\,\mathrm{d}x \\
&+ \int_0^\infty x^{m+n+g+j+1}e^{-[\lambda_1(k+p+f+1)+\lambda_2 i]x}\,\mathrm{d}x \\
&= \frac{(m+n+g+j)!}{[\lambda_1(k+p+f+1)+\lambda_2 i]^{m+n+g+j+1}} \\
&\times \left[1+\frac{m+n+g+j+1}{\lambda_1(k+p+f+1)+\lambda_2 i}\right],
\end{aligned}$$

to get

$$\begin{aligned} R &= \delta_1 b_1 \theta_1 \sum_{k,l,p,q,e,f,t,h,i=0}^{\infty} \sum_{m=0}^{k} \sum_{n=0}^{p} \sum_{g=0}^{f} \sum_{j=0}^{i} \binom{\theta_1 - 1}{k} \binom{k}{m} \binom{b_1 - 1}{l} \binom{\theta_1 l}{p} \binom{p}{n} \\ &\times \binom{\delta_1 - 1}{q} \binom{b_1 q}{e} \binom{\theta_1 e}{f} \binom{f}{g} \binom{\delta_2}{t} \binom{b_2 t}{h} \binom{\theta_2 h}{i} \binom{i}{j} \\ &\times \frac{(-1)^{k+l+p+q+e+f+t+h+i} \lambda_1^{m+n+g+2} \lambda_2^{j}}{(1+\lambda_1)^{m+n+g+1}(1+\lambda_2)^{j}} \frac{(m+n+g+j)!}{[\lambda_1(k+p+f+1)+\lambda_2 i]^{m+n+g+j+1}} \\ &\times \left[1 + \frac{m+n+g+j+1}{\lambda_1(k+p+f+1)+\lambda_2 i}\right]. \end{aligned}$$

6 Quantile Function

The quantile function, say $Q(p)$, is defined by $F(Q(p)) = p$. Now, from the cdf of the EKL distribution, we have

$$F_{EKL}(Q(p)) = \left\{ 1 - \left\{ 1 - \left[1 - \frac{1+\lambda+\lambda Q(p)}{1+\lambda} \exp(-\lambda Q(p)) \right]^{\theta} \right\}^{b} \right\}^{\delta} = p,$$

and we can obtain $Q(p)$ as the root of the following equation

$$-\frac{1+\lambda+\lambda Q(p)}{1+\lambda} \exp(-\lambda Q(p)) = \left[1 - (1 - p^{\frac{1}{\delta}})^{\frac{1}{b}}\right]^{\frac{1}{\theta}} - 1, \quad (32)$$

for $0 < p < 1$. Substituting $Z(p) = -(1+\lambda+\lambda Q(p))$, we can rewrite equation (32) as

$$\frac{Z(p)}{1+\lambda} \exp(1+\lambda+Z(p)) = \left[1 - (1 - p^{\frac{1}{\delta}})^{\frac{1}{b}}\right]^{\frac{1}{\theta}} - 1,$$

so that

$$Z(p) \exp(Z(p)) = (1+\lambda) \exp(-1-\lambda) \left\{ \left[1 - (1 - p^{\frac{1}{\delta}})^{\frac{1}{b}}\right]^{\frac{1}{\theta}} - 1 \right\},$$

for $0 < p < 1$. As the defining equation for Lambert W function $W(x)$ is $x = W(x) \exp(W(x))$, we get

$$Z(p) = W\left((1+\lambda) \exp(-1-\lambda) \left\{ \left[1 - (1 - p^{\frac{1}{\delta}})^{\frac{1}{b}}\right]^{\frac{1}{\theta}} - 1 \right\} \right),$$

for $0 < p < 1$. Then, we obtain

$$Q(p) = -1 - \frac{1}{\lambda} - \frac{W\left((1+\lambda) \exp(-1-\lambda) \left\{ \left[1 - (1 - p^{\frac{1}{\delta}})^{\frac{1}{b}}\right]^{\frac{1}{\theta}} - 1 \right\} \right)}{\lambda}, \quad (33)$$

for $0 < p < 1$. Since the Taylor series of $W(x)$ around $x = 0$ is given by $W(x) = \sum_{i=1}^{\infty} \frac{(-i)^{i-1}}{i!} x^i$, a series expansion for equation (33) around $p = 1$ can be obtained as

$$Q(p) = -1 - \frac{1}{\lambda} - \frac{1}{\lambda} \sum_{i=1}^{\infty} \frac{(-i)^{i-1}}{i!} (1+\lambda)^i \exp(-i - \lambda i) \left\{ \left[1 - (1 - p^{\frac{1}{\delta}})^{\frac{1}{b}}\right]^{\frac{1}{\theta}} - 1 \right\}^i . \tag{34}$$

7 Mean Deviations

The mean deviation about the mean and the mean deviation about the median are defined by

$$\delta_1(X) = \int_0^\infty |x - \mu| f(x)\,\mathrm{d}x, \quad \text{and} \quad \delta_2(X) = \int_0^\infty |x - M| f(x)\,\mathrm{d}x,$$

respectively, where $\mu = E(X)$, and $M = Median(X)$ denotes the median. The measures $\delta_1(X)$ and $\delta_2(X)$ can be calculated as follows:

$$\delta_1(X) = 2\mu F(\mu) - 2\mu + 2\int_\mu^\infty x f(x)\,\mathrm{d}x, \tag{35}$$

and

$$\delta_2(X) = -\mu + 2\int_M^\infty x f(x)\,\mathrm{d}x, \tag{36}$$

respectively. By using the moments for EKL distribution and the results in Lemma 2 (Nadarajah et al. [12]), we can calculate equations (35) and (36). Note that

$$K(m,n,p,q) = \int_0^\infty x^p (1+x) \left[1 - \frac{1+n+nx}{1+n} \exp(-nx)\right]^{m-1} \exp(-qx)\,\mathrm{d}x,$$

and

$$L(m,n,p,q,t) = \int_t^\infty x^p (1+x) \left[1 - \frac{1+n+nx}{1+n} \exp(-nx)\right]^{m-1} \exp(-qx)\,\mathrm{d}x.$$

We consider the case when b and δ are real non-integer. From equation (24) and Lemma 2 (Nadarajah et al. [12]), we know that

$$\mu = \frac{\delta b \theta \lambda^2}{1+\lambda} \sum_{i,j,r=0}^{\infty} (-1)^{i+j+r} \binom{b-1}{i} \binom{\delta-1}{j} \binom{bj}{r} K(\theta(i+r+1), \lambda, 1, \lambda),$$

$$\int_{\mu}^{\infty} xf(x)\,\mathrm{d}x = \frac{\delta b\theta\lambda^2}{1+\lambda}\sum_{i,j,r=0}^{\infty}(-1)^{i+j+r}\binom{b-1}{i}\binom{\delta-1}{j}\binom{bj}{r} \times L(\theta(i+r+1),\lambda,1,\lambda,\mu), \quad (37)$$

and

$$\int_{M}^{\infty} xf(x)\,\mathrm{d}x = \frac{\delta b\theta\lambda^2}{1+\lambda}\sum_{i,j,r=0}^{\infty}(-1)^{i+j+r}\binom{b-1}{i}\binom{\delta-1}{j}\binom{bj}{r} \times L(\theta(i+r+1),\lambda,1,\lambda,M),$$

so that

$$\begin{aligned}\delta_1(X) &= 2\mu F(\mu) - 2\mu \\ &+ \frac{2\delta b\theta\lambda^2}{1+\lambda}\sum_{i,j,r=0}^{\infty}(-1)^{i+j+r}\binom{b-1}{i}\binom{\delta-1}{j}\binom{bj}{r} \\ &\times L(\theta(i+r+1),\lambda,1,\lambda,\mu),\end{aligned}$$

and

$$\begin{aligned}\delta_2(X) &= -\mu\frac{2\delta b\theta\lambda^2}{1+\lambda}\sum_{i,j,r=0}^{\infty}(-1)^{i+j+r}\binom{b-1}{i}\binom{\delta-1}{j}\binom{bj}{r} \\ &\times L(\theta(i+r+1),\lambda,1,\lambda,M).\end{aligned}$$

Note here that we have considered the case when b and δ are non-integer, however the other cases can be similarly derived.

