Solution of the Spectrum Coincides with the Poles and Zeros of the Compensation Approach to the Synthesis of Matrix Feedback

E.G. Kreventsov

Research Institute of Measuring Instruments, Russia

Abstract.

The problem of finding an algorithm for matrix feedback, providing the specified zeros individual transfer functions, until finally solved in control theory of linear stationary systems. The siting of individual zeros of transfer functions has been widely discussed. As for a multidimensional system matrix system provides a predetermined range of the poles is not unique , then, generally speaking , this is not the only can be used to organize the above mentioned zeros , followed by the possibility of compensation. However, procedures that would allow arbitrarily select all the zeros of the transfer functions of individual multidimensional system do not yet exist. Attempt to solve this problem and the subject of this article.

Keywords: **M**atrix calculation feedback in the original basis, Providing a full range of poles of the system, Compensation zeros of the transfer matrix method for calculating the matrix modification feedback.

1. Introduction

The modern theory of linear systems based on state-space methods [1-3, 6, 7, 11, 12], has a vast array of feedback systems [3, 5, 12, 15]. However, the matrix of feedback (SF) on the state vector satisfies only one requirement - placing the roots of the characteristic equation of the system. The rest of the requirements for the control system (compensation zeros of the transfer matrix) are ignored in the calculation of the matrix. The theory of linear systems generates explicit and implicit algorithms for computing the coefficients of the matrix [2, 5, 6, 7, 14]. Under the explicit algorithms are understood such calculation algorithms, which can be made non-linear equations for the unknown coefficients of the matrix SF, and their number is equal to the number of unknowns [2, 11, 14]. Among such algorithms can also be attributed, and such algorithms in which the transition from the original basis for a new, more appropriate to calculate the coefficients of the matrix [2, 11]. By the implicit algorithms should include such algorithms, which formed the system of nonlinear equations with a number greater than the number of unknown coefficients compiled equations. In such cases, part of the "free" matrix coefficients specified arbitrarily, or from which, or the requirements for the system [1], and thus, the original is not redefining the system of nonlinear equations becomes a system with an equal number of unknowns and equations.

However, in all known methods of calculating the matrix SF is not fully resolved the issue of compensation zeros of the transfer matrix [1, 6, 7, 11, 14]. In domestic and foreign literature payment methods such zeros and management is not fully developed, since when the spectrum of individual zeros of the transfer matrix is changed and range poles. Uncompensated zeros distort the transient response of the system [14]. Such systems typically provide only shift the poles, leaving unchanged the number of dominant zeros. Similar problems in the modern control theory of linear systems are solved by selecting such an arbitrary spectrum of poles at which the transient quality output meets the specified requirements. Avoid such drawbacks modifications allow various kinds of methods of calculation of the matrix feedback, in which the result set of zeros of the transfer matrix offset without degrading the quality of the remaining indicators.

2. Statement of the Problem

Consider multivariate, fully control and observe a linear stationary system in which the dimension of the vectors of input and output matching:

$$\begin{cases} \dot{\mathbf{x}} = \left(\mathbf{A} + \mathbf{BK}\right)\mathbf{x} + \mathbf{Bu}, \\ \mathbf{y} = \mathbf{Cx}, \end{cases} \tag{1}$$

where $\mathbf{x}$ – n - dimensional vector of state coordinates, $\mathbf{u}$ – m - dimensional vector of control impacts, $\mathbf{y}$ – m - dimensional vector of output variables, $n \geq m$, $\mathbf{A}$ – own matrix system, $\mathbf{B}$ – matrix controls, $\mathbf{C}$ – matrix output, $\mathbf{K}$ – feedback matrix vector coordinates state. Dimension of these vectors and matrices as follows: $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^m$, $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times m}$, $\mathbf{C}, \mathbf{K} \in \mathbb{R}^{m \times n}$. It is assumed that the vector of the state coordinates $\mathbf{x}$ is fully accessible to direct measurement.

The procedure for calculating the matrix $\mathbf{K}$ implies finding the transformation matrix from the initial basis of the model (1), for a given basis [1, 11]. Require that it was possible to calculate the feedback matrix $\mathbf{K}$, through the matrix $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, i.e., in the original basis (the invariance of the algorithm for calculating the matrix $\mathbf{K}$ on the basis of which is represented by the system (1)), to simplify the implementation of this relation, the matrix $\mathbf{K}$, on a computer. Require that the algorithm for calculating the matrix feedback provides a set of full range of poles of the system (1). Poles in the system (1) and define matching lying at the point Ω, see Fig. 1, on the left of the real axis of the complex plane. This ensures the monotony of the transient response, i.e., the absence of the vibrational nature of the transition process.