8 Bonferroni and Lorenz Curves

Bonferroni and Lorenz curves are defined by

$$B(p) = \frac{1}{p\mu}\int_0^q xf(x)\,\mathrm{d}x, \quad \text{and} \quad L(p) = \frac{1}{\mu}\int_0^q xf(x)\,\mathrm{d}x,$$

respectively, where $\mu = E(X)$, and $q = F^{-1}(p)$.

Now, we obtain Bonferroni and Lorenz curves for EKL distribution as follows: If b and δ are real non-integer, then from equation (37), we have

$$\begin{aligned}B(p) &= \frac{1}{p} - \frac{\delta b\theta\lambda^2}{p\mu(1+\lambda)}\sum_{i,j,r=0}^{\infty}(-1)^{i+j+r}\binom{b-1}{i}\binom{\delta-1}{j}\binom{bj}{r} \\ &\times L(\theta(i+r+1),\lambda,1,\lambda,q),\end{aligned}$$

and

$$\begin{aligned} L(p) &= 1-\frac{\delta b\theta\lambda^2}{\mu(1+\lambda)}\sum_{i,j,r=0}^{\infty}(-1)^{i+j+r}\binom{b-1}{i}\binom{\delta-1}{j}\binom{bj}{r} \\ &\times \; L(\theta(i+r+1),\lambda,1,\lambda,q), \end{aligned}$$

respectively. Note here that we have considered the case when b and δ are non-integer, however the other cases can be similarly derived.

The equivalent definitions of Bonferroni and Lorenz curves are as follows:

$$B(p)=\frac{1}{p\mu}\int_0^p F^{-1}(x)\,\mathrm{d}x, \quad \text{and} \quad L(p)=\frac{1}{\mu}\int_0^p F^{-1}(x)\,\mathrm{d}x,$$

respectively, where $\mu = E(X)$. From equation (34), we know that

$$F^{-1}(x) = -1-\frac{1}{\lambda}-\frac{1}{\lambda}\sum_{i=1}^{\infty}\frac{(-i)^{i-1}}{i!}(1+\lambda)^i e^{-i(1+\lambda)}\left\{\left[1-(1-x^{\frac{1}{\delta}})^{\frac{1}{b}}\right]^{\frac{1}{\theta}}-1\right\}^i. \tag{38}$$

By using the series expansion

$$\left\{\left[1-(1-x^{\frac{1}{\delta}})^{\frac{1}{b}}\right]^{\frac{1}{\theta}}-1\right\}^i = (-1)^i\sum_{j=0}^{i}\sum_{k,m=0}^{\infty}\binom{i}{j}\binom{\frac{j}{\theta}}{k}\binom{\frac{k}{b}}{m}(-1)^{j+k+m}x^{\frac{m}{\delta}}, \tag{39}$$

and substituting equation (39) into equation (38), we get

$$\begin{aligned} F^{-1}(x) &= -1-\frac{1}{\lambda}+\frac{1}{\lambda}\sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k,m=0}^{\infty}\binom{i}{j}\binom{\frac{j}{\theta}}{k}\binom{\frac{k}{b}}{m} \\ &\times \frac{i^{i-1}(-1)^{j+k+m}(1+\lambda)^i e^{-i(1+\lambda)}}{i!}x^{\frac{m}{\delta}}, \end{aligned}$$

so that

$$\begin{aligned} B(p) &= \frac{1}{p\mu}\int_0^p\left(-1-\frac{1}{\lambda}\right)\mathrm{d}x \\ &+ \frac{1}{p\mu\lambda}\sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k,m=0}^{\infty}\binom{i}{j}\binom{\frac{j}{\theta}}{k}\binom{\frac{k}{b}}{m} \\ &\times \frac{i^{i-1}(-1)^{j+k+m}(1+\lambda)^i e^{-i(1+\lambda)}}{i!}\int_0^p x^{\frac{m}{\delta}}\,\mathrm{d}x \\ &= \frac{1}{\mu}\left(-1-\frac{1}{\lambda}\right) \\ &+ \frac{1}{\mu\lambda}\sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k,m=0}^{\infty}\binom{i}{j}\binom{\frac{j}{\theta}}{k}\binom{\frac{k}{b}}{m}\frac{i^{i-1}(-1)^{j+k+m}(1+\lambda)^i e^{-i(1+\lambda)}p^{\frac{m}{\delta}}}{i!(\frac{m}{\delta}+1)}, \end{aligned} \tag{40}$$

and

$$\begin{aligned} L(p) &= \frac{p}{\mu}\left(-1-\frac{1}{\lambda}\right) \\ &+ \frac{p}{\mu\lambda}\sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k,m=0}^{\infty}\binom{i}{j}\binom{\frac{j}{\theta}}{k}\binom{\frac{k}{b}}{m}\frac{i^{i-1}(-1)^{j+k+m}(1+\lambda)^i e^{-i(1+\lambda)}p^{\frac{m}{\delta}}}{i!(\frac{m}{\delta}+1)}. \end{aligned} \tag{41}$$

The Bonferroni and Gini indices are defined by

$$B = 1-\int_0^1 B(p)\,\mathrm{d}p, \quad \text{and} \quad G = 1-2\int_0^1 L(p)\,\mathrm{d}p,$$

respectively. By using equations (40) and (41), we obtain

$$\begin{aligned} B &= 1-\frac{1}{\mu}\left(-1-\frac{1}{\lambda}\right) \\ &- \frac{1}{\mu\lambda}\sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k,m=0}^{\infty}\binom{i}{j}\binom{\frac{j}{\theta}}{k}\binom{\frac{k}{b}}{m}\frac{i^{i-1}(-1)^{j+k+m}(1+\lambda)^i e^{-i(1+\lambda)}}{i!(\frac{m}{\delta}+1)^2}, \end{aligned}$$

and

$$\begin{aligned} G &= 1-\frac{1}{\mu}\left(-1-\frac{1}{\lambda}\right) \\ &- \frac{2}{\mu\lambda}\sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k,m=0}^{\infty}\binom{i}{j}\binom{\frac{j}{\theta}}{k}\binom{\frac{k}{b}}{m}\frac{i^{i-1}(-1)^{j+k+m}(1+\lambda)^i e^{-i(1+\lambda)}}{i!(\frac{m}{\delta}+1)(\frac{m}{\delta}+2)}. \end{aligned}$$

9 Order Statistics, Measures of Uncertainty, and Information

In this section, the distribution of the k^{th} order statistic, measures of uncertainty, and information for the EKL distribution are presented. The concept of entropy plays a vital role in information theory. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty.

9.1 Distribution of Order Statistics

Suppose that $X_1, \cdots, X_n$ is a random sample of size n from a continuous pdf, $f(x)$. Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the corresponding order statistics. If $X_1, \cdots, X_n$ is a random sample from EKL distribution, it follows from the equations (11) and (12) that the pdf of the k^{th} order statistic, say $Y_k = X_{k:n}$, is given by

$$\begin{aligned} f_k(y_k) &= \frac{n!}{(k-1)!(n-k)!}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l[F_{EKL}(y_k)]^{k-1+l}f_{EKL}(y_k) \\ &= \frac{\delta b\theta\lambda^2 n!(1+y_k)\exp(-\lambda y_k)}{(1+\lambda)(k-1)!(n-k)!}\sum_{l=0}^{n-k}\sum_{p,q,i,j,r=0}^{\infty}\binom{n-k}{l}\binom{\delta(k-1+l)}{p} \\ &\times \binom{bp}{q}\binom{b-1}{i}\binom{\delta-1}{j}\binom{bj}{r}(-1)^{l+p+q+i+j+r}[V(y_k)]^{\theta(q+i+r+1)-1}, \end{aligned}$$

where $V(y_k) = 1 - \frac{1+\lambda+\lambda y_k}{1+\lambda}\exp(-\lambda y_k)$. The corresponding cdf of Y_k is

$$\begin{aligned} F_k(y_k) &= \sum_{j=k}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l\sum_{p,q=0}^{\infty}\binom{\delta(j+l)}{p}\binom{bp}{q}(-1)^{p+q}[V(y_k)]^{\theta q} \\ &= \sum_{j=k}^{n}\sum_{l=0}^{n-j}\sum_{p,q=0}^{\infty}\binom{n}{j}\binom{n-j}{l}\binom{\delta(j+l)}{p}\binom{bp}{q}(-1)^{l+p+q}[V(y_k)]^{\theta q}. \end{aligned}$$

The s^{th} moment of the k^{th} order statistic Y_k from EKL distribution is obtained as follows: If b and δ are real non-integer, then

$$\begin{aligned} E(Y_k^s) &= \int_0^\infty y_k^s f_k(y_k;\lambda,\theta,b,\delta)\,\mathrm{d}y_k \\ &= \frac{\delta b\theta\lambda^2 n!}{(1+\lambda)(k-1)!(n-k)!}\sum_{l=0}^{n-k}\sum_{p,q,i,j,r=0}^{\infty}\binom{n-k}{l}\binom{\delta(k-1+l)}{p} \\ &\times \binom{bp}{q}\binom{b-1}{i}\binom{\delta-1}{j}\binom{bj}{r}(-1)^{l+p+q+i+j+r} \\ &\times K(\theta(q+i+r+1),\lambda,s,\lambda). \end{aligned}$$

Note here that we have considered the case when b and δ are non-integer, however the other cases can be similarly derived.

9.2 Renyi Entropy

Renyi entropy [15] is an extension of Shannon entropy. Renyi entropy is defined to be $H_\gamma(f_{EKL}(x)) = H_\gamma(f_{EKL}(x;\lambda,\theta,b,\delta)) = \frac{\log\left(\int_0^\infty f_{EKL}^\gamma(x;\lambda,\theta,b,\delta)\,dx\right)}{1-\gamma}$, where

$\gamma > 0$, and $\gamma \neq 1$. Renyi entropy tends to Shannon entropy as $\gamma \to 1$. Now,

$$\begin{aligned}\int_0^\infty f_{EKL}^{\gamma}(x)\,\mathrm{d}x &= \left(\frac{\delta b\theta\lambda^2}{1+\lambda}\right)^{\gamma} \\ &\times \int_0^\infty (1+x)^{\gamma}\exp(-\lambda\gamma x)[V(x)]^{\theta\gamma-\gamma}[1-[V(x)]^{\theta}]^{b\gamma-\gamma} \\ &\times \left\{1-[1-[V(x)]^{\theta}]^{b}\right\}^{\delta\gamma-\gamma}\,\mathrm{d}x. \end{aligned} \tag{42}$$

Note that

$$\begin{aligned}[V(x)]^{\theta\gamma-\gamma} &= \left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\theta\gamma-\gamma} \\ &= \sum_{k=0}^{\infty}(-1)^k\binom{\theta\gamma-\gamma}{k}\frac{\sum_{j=0}^{k}\binom{k}{j}\lambda^j(1+x)^j}{(1+\lambda)^k}\exp(-\lambda kx), \end{aligned} \tag{43}$$

$$\begin{aligned}[1-[V(x)]^{\theta}]^{b\gamma-\gamma} &= \sum_{m=0}^{\infty}(-1)^m\binom{b\gamma-\gamma}{m} \\ &\times \sum_{n=0}^{\infty}(-1)^n\binom{\theta m}{n}\frac{\sum_{e=0}^{n}\binom{n}{e}\lambda^e(1+x)^e}{(1+\lambda)^n}\exp(-\lambda nx), \end{aligned} \tag{44}$$

and

$$\begin{aligned}\left\{1-[1-[V(x)]^{\theta}]^{b}\right\}^{\delta\gamma-\gamma} &= \sum_{q=0}^{\infty}(-1)^q\binom{\delta\gamma-\gamma}{q}\sum_{p=0}^{\infty}(-1)^p\binom{bq}{p} \\ &\times \sum_{l=0}^{\infty}(-1)^l\binom{\theta p}{l}\frac{\sum_{t=0}^{l}\binom{l}{t}\lambda^t(1+x)^t}{(1+\lambda)^l}\exp(-\lambda lx). \end{aligned} \tag{45}$$

Substituting equations (43), (44) and (45) into equation (42), we get

$$\begin{aligned}\int_0^\infty f_{KGL}^{\gamma}(x)\,\mathrm{d}x &= \left(\frac{\delta b\theta\lambda^2}{1+\lambda}\right)^{\gamma} \\ &\times \sum_{k,m,n,q,p,l=0}^{\infty}\sum_{j=0}^{k}\sum_{e=0}^{n}\sum_{t=0}^{l}\binom{\theta\gamma-\gamma}{k}\binom{k}{j}\binom{b\gamma-\gamma}{m}\binom{\theta m}{n}\binom{n}{e} \\ &\times \binom{\delta\gamma-\gamma}{q}\binom{bq}{p}\binom{\theta p}{l}\binom{l}{t} \\ &\times \frac{(-1)^{k+m+n+q+p+l}\exp(\lambda(\gamma+k+n+l))}{(1+\lambda)^{k+n+l}(\gamma+k+n+l)^{\gamma+j+t+1}\lambda^{\gamma+1}} \\ &\times \Gamma(\gamma+j+e+t+1,\lambda(\gamma+k+n+l)). \end{aligned} \tag{46}$$

Consequently, Renyi entropy for EKL distribution reduces to :

$$\begin{aligned}
H_\gamma(f_{EKL}(x)) &= \frac{\gamma}{1-\gamma}\log\left(\frac{\delta b\theta\lambda^2}{1+\lambda}\right)\\
&+ \frac{1}{1-\gamma}\log\Bigg\{\sum_{k,m,n,q,p,l=0}^{\infty}\sum_{j=0}^{k}\sum_{e=0}^{n}\sum_{t=0}^{l}\binom{\theta\gamma-\gamma}{k}\binom{k}{j}\binom{b\gamma-\gamma}{m}\\
&\times \binom{\theta m}{n}\binom{n}{e}\binom{\delta\gamma-\gamma}{q}\binom{bq}{p}\binom{\theta p}{l}\binom{l}{t}\\
&\times \frac{(-1)^{k+m+n+q+p+l}\exp(\lambda(\gamma+k+n+l))}{(1+\lambda)^{k+n+l}(\gamma+k+n+l)^{\gamma+j+t+1}\lambda^{\gamma+1}}\\
&\times \Gamma(\gamma+j+e+t+1,\lambda(\gamma+k+n+l))\Bigg\},
\end{aligned}$$

for $\gamma > 0$, and $\gamma \neq 1$.

From equation (16), we obtain Shannon entropy for EKL distribution as follows:

$$\begin{aligned}
E[-\log f_{EKL}(X;\lambda,\theta,b,\delta)] &= -\log\left(\frac{\delta b\theta\lambda^2}{1+\lambda}\right) - E[\log(1+X)] + \lambda E(X)\\
&+ (1-\theta)E[\log V(X)]\\
&+ (1-b)E[\log(1-[V(X)]^\theta)]\\
&+ (1-\delta)E[\log\left\{1-[1-[V(X)]^\theta]^b\right\}].
\end{aligned}$$

Now, as $0 < [V(x)]^\theta < 1$,

$$\log(1-[V(x)]^\theta) = -\sum_{k=1}^{\infty}\frac{[V(x)]^{\theta k}}{k},$$

$$\log[V(x)] = -\sum_{k=1}^{\infty}\sum_{l=0}^{k}\binom{k}{l}\frac{\lambda^l}{k(1+\lambda)^l}x^l\exp(-\lambda kx),$$

and

$$\log\left\{1-[1-[V(x)]^\theta]^b\right\} = -\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\binom{bn}{m}\frac{(-1)^m}{n}[V(x)]^{\theta m},$$

we have

$$E[\log(1-[V(X)]^\theta)] = -\sum_{k=1}^{\infty}\frac{1}{k}E([V(X)]^{\theta k}), \tag{47}$$

$$E[\log(V(X))] = -\sum_{k=1}^{\infty}\sum_{l=0}^{k}\binom{k}{l}\frac{\lambda^l}{k(1+\lambda)^l}E[X^l\exp(-\lambda kX)], \quad (48)$$

and

$$E[\log\{1-[1-[V(X)]^{\theta}]^{b}\}] = -\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\binom{bn}{m}\frac{(-1)^m}{n}E[[V(X)]^{\theta m}]. \quad (49)$$