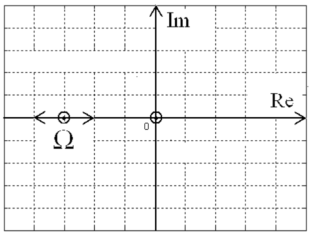

Fig. 1. Spectrum of poles of the system (1)

Require that the individual poles and zeros of the transfer functions were coincident.

3. The method of solution

Represent the output in equation (1) as $\left[y_i\right]_{i=\overline{1,m}} = \left[\mathbf{c}_i\left(\mathbf{A}+\mathbf{BK}\right)^0\right]_{i=\overline{1,m}}\mathbf{x}$,

where - $\mathbf{c}_i$ - i-th row, $i=\overline{1,m}$, the matrix $\mathbf{C}$ in the model, we have:

$$\left[y_i^{(j_i+1)}\right]_{i=\overline{1,m}} = \left[\mathbf{c}_i\left(\mathbf{A}+\mathbf{BK}\right)^{j_i+1}\right]_{i=\overline{1,m}}\mathbf{x} + \left[\mathbf{c}_i\mathbf{A}^{j_i}\mathbf{B}\right]_{i=\overline{1,m}}\mathbf{u}\,. \quad (2)$$

Differentiation will continue as long as the block matrix $\left[\mathbf{c}_i\mathbf{A}^{j_i}\mathbf{B}\right]_{i=\overline{1,m}}$ will not block the zero line $\mathbf{c}_i\mathbf{A}^{j_i}\mathbf{B}$ to $\forall\ i,\in\ [1,\,m]$, for $i\in\mathbb{R}$. Given a whole, positive numbers $j_1, j_2,\ldots, j_m$, are the minimal indices of the matrix product "$\mathbf{c}_i\mathbf{A}^j\mathbf{B}$, $i=\overline{1,m}$, $j=\overline{0,n-1}$: (3)

$$j_i = \min j \;\middle|\; \begin{array}{l} \mathbf{c}_i\mathbf{A}^j\mathbf{B}\neq[\mathbf{0}]^{1\times m} \\ j=\overline{0,n-1},\ i=\overline{1,m}. \end{array}$$

Given (2) introduce the matrix $\mathbf{D}$, and the matrix $\mathbf{D}$-non-degenerate $\mathbf{D}\in\mathbb{R}^{m\times m}$:

$$\mathbf{D} = \left(\left[\mathbf{c}_1\mathbf{A}^{j_1}\mathbf{B}\right]\ \left[\mathbf{c}_2\mathbf{A}^{j_2}\mathbf{B}\right]\ \cdots\ \left[\mathbf{c}_m\mathbf{A}^{j_m}\mathbf{B}\right]\right)^T, \quad (4)$$

Continuing to differentiation of (2) we obtain:

$$\left[y_i^{(n)}\right]_{i=\overline{1,m}} = \left[\mathbf{c}_i\left(\mathbf{A}+\mathbf{BK}\right)^n\right]_{i=\overline{1,m}}\mathbf{x} + \left[\mathbf{c}_i\left(\mathbf{A}+\mathbf{BK}\right)^{n-1}\mathbf{B}\right]_{i=\overline{1,m}}\mathbf{u} + \ldots + \mathbf{D}\mathbf{u}^{(n-j_i-1)}. \quad (5)$$

Calculating the product of matrices $\mathbf{c}_i\mathbf{A}^j\mathbf{B}$, gives

$$\mathbf{c}_i\left(\mathbf{A}+\mathbf{BK}\right)^k \Big|_{k=\overline{0,j_i},} = \mathbf{c}_i\mathbf{A}^k, \qquad \mathbf{c}_i\left(\mathbf{A}+\mathbf{BK}\right)^k \Big|_{k=\overline{(j_i+1),n}.} = \mathbf{c}_i\mathbf{A}^{j_i}\left(\mathbf{A}+\mathbf{BK}\right)^{k-j_i}, \quad (6)$$

Taking into account (6) we introduce the matrix $\mathbf{A}^*$, $\mathbf{A}^* \in \mathbb{R}^{m\times n}$:

$$\mathbf{A}^* = \left(\left[\mathbf{c}_1 \mathbf{A}^{j_1+1} \right] \quad \left[\mathbf{c}_2 \mathbf{A}^{j_2+1} \right] \quad \cdots \quad \left[\mathbf{c}_m \mathbf{A}^{j_m+1} \right] \right)^T . \qquad (7)$$