By using the expansion of density for EKL distribution and the results in Lemma 1 (Nadarajah et al. [12]), we can calculate equations (47), (48), and (49). Now, we obtain Shannon entropy for EKL distribution as follows: When b and δ are real non-integer, we have

$$\begin{aligned} E[-\log f_{EKL}(X;\lambda,\theta,b,\delta)] &= -\log\left(\frac{\delta b\theta\lambda^2}{1+\lambda}\right)+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}E(X^n)+\lambda E(X) \\ &+ (\theta-1)\sum_{k=1}^{\infty}\sum_{l=0}^{k}\binom{k}{l}\frac{\lambda^l}{k(1+\lambda)^l}E[X^l\exp(-\lambda kX)] \\ &+ (b-1)\sum_{k=1}^{\infty}\frac{1}{k}E([V(X)]^{\theta k}) \\ &+ (\delta-1)\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\binom{bn}{m}\frac{(-1)^m}{n}E[[V(X)]^{\theta m}], \end{aligned}$$

where

$$E(X^n) = \frac{\delta b\theta\lambda^2}{1+\lambda}\sum_{i,j,r=0}^{\infty}(-1)^{i+j+r}\binom{b-1}{i}\binom{\delta-1}{j}\binom{bj}{r}K(\theta(i+r+1),\lambda,n,\lambda),$$

$$E(X) = \frac{\delta b\theta\lambda^2}{1+\lambda}\sum_{i,j,r=0}^{\infty}(-1)^{i+j+r}\binom{b-1}{i}\binom{\delta-1}{j}\binom{bj}{r}K(\theta(i+r+1),\lambda,1,\lambda),$$

$$\begin{aligned} E[X^l\exp(-\lambda kX)] &= \frac{\delta b\theta\lambda^2}{1+\lambda}\sum_{i,j,r=0}^{\infty}(-1)^{i+j+r}\binom{b-1}{i}\binom{\delta-1}{j}\binom{bj}{r} \\ &\times K(\theta(i+r+1),\lambda,l,\lambda(1+k)), \end{aligned}$$

$$\begin{aligned} E([V(X)]^{\theta k}) &= \frac{\delta b\theta\lambda^2}{1+\lambda}\sum_{i,j,r=0}^{\infty}(-1)^{i+j+r}\binom{b-1}{i}\binom{\delta-1}{j}\binom{bj}{r} \\ &\times K(\theta(i+r+k+1),\lambda,0,\lambda), \end{aligned}$$

and

$$\begin{aligned} E([V(X)]^{\theta m}) &= \frac{\delta b\theta\lambda^2}{1+\lambda}\sum_{i,j,r=0}^{\infty}(-1)^{i+j+r}\binom{b-1}{i}\binom{\delta-1}{j}\binom{bj}{r} \\ &\times K(\theta(i+r+m+1),\lambda,0,\lambda). \end{aligned}$$

Note here that we have considered the case when b and δ are non-integer, however the other cases can be similarly derived.

9.3 s-Entropy

The s-entropy for EKL distribution is defined by

$$H_s(f_{EKL}(x;\lambda,\theta,b,\delta)) = \begin{cases} \frac{1}{s-1}[1-\int_0^\infty f_{EKL}^s(x;\lambda,\theta,b,\delta)\,\mathrm{d}x] & \text{if } s\neq 1,\ s>0, \\ E[-\log f(X)] & \text{if } s=1. \end{cases}$$

Consequently, if $s\neq 1, s>0$, then from equation (46), we have

$$\begin{aligned} H_s(f_{EKL}(x)) &= \frac{1}{s-1}-\frac{1}{s-1}\Bigg(\left(\frac{\delta b\theta\lambda^2}{(1+\lambda)}\right)^s \\ &\times \sum_{k,m,n,q,p,l=0}^{\infty}\sum_{j=0}^{k}\sum_{e=0}^{n}\sum_{t=0}^{l}\binom{\theta s-s}{k}\binom{k}{j}\binom{bs-s}{m}\binom{\theta m}{n}\binom{n}{e} \\ &\times \binom{\delta s-s}{q}\binom{bq}{p}\binom{\theta p}{l}\binom{l}{t} \\ &\times \frac{(-1)^{k+m+n+q+p+l}\exp(\lambda(s+k+n+l))}{(1+\lambda)^{k+n+l}(s+k+n+l)^{s+j+t+1}\lambda^{s+1}} \\ &\times \Gamma(s+j+e+t+1,\lambda(s+k+n+l))\Bigg). \end{aligned}$$

If $s=1$, then s-entropy is Shannon entropy.

9.4 Fisher Information Matrix

This section presents a measure for the amount of information. This information measure can be used to obtain bounds on the variance of estimators, and as well as approximate the sampling distribution of an estimator obtained from a large sample. Furthermore, it can be used to obtain approximate confidence intervals in case of large sample.

Let X be a random variable (rv) with the EKL pdf $f_{EKL}(.;\mathbf{\Theta})$, where $\mathbf{\Theta}=(\theta_1,\theta_2,\theta_3,\theta_4)^T=(\lambda,\theta,b,\delta)^T$. Then, Fisher information matrix (FIM) is the 4×4 symmetric matrix with elements:

$$\mathbf{I}_{ij}(\mathbf{\Theta}) = E_{\mathbf{\Theta}}\left[\frac{\partial\log(f_{EKL}(X;\mathbf{\Theta}))}{\partial\theta_i}\frac{\partial\log(f_{EKL}(X;\mathbf{\Theta}))}{\partial\theta_j}\right].$$

If the density $f_{EKL}(.;\mathbf{\Theta})$ has a second derivative for all i and j, then an alternative expression for $\mathbf{I}_{ij}(\mathbf{\Theta})$ is

$$\mathbf{I}_{ij}(\mathbf{\Theta}) = -E_{\mathbf{\Theta}}\left[\frac{\partial^2 \log(f_{EKL}(X;\mathbf{\Theta}))}{\partial\theta_i\partial\theta_j}\right]. \tag{50}$$

For the EKL distribution, all second derivatives exist, therefore the formula above is appropriate and simplifies the computations. The elements of the observed information matrix of the EKL distribution are given in Appendix A.

10 Maximum Likelihood Estimation

In this section, the maximum likelihood estimates (MLEs) of the parameters λ, θ, b, and δ of the EKL distribution are presented. If $x_1, \cdots, x_n$ is a random sample from EKL distribution, then the log-likelihood function is given by

$$\begin{aligned}\log(L(\lambda,\theta,b,\delta)) &= n\log\left(\frac{\delta b\theta\lambda^2}{1+\lambda}\right) + \sum_{i=1}^{n}\log(1+x_i) - \lambda\sum_{i=1}^{n}x_i \\ &+ (\theta-1)\sum_{i=1}^{n}\log(V(x_i)) + (b-1)\sum_{i=1}^{n}\log[1-[V(x_i)]^{\theta}] \\ &+ (\delta-1)\sum_{i=1}^{n}\log\left\{1-[1-[V(x_i)]^{\theta}]^{b}\right\}.\end{aligned}$$

The partial derivatives of $\log L(\lambda,\theta,b,\delta)$ with respect to the parameters λ, θ, b and δ are:

$$\begin{aligned}\frac{\partial\log L(\lambda,\theta,b,\delta)}{\partial\theta} &= \frac{n}{\theta} + \sum_{i=1}^{n}\log(V(x_i)) + (1-b)\sum_{i=1}^{n}\frac{[V(x_i)]^{\theta}\log(V(x_i))}{1-[V(x_i)]^{\theta}} \\ &+ b(\delta-1)\sum_{i=1}^{n}\frac{[1-[V(x_i)]^{\theta}]^{b-1}[V(x_i)]^{\theta}\log(V(x_i))}{1-[1-[V(x_i)]^{\theta}]^{b}},\end{aligned}$$

$$\begin{aligned}\frac{\partial\log L(\lambda,\theta,b,\delta)}{\partial b} &= \frac{n}{b} + \sum_{i=1}^{n}\log[1-[V(x_i)]^{\theta}] \\ &+ (1-\delta)\sum_{i=1}^{n}\frac{[1-[V(x_i)]^{\theta}]^{b}\log[1-[V(x_i)]^{\theta}]}{1-[1-[V(x_i)]^{\theta}]^{b}},\end{aligned}$$

$$\frac{\partial\log L(\lambda,\theta,b,\delta)}{\partial\delta} = \frac{n}{\delta} + \sum_{i=1}^{n}\log\left\{1-[1-[V(x_i)]^{\theta}]^{b}\right\},$$

and

$$\begin{aligned}\frac{\partial \log L(\lambda,\theta,b,\delta)}{\partial\lambda} &= n\frac{1+\lambda}{\delta b\theta\lambda^2}\frac{\partial\left(\frac{\delta b\theta\lambda^2}{1+\lambda}\right)}{\partial\lambda} - \sum_{i=1}^{n} x_i + (\theta-1)\sum_{i=1}^{n}\frac{\partial V(x_i)/\partial\lambda}{V(x_i)}\\ &+ \theta(1-b)\sum_{i=1}^{n}\frac{[V(x_i)]^{\theta-1}(\partial V(x_i)/\partial\lambda)}{1-V(x_i)^{\theta}}\\ &+ b\theta(\delta-1)\sum_{i=1}^{n}\frac{[1-[V(x_i)]^{\theta}]^{b-1}[V(x_i)]^{\theta-1}(\partial V(x_i)/\partial\lambda)}{1-[1-[V(x_i)]^{\theta}]^{b}}.\end{aligned} \tag{51}$$

Note that

$$\frac{\partial\left(\frac{\delta b\theta\lambda^2}{1+\lambda}\right)}{\partial\lambda} = \delta b\theta\frac{2\lambda+\lambda^2}{(1+\lambda)^2}, \tag{52}$$

and

$$\frac{\partial V(x_i)}{\partial\lambda} = \left[\frac{1+\lambda+\lambda x_i}{1+\lambda} - \frac{1}{(1+\lambda)^2}\right] x_i\exp(-\lambda x_i). \tag{53}$$

Substituting equations (52) and (53) into equation (51), we get

$$\begin{aligned}\frac{\partial \log L(\lambda,\theta,b,\delta)}{\partial\lambda} &= \frac{n(2+\lambda)}{\lambda(1+\lambda)} - \sum_{i=1}^{n} x_i\\ &+ \sum_{i=1}^{n}\frac{\lambda(2+\lambda+x_i+\lambda x_i)x_i\exp(-\lambda x_i)}{1+\lambda}\\ &\times \left[\frac{\theta-1}{(1+\lambda)V(x_i)} + \frac{\theta(1-b)[V(x_i)]^{\theta-1}}{(1+\lambda)(1-[V(x_i)]^{\theta})}\right]\\ &+ b\theta(\delta-1)\sum_{i=1}^{n}\frac{[1-[V(x_i)]^{\theta}]^{b-1}[V(x_i)]^{\theta-1}x_i\exp(-\lambda x_i)}{1-[1-[V(x_i)]^{\theta}]^{b}}\\ &\times \left[\frac{1+\lambda+\lambda x_i}{1+\lambda} - \frac{1}{(1+\lambda)^2}\right].\end{aligned}$$

When all the parameters are unknown, numerical methods must be used to obtain estimates of the model parameters since the system does not admit any explicit solution, therefore the MLE $(\hat{\lambda},\hat{\theta},\hat{b},\hat{\delta})$ of $(\lambda,\theta,b,\delta)$ can be obtained only by means of numerical procedures. The MLEs of the parameters, denoted by $\hat{\mathbf{\Theta}}$ is obtained by solving the nonlinear equation $(\frac{\partial \log L}{\partial\lambda}, \frac{\partial \log L}{\partial\theta}, \frac{\partial \log L}{\partial b}, \frac{\partial \log L}{\partial\delta})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The Fisher information matrix given by $\mathbf{I}(\mathbf{\Theta}) = [\mathbf{I}_{\theta_i,\theta_j}]_{4X4} = E(-\frac{\partial^2 \log L}{\partial\theta_i\partial\theta_j})$, $i,j = 1,2,3,4$, can be numerically obtained by MATLAB, MAPLE or R software. The total Fisher information matrix $\mathbf{I}_n(\boldsymbol{\Theta}) = n\mathbf{I}(\boldsymbol{\Theta})$ can be approximated by

$$\mathbf{J}_n(\hat{\boldsymbol{\Theta}}) \approx \left[-\frac{\partial^2 \log L}{\partial\theta_i \partial\theta_j}\bigg|_{\boldsymbol{\Theta}=\hat{\boldsymbol{\Theta}}} \right]_{4X4}, \quad i,j = 1,2,3,4. \tag{54}$$

For real data, the matrix given in equation (54) is obtained after the convergence of the Newton-Raphson procedure in MATLAB or R software.

10.1 Asymptotic Confidence Intervals

In this section, we present the asymptotic confidence intervals for the parameters of the EKL distribution. The expectations in the FIM can be obtained numerically. Let $\hat{\boldsymbol{\Theta}} = (\hat{\lambda}, \hat{\theta}, \hat{b}, \hat{\delta})^T$ be the MLE of $\boldsymbol{\Theta} = (\lambda, \theta, b, \delta)^T$. Under the conditions that the parameters are in the interior of the parameter space, but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\boldsymbol{\Theta}} - \boldsymbol{\Theta})$ is $N_4(\mathbf{0}, \mathbf{I}^{-1}(\boldsymbol{\Theta}))$.

The multivariate normal distribution with mean vector $(0,0,0,0)^T$ and covariance matrix $\mathbf{I}^{-1}(\boldsymbol{\Theta})$ can be used to construct confidence intervals for the model parameters. That is, the approximate $100(1-\eta)\%$ two-sided confidence intervals for λ, θ, b and δ are given by:

$$\widehat{\lambda} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{\lambda\lambda}^{-1}(\widehat{\boldsymbol{\Theta}})}, \quad \widehat{\theta} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{\theta\theta}^{-1}(\widehat{\boldsymbol{\Theta}})}, \quad \widehat{b} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{bb}^{-1}(\widehat{\boldsymbol{\Theta}})}, \quad \text{and} \quad \widehat{\delta} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{\delta\delta}^{-1}(\widehat{\boldsymbol{\Theta}})},$$

respectively, where $\mathbf{I}_{\lambda\lambda}^{-1}(\widehat{\boldsymbol{\Theta}})$, $\mathbf{I}_{\theta\theta}^{-1}(\widehat{\boldsymbol{\Theta}})$, $\mathbf{I}_{bb}^{-1}(\widehat{\boldsymbol{\Theta}})$, and $\mathbf{I}_{\delta\delta}^{-1}(\widehat{\boldsymbol{\Theta}})$ are the diagonal elements of $\mathbf{I}_n^{-1}(\hat{\boldsymbol{\Theta}}) = (n\mathbf{I}(\hat{\boldsymbol{\Theta}}))^{-1}$, and $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}^{th}$ percentile of a standard normal distribution.

11 Applications

In this section, applications of the EKL distribution including the estimation of the parameters via the method of maximum likelihood and likelihood ratio (LR) test for comparison of the EKL distribution with its sub-models for given sets of data are presented. The examples illustrate the flexibility of the EKL distribution in contrast to other models including the Kumaraswamy Lindley (KL), GL, L, Kumaraswamy Weibull (KW), weighted generalized gamma (WGG) and gamma (GAM) distributions for data modeling. The pdf of the four-parameter weighted generalized gamma distribution with the weight function $w(x) = x^a$ used in the comparisons is given by

$$f_{WGG}(x;\lambda,\alpha,a,b) = \frac{b\lambda^{b\alpha+a}}{\Gamma(\alpha + a/b)} x^{b\alpha+a-1} e^{-(\lambda x)^b}, \text{for } \lambda > 0,\ \alpha > 0,\ a > 0,\ b > 0.$$

The KW pdf [3] was also used to model these data. The KW pdf is given by

$$\begin{aligned} f_{KW}(x;\lambda,\alpha,a,b) &= ab\alpha\lambda^{\alpha}x^{\alpha-1}\exp(-(\lambda x)^{\alpha})[1-\exp(-(\lambda x)^{\alpha})]^{a-1} \\ &\times \quad [1-(1-\exp(-(\lambda x)^{\alpha}))^{a}]^{b-1}, \end{aligned}$$

where $a, b > 0$ are additional shape parameters that pertains to skewness and kurtosis.