By Theorem Kelly Hamilton have:

$$\left[\mathbf{c}_i \left(\mathbf{A} + \mathbf{BK} \right)^n \right]_{i=\overline{1,m}} = -\left[\mathbf{c}_i \sum_{k=0}^{n-1} \binom{n}{n-k} \Omega^{n-k} \left(\mathbf{A} + \mathbf{BK} \right)^k \right]_{i=\overline{1,m}}, \qquad (8)$$

where Ω-square geometric average, $\binom{n}{k}$-binomial coefficients. Write (2) taking into account (4) and (7):

$$\left[y_i^{(j_i+1)} \right]_{i=\overline{1,m}} = \left(\mathbf{A}^* + \mathbf{DK} \right) \mathbf{x} + \mathbf{Du} . \qquad (9)$$

Feedback matrix can be found as follows: $\mathbf{K} = -\mathbf{D}^{-1} \cdot \mathbf{A}^*$. Since in this case the condition - $\mathbf{DK} = -\mathbf{D} \cdot \mathbf{D}^{-1} \cdot \mathbf{A}^* = -\mathbf{A}^*$, then

$$\left(\mathbf{c}_1 \left(\mathbf{A} + \mathbf{BK} \right)^{j_1+k} \quad \cdots \quad \mathbf{c}_m \left(\mathbf{A} + \mathbf{BK} \right)^{j_m+k} \right)^T = \left[\mathbf{0} \right]^{1\times n} . \qquad (10)$$

The resulting expression is valid for any positive integer k, $k \in \mathbb{R}_{>0}$. As a result, (5) can be represented as:

$$\left[\mathbf{y}_i^{(n)} \right] - \left[\sum_{k=0}^{n-1} \binom{n}{n-k} \Omega^{n-k} \mathbf{y}_i^{(k)} \right] = Sp\left(\mathbf{L\Theta} \right), \qquad (11)$$

where $Sp(\bullet)$ - denotes the trace of the matrix. Matrix $\mathbf{L}$ and $\mathbf{\Theta}$, $\mathbf{L} \in \mathbb{R}^{n\times m}$, $\mathbf{\Theta} \in \mathbb{R}^{m\times n}$ in (11) has the form:

$$\mathbf{L} = \left(0 \quad 0 \quad \ldots \quad \left[\mathbf{c}_i \mathbf{A}^{j_i} \mathbf{B} \right] \quad [0] \right)^T, \qquad \mathbf{\Theta} = \left(\mathbf{u} \middle| \quad \mathbf{u}^{(1)} \middle| \quad \cdots \quad \middle| \mathbf{u}^{(n-1)} \right). \qquad (12)$$

Matrix $\mathbf{L}$ contains a single non-zero block string - $\left[\mathbf{L}_i \right] = \left[\mathbf{c}_i \mathbf{A}^{j_i} \mathbf{B} \right]$, $i = \overline{1,m}$ then $rank\left[\mathbf{L}_i \right] = 1$. Rewrite (9), using (8):

$$\left[\sum_{k=0}^{j_i+1} \binom{j_i+1}{j_i+1-k} \Omega^{j_i+1-k} y_i^{(k)} \right]_{i=\overline{1,m}} = \left(\mathbf{A}^* + \left[\mathbf{c}_i \sum_{k=0}^{j_i} \binom{j_i+1}{j_i+1-k} \Omega^{j_i+1-k} \mathbf{A}^k \right]_{i=\overline{1,m}} + \mathbf{DK} \right) \mathbf{x} + \mathbf{Du}. \qquad (13)$$

Given (13) define a matrix of poles $\mathbf{K_P}$, $\mathbf{K_P} \in \mathbb{R}^{m\times n}$:

$$\mathbf{K_P} = \left[\mathbf{c}_i \sum_{k=0}^{j_i} \binom{j_i+1}{j_i+1-k} \Omega^{j_i+1-k} \mathbf{A}^k \right]_{i=\overline{1,m}}, \qquad (14)$$

Using (10) we have $\left(\mathbf{A}^* + \mathbf{K_P} + \mathbf{D}\cdot\mathbf{K}\right) = [0]$. The last expression can be found matrix $\mathbf{K}$, given the notation (4), (7), (14) - $\mathbf{K} = -\left(\mathbf{D}^{-1}\mathbf{A}^* + \mathbf{D}^{-1}\mathbf{K_P}\right)$:

$$\mathbf{K} = -\left(\left(\mathbf{D}^{-1}\mathbf{A}^*\right) + \left(\mathbf{D}^{-1}\cdot\left[\mathbf{c}_i \sum_{k=0}^{j_i} \Omega^{j_i+1-k}\cdot\left(\frac{(j_i+1)!}{(j_i+1-k)!\cdot k!}\right)\mathbf{A}^k\right]_{i=\overline{1,m}}\right)\right). \tag{15}$$