The MLEs of the EKL parameters λ, θ, b and δ are computed by maximizing the objective function via the subroutine NLMIXED in SAS. The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic, Akaike Information Criterion, $AIC = 2p - 2\log(L)$, Bayesian Information Criterion, $BIC = p\log(n) - 2\log(L)$, and Consistent Akaike Information Criterion, $AICC = AIC + 2\frac{p(p+1)}{n-p-1}$, where $L = L(\hat{\Theta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are presented in Table 3. The EKL distribution is fitted to the data sets and these fits are compared to the fits using the KL, GL, L, KW, WGG and GAM distributions.

We can use the LR test to compare the fit of the EKL distribution with its sub-models for a given data set. For example, to test $a = b = 1$, the LR statistic is $\omega = 2[\log(L(\hat{\lambda}, \hat{\theta}, \hat{b}, \hat{\delta})) - \log(L(\tilde{\lambda}, \tilde{\theta}, 1, 1))]$, where $\hat{\lambda}$, $\hat{\theta}$, $\hat{b}$, and $\hat{\delta}$, are the unrestricted estimates, and $\tilde{\lambda}$, and $\tilde{\theta}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi^2_d$, where χ^2_d denotes the upper $100d\%$ point of the χ^2 distribution with 2 degrees of freedom.

Specifically, we consider two data sets. The first set of data comes from R software package "faraway" data. It is given in Table 1. The second set of data consists of the estimated time since given growth hormone medication until the children reached the target age in the *Programa Hormonal de Secretaria de Saude de Minas Gerais* [1]. The data set is given in Table 2. The MLEs of the parameters with standard errors in parenthesis and the values of the statistics (-2ln(L), AIC, AICC and BIC) are given in Table 3. The starting points of the iterative processes for the data sets I and II for the $EKL(\lambda, \theta, b, \delta)$ distribution are (0.476, 0.53, 0.061, 2) and (0.476, 0.051, 0.061, 2), respectively.

3.1	3.5	3.5	3.5	3.6	4.0	4.0	4.0	4.1	4.4	4.4
4.4	4.5	4.5	4.5	4.5	4.5	4.7	5.0	5.0	5.3	5.5
5.6	6.0	6.2	6.5	7.0	7.5	8.1	8.5	8.5	8.5	8.8
9.0	11.2	11.5	17.5	17.5	-	-	-	-	-	-

Table 1: R software package "faraway" data

Probability plots (Chambers et al [2]) consists of plots of the observed probabilities against the probabilities predicted by the fitted model are also

2.15	2.20	2.55	2.56	2.63	2.74	2.81	2.90	3.05	3.41	3.43
3.43	3.84	4.16	4.18	4.36	4.42	4.51	4.60	4.61	4.75	5.03
5.10	5.44	5.90	5.96	6.77	7.82	8.00	8.16	8.21	8.72	10.40
13.20	13.70	-	-	-	-	-	-	-	-	-

Table 2: Growth Hormone Data [1]

Data set	Model	λ	θ	δ	b	$-2\ln(L)$	AIC	$AICC$	BIC	SS
Data set	Model	λ	θ	δ	b	$-2\ln(L)$	AIC	$AICC$	BIC	SS
I (n=38)	$EKL(\lambda,\theta,b,\delta)$	2.2252	2318.58	0.4581	0.1127	164.1	172.1	173.3	178.7	0.04981686
		(0.03933)	(1087.36)	(0.1322)	(0.03264)					
	$KL(\lambda,\theta,b,1)$	2.2429	613.23	1	0.1610	167.8	173.8	174.5	178.7	0.09244024
		(0.04603)	(249.44)		(0.02759)					
	$GL(\lambda,\theta,1,1)$	0.5749	5.9602	1	1	179.0	183.0	183.3	186.3	0.2372781
		(0.08189)	(2.2876)							
	$L(\lambda,1,1,1)$	0.2793	1	1	1	202.0	204.0	204.2	205.7	0.672952
		(0.03243)								
		λ	α	a	b					
	$KW(\lambda,\alpha,a,b)$	0.3093	2.1075	1.6332	0.1967	187.4	195.4	196.6	201.9	0.3868952
		(0.05690)	(0.2275)	(0.7170)	(0.03991)					
	$WGG(\lambda,\alpha,a,b)$	57.4440	7.2804	6.2870	0.5089	179.0	187.0	188.3	193.6	0.2197302
		(130.85)	(1.9494)	(2.6966)	(0.1185)					
	$GAM(\lambda,\alpha)$	0.7829	3.9939	1	1	182.2	186.2	186.6	189.5	0.2602691
		(0.1830)	(1.1095)							
I (n=35)	$EKL(\lambda,\theta,b,\delta)$	2.8396	338.10	0.8768	0.1202	152.9	160.9	162.2	167.1	0.03427661
		(0.05063)	(181.58)	(0.2493)	(0.02854)					
	$KL(\lambda,\theta,b,1)$	2.1379	75.0012	1	0.1759	153.8	159.8	160.6	164.5	0.03290961
		(0.07884)	(33.6119)		(0.03324)					
	$GL(\lambda,\theta,1,1)$	0.5926	3.9478	1	1	159.3	163.3	163.7	166.4	0.0901124
		(0.0878)	(1.3624)							
	$L(\lambda,1,1,1)$	0.3303	1	1	1	174.9	176.9	177.1	178.5	0.3855236
		(0.04009)								
		λ	α	a	b					
	$KW(\lambda,\alpha,a,b)$	1.2868	1.2417	54.9279	0.1370	153.8	161.8	163.2	168.1	0.03925958
		(0.2911)	(0.09659)	(35.3591)	(0.02506)					
	$WGG(\lambda,\alpha,a,b)$	3210.85	0.02566	11.4986	0.3589	157.9	165.9	167.3	172.2	0.06756737
		(0.000473)	(0.7120)	(1.9839)	(0.01570)					
	$GAM(\lambda,\alpha)$	0.7798	4.1381	1	1	160.2	164.2	164.6	167.3	1.13915
		(0.1907)	(0.9520)							

Table 3: Parameters Estimates, Log-likelihood, AIC, AICC, BIC, and SS

presented in Figures 3 and 4. For the EKL distribution, we plotted for example,

$$F_{EKL}(y_k;\hat{\lambda},\hat{\theta},\hat{b},\hat{\delta}) = \left\{1-\left\{1-\left[1-\frac{1+\hat{\lambda}+\hat{\lambda}y_k}{1+\hat{\lambda}}\exp(-\hat{\lambda}y_k)\right]^{\hat{\theta}}\right\}^{\hat{b}}\right\}^{\hat{\delta}}, \quad (55)$$

against $\frac{k-0.375}{n+0.25}$, $k=1,2,\cdots,n$, where y_k are the ordered values of the observed data. A measure of closeness of the plot to the diagonal line given by the sum of squares

$$SS=\sum_{k=1}^{n}\left[F_{EKL}(y_k;\hat{\lambda},\hat{\theta},\hat{b},\hat{\delta})-\left(\frac{k-0.375}{n+0.25}\right)\right]^2,$$

was calculated for each plot. The plot with the smallest SS corresponds to the model with points that are closer to the diagonal line. The EKL model performs very well in this regard.

For the "faraway" data, the LR statistics for the test of the hypotheses $H_0: L(\lambda,1,1,1)$ against $H_a: KL(\lambda,\theta,b,1)$, and $H_0: GL(\lambda,\theta,1,1)$ against $H_a: KL(\lambda,\theta,b,1)$ are 34.2 ($p-value < 1\times10^{-7}$) and 11.2 ($p-value < 0.001$), respectively. Consequently, we reject the null hypothesis in favor of the KL distribution. The test of the hypotheses $H_0: KL(\lambda,\theta,b,1)$ against $H_a: EKL(\lambda,\theta,b,\delta)$ is 3.7 ($p-value = 0.0544$), so we reject the null hypothesis in favor of the EKL distribution at 5.5% level. We conclude that the EKL distribution is significantly better than KL, GL and L distributions based on the LR statistic. The EKL distribution is also better than the KW, WGG and GAM distributions based on the values of the statistics AIC, AICC and BIC. The plots of the fitted EKL distribution and sub-models are shown in Figure 4. Also, the value of sum of squares for EKL distribution is SS=0.04981686, which is the smallest.

For the growth hormone data, the LR statistics for the test of the hypotheses $H_0: L(\lambda,1,1,1)$ against $H_a: EKL(\lambda,\theta,b,\delta)$, and $H_0: GL(\lambda,\theta,1,1)$ against $H_a: EKL(\lambda,\theta,b,\delta)$ are 22 ($p-value < 0.0001$) and 6.4 ($p-value < 0.05$), respectively. Consequently, we reject the null hypothesis in favor of the EKL distribution. However, there is no difference between the $EKL(\lambda,\theta,b,\delta)$ distribution and $KL(\lambda,\theta,b,1)$ distribution. We conclude that the KL distribution is significantly better than the GL and L distributions based on the LR statistic. The EKL and KL distributions are also better than the KW, WGG and GAM distributions based on the values of the statistics AIC, AICC and BIC. The plots of the fitted EKL distribution and sub-models are shown in Figure 3. Also, the values of sum of squares, are SS=0.03427661 and SS=0.03290961 for EKL and KL distributions, respectively.

Based on the values of these statistics, we conclude that the EKL and KL distributions provide better fit than the generalized gamma, Kumaraswamy Weibull, and generalized Lindley models. In the cases considered, the EKL, and KL performed far better than the generalized Lindley, Lindley, Kumaraswamy Weibull, and gamma distributions.

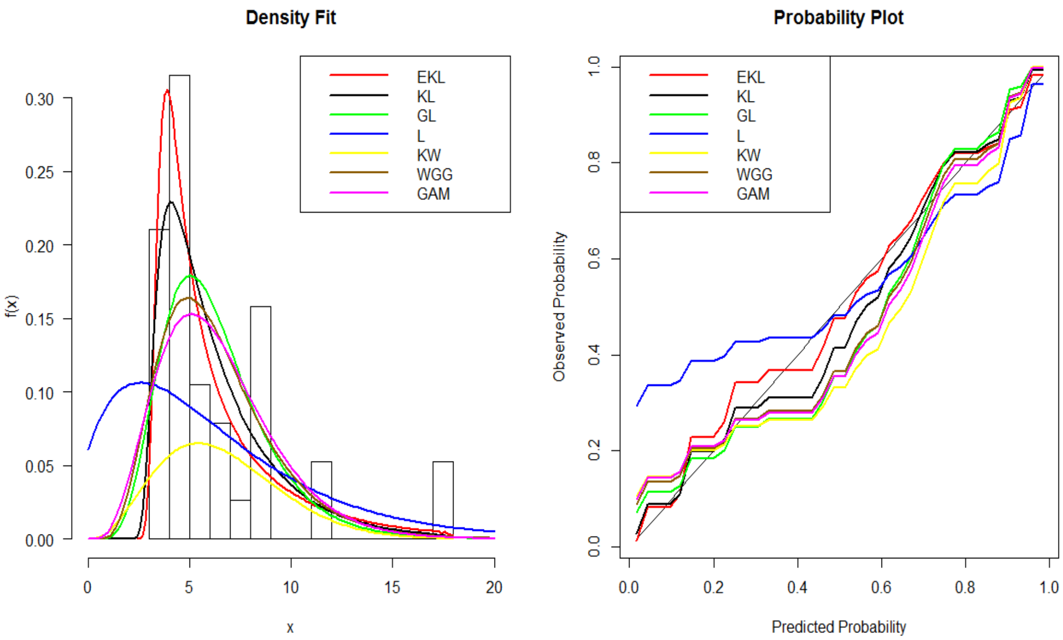

Figure 3: Fitted densities and probability plots of EKL distribution and sub-models for "faraway" data

12 Concluding Remarks

A new class of generalized Lindley distribution referred to as exponentiated Kumaraswamy Lindley (EKL) distribution with flexible and desirable properties is proposed. Properties of the EKL distribution and sub-distributions were presented. The pdf, cdf, moments, hazard function, reverse hazard function, reliability, quantile function, mean deviations, Bonferroni and Lorenz curves were presented. Entropy measures including Renyi entropy, s- entropy as well as Fisher information matrix for EKL distribution were also derived. Estimate of the model parameters via the method of maximum likelihood obtained and applications to illustrate the usefulness of the proposed model to real data given.

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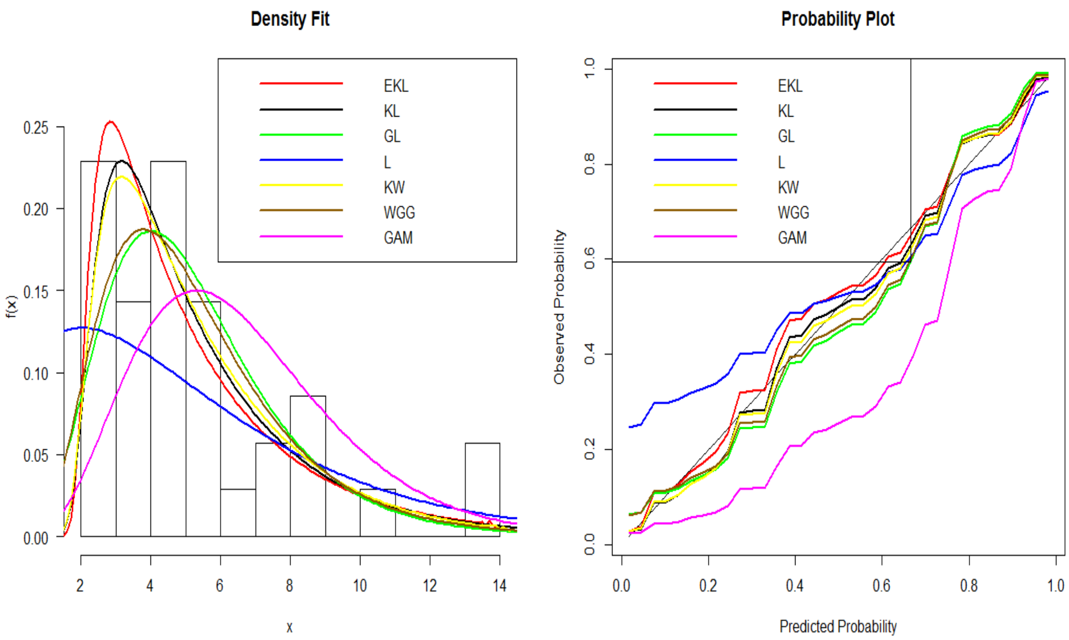

Figure 4: Fitted densities and probability plots of EKL distribution and sub-models for growth hormone data

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Appendix A FIM for EKL distribution

Let $\ell = L(\alpha, \lambda, a, b)$, and $V(x) = G_L(x;\lambda) = 1 - \frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)$. We note here that we have considered the case when $b > 0$ and $\delta > 0$ are non-integer, however the other cases can be similarly derived. Elements of the observed information matrix of the EKL distribution are given by

$$\begin{aligned}\frac{\partial^2\ell}{\partial\delta\partial\lambda} &= b\theta\sum_{i=1}^{n}\frac{[1-[V(x_i)]^\theta]^{b-1}[V(x_i)]^{\theta-1}x_i\exp(-\lambda x_i)}{1-[1-[V(x_i)]^\theta]^b}\\ &\times\left[\frac{1+\lambda+\lambda x_i}{1+\lambda}-\frac{1}{(1+\lambda)^2}\right],\end{aligned}$$

$$\frac{\partial^2\ell}{\partial\delta\partial\theta} = b\sum_{i=1}^{n}\frac{[1-[V(x_i)]^\theta]^{b-1}[V(x_i)]^\theta\log(V(x_i))}{1-[1-[V(x_i)]^\theta]^b},$$

$$\frac{\partial^2\ell}{\partial\delta\partial b} = -\sum_{i=1}^{n}\frac{[1-[V(x_i)]^\theta]^b\log[1-[V(x_i)]^\theta]}{1-[1-[V(x_i)]^\theta]^b},\quad \frac{\partial^2\ell}{\partial\delta^2} = -\frac{n}{\delta^2},$$

$$\begin{aligned}\frac{\partial^2\ell}{\partial b\partial\lambda} &= -\theta\sum_{i=1}^{n}\frac{[V(x_i)]^{\theta-1}x_i\exp(-\lambda x_i)}{1-[V(x_i)]^\theta}\left[\frac{1+\lambda+\lambda x_i}{1+\lambda}-\frac{1}{(1+\lambda)^2}\right]\\ &+ \theta(\delta-1)\sum_{i=1}^{n}\frac{[1-[V(x_i)]^\theta]^{b-1}[V(x_i)]^{\theta-1}x_i\exp(-\lambda x_i)}{[1-[1-[V(x_i)]^\theta]^b]^2}\\ &\times\left[\frac{1+\lambda+\lambda x_i}{1+\lambda}-\frac{1}{(1+\lambda)^2}\right]\left\{b\log[1-[V(x_i)]^\theta]+1-[1-[V(x_i)]^\theta]^b\right\},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\ell}{\partial b\partial\theta} &= -\sum_{i=1}^{n}\frac{[V(x_i)]^\theta\log(V(x_i))}{1-[V(x_i)]^\theta}\\ &+ (\delta-1)\sum_{i=1}^{n}\frac{[1-[V(x_i)]^\theta]^{b-1}[V(x_i)]^\theta\log(V(x_i))}{[1-[1-[V(x_i)]^\theta]^b]^2}\\ &\times\left\{b\log[1-[V(x_i)]^\theta]+1-[1-[V(x_i)]^\theta]^b\right\},\end{aligned}$$

$$\frac{\partial^2\ell}{\partial b^2} = -\frac{n}{b^2}+(1-\delta)\sum_{i=1}^{n}\frac{[1-[V(x_i)]^\theta]^b[\log[1-[V(x_i)]^\theta]]^2}{[1-[1-[V(x_i)]^\theta]^b]^2},$$

$$
\begin{aligned}
\frac{\partial^2 \ell}{\partial \lambda^2} &= -n\frac{\lambda^2+4\lambda+2}{\lambda^2(1+\lambda)^2} \\
&+ \sum_{i=1}^{n} \frac{(2+x_i+2\lambda-x_i^2\lambda+\lambda^2-2x_i\lambda^2-2x_i^2\lambda^2-x_i\lambda^3-x_i^2\lambda^3)x_i\exp(-\lambda x_i)}{(1+\lambda)^2} \\
&\times \left[\frac{\theta-1}{(1+\lambda)V(x_i)}+\frac{\theta(1-b)[V(x_i)]^{\theta-1}}{(1+\lambda)(1-[V(x_i)]^{\theta})}\right] \\
&+ (1-\theta)\lambda\sum_{i=1}^{n}\frac{(2+\lambda+x_i+\lambda x_i)x_i\exp(-\lambda x_i)}{(1+\lambda)^2[V(x_i)]^2}\left[\frac{1+\lambda+\lambda x_i}{1+\lambda}-\frac{1}{(1+\lambda)^2}\right] \\
&+ (1-\theta)\lambda\sum_{i=1}^{n}\frac{(2+\lambda+x_i+\lambda x_i)x_i\exp(-\lambda x_i)}{(1+\lambda)^3V(x_i)} \\
&+ \theta(1-b)\lambda\sum_{i=1}^{n}\frac{[V(x_i)]^{\theta-2}[\theta-1+[V(x_i)]^{\theta}](2+\lambda+x_i+\lambda x_i)x_i^2\exp(-2\lambda x_i)}{(1+\lambda)^2(1-[V(x_i)]^{\theta})^2} \\
&\times \left[\frac{1+\lambda+\lambda x_i}{1+\lambda}-\frac{1}{(1+\lambda)^2}\right] \\
&- \theta(1-b)\lambda\sum_{i=1}^{n}\frac{[V(x_i)]^{\theta-1}[1-[V(x_i)]^{\theta}](2+\lambda+x_i+\lambda x_i)x_i\exp(-\lambda x_i)}{(1+\lambda)^3(1-[V(x_i)]^{\theta})^2} \\
&+ b(1-b)\theta^2(\delta-1)\sum_{i=1}^{n}\frac{[1-[V(x_i)]^{\theta}]^{b-2}[V(x_i)]^{2\theta-2}x_i^2\exp(-2\lambda x_i)}{[1-[1-[V(x_i)]^{\theta}]^{b}]^2} \\
&\times \left[\frac{1+\lambda+\lambda x_i}{1+\lambda}-\frac{1}{(1+\lambda)^2}\right]^2 \\
&- b\theta^2(\delta-1)\sum_{i=1}^{n}\frac{[1-[V(x_i)]^{\theta}]^{2b-2}[V(x_i)]^{2\theta-2}x_i^2\exp(-2\lambda x_i)}{[1-[1-[V(x_i)]^{\theta}]^{b}]^2} \\
&\times \left[\frac{1+\lambda+\lambda x_i}{1+\lambda}-\frac{1}{(1+\lambda)^2}\right]^2 \\
&+ b\theta(\theta-1)(\delta-1)\sum_{i=1}^{n}\frac{[1-[V(x_i)]^{\theta}]^{b-1}[V(x_i)]^{\theta-2}x_i^2\exp(-2\lambda x_i)}{[1-[1-[V(x_i)]^{\theta}]^{b}]^2} \\
&\times \left[\frac{1+\lambda+\lambda x_i}{1+\lambda}-\frac{1}{(1+\lambda)^2}\right]^2 \\
&- b\theta(\delta-1)\sum_{i=1}^{n}\frac{[1-[V(x_i)]^{\theta}]^{b-1}[V(x_i)]^{\theta-1}x_i^2\exp(-\lambda x_i)}{[1-[1-[V(x_i)]^{\theta}]^{b}]^2} \\
&\times \left[\frac{1+\lambda+\lambda x_i}{1+\lambda}-\frac{1}{(1+\lambda)^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad b\theta(1-\theta)(\delta-1)\sum_{i=1}^{n}\frac{[1-[V(x_i)]^{\theta}]^{2b-1}[V(x_i)]^{\theta-2}x_i^2\exp(-2\lambda x_i)}{[1-[1-[V(x_i)]^{\theta}]^{b}]^2}\\
&\times \quad \left[\frac{1+\lambda+\lambda x_i}{1+\lambda}-\frac{1}{(1+\lambda)^2}\right]^2\\
&+ \quad b\theta(\delta-1)\sum_{i=1}^{n}\frac{[1-[V(x_i)]^{\theta}]^{2b-1}[V(x_i)]^{\theta-1}x_i^2\exp(-\lambda x_i)}{[1-[1-[V(x_i)]^{\theta}]^{b}]^2}\\
&\times \quad \left[\frac{1+\lambda+\lambda x_i}{1+\lambda}-\frac{1}{(1+\lambda)^2}\right]\\
&+ \quad b\theta(\delta-1)\sum_{i=1}^{n}\frac{[1-[V(x_i)]^{\theta}]^{b-1}[V(x_i)]^{\theta-1}x_i\exp(-\lambda x_i)[x_i(1+\lambda)+2]}{[1-[1-[V(x_i)]^{\theta}]^{b}](1+\lambda)^3},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2\ell}{\partial\theta\partial\lambda} &= \sum_{i=1}^{n}\frac{x_i\exp(-\lambda x_i)}{V(x_i)}\left[\frac{1+\lambda+\lambda x_i}{1+\lambda}-\frac{1}{(1+\lambda)^2}\right]\\
&+ \quad (1-b)\sum_{i=1}^{n}\frac{[V(x_i)]^{\theta-1}\left[\theta\log(V(x_i))+1-[V(x_i)]^{\theta}\right]x_i\exp(-\lambda x_i)}{[1-[V(x_i)]^{\theta}]^2}\\
&\times \quad \left[\frac{1+\lambda+\lambda x_i}{1+\lambda}-\frac{1}{(1+\lambda)^2}\right]\\
&+ \quad b(\delta-1)\sum_{i=1}^{n}\frac{[1-[V(x_i)]^{\theta}]^{b-2}[V(x_i)]^{\theta-1}x_i\exp(-\lambda x_i)}{[1-[1-[V(x_i)]^{\theta}]^{b}]^2}\\
&\times \quad \left[\frac{1+\lambda+\lambda x_i}{1+\lambda}-\frac{1}{(1+\lambda)^2}\right]\\
&\times \quad \Bigg\{-\theta b[V(x_i)]^{\theta}\log(V(x_i))\\
&+ \quad \left[\theta\log(V(x_i))+1-[V(x_i)]^{\theta}\right]\left[1-[1-[V(x_i)]^{\theta}]^{b}\right]\Bigg\}.
\end{aligned}
$$