As a result, the ratio obtained for the matrix $\mathbf{K}$, which is calculated directly from the matrix $\mathbf{A}, \mathbf{B}, \mathbf{C}$ (1), in the original basis. According to (15), the matrix $\mathbf{K}$ can be "easily" calculated by computer. It remains an open question - what class of controlled objects of the form (1) expression (15) can be calculated? To do this, we consider the transfer matrix of a closed system:

$$\mathbf{W}(p) = \left[\frac{\mathbf{c}_i\mathbf{A}^{j_i}\mathbf{B}}{\sum_{k=0}^{j_i+1}\begin{pmatrix} j_i+1 \\ j_i+1-k\end{pmatrix}\Omega^{j_i+1-k}p^k}\right]_{i=\overline{1,m}}, \tag{16}$$

where $\mathbf{W}(p) \in \mathbb{R}[p]^{m\times m}$, p - Laplace operator. From the right side of (16) we see that the zeros of the transfer functions of the individual and the poles are contractile.

Considering the method of calculating the complete set of zeros [13] for control and observe system having an equal number of inputs and outputs, we have:

$$\psi(p) = \sum_{k=0}^{\mu}\psi_k p^k = \det\begin{pmatrix}[p\mathbf{I}_n - \mathbf{A}] & -[\mathbf{B}] \\ [\mathbf{C}] & [0]\end{pmatrix} = \det\begin{pmatrix}[p\mathbf{I}_n - \mathbf{A} - \mathbf{BK}] & -[\mathbf{B}] \\ [\mathbf{C}] & [0]\end{pmatrix}, \tag{17}$$

where $\psi(p)$ - the zero polynomial, $\psi(p) \in \mathbb{R}[p]^{\mu}$, μ - number of finite zeros (1), $\mu \le (n-m)$. Expression (17) holds for $\forall$ $\mathbf{K}$, and in particular for $\mathbf{K} = [0]$. This follows directly from the principle of invariance of the set of trailing zeros for the matrix $\mathbf{K}$. Block matrix, standing under the sign of the determinant in (17) is called the "Rosenbrock matrix" or "matrix system" [13], for which we introduce the following notation - $\mathbf{P}(p)$, $\mathbf{P}(p) \in \mathbb{R}[p]^{(n+m)\times(n+m)}$. Given (16) and opening qualifier for the block matrix (17), we obtain:

$$\psi(p) = \frac{\det\left[\mathbf{c}_i\mathbf{A}^{j_i}\left(\sum_{k=0}^{n-j_i-1}\begin{pmatrix} n-j_i-1 \\ n-j_i-1-k\end{pmatrix}\Omega^{n-j_i-1-k}p^k\right)\mathbf{B}\right]_{i=\overline{1,m}}}{\left(\sum_{k=0}^{n}\begin{pmatrix} n \\ n-k\end{pmatrix}\Omega^{n-k}p^k\right)^{m-1}}. \tag{18}$$

Rewrite the expression (18):

$$\det\left[\mathbf{c}_i\mathbf{A}^{j_i}\left(\sum_{k=0}^{n-j_i-1}\binom{n-j_i-1}{n-j_i-1-k}\Omega^{n-j_i-1-k}p^k\right)\mathbf{B}\right]_{i=\overline{1,m}} =$$

$$=\left(\sum_{k=0}^{\mu}\psi_k p^k\right)\cdot\left(\sum_{k=0}^{n}\binom{n}{n-k}\Omega^{n-k}p^k\right)^{m-1}.$$

Consider the degree of the polynomial in the left and right sides.

$$mn-\left(\sum_{i=1}^{m} j_i+m\right)=\mu-n(m-1) \text{ or } (n-\mu)=\left(\sum_{i=1}^{m} j_i+m\right)=\tau_l\,. \qquad (19)$$

If the system is μ zeros with the number of "free" poles τ_l, $\tau_l \le n$ to μ times less. As a result, μ are not subject to the poles shear and are invariant under feedback. It is therefore advisable to choose a class of systems (1) for which the $\mu = 0$. In this all-pole system with feedback and a matrix (15) are free and may have the desired spectrum. If the matrix is unimodal Rosenbrock:

$$\begin{vmatrix}[p\mathbf{I}_n-\mathbf{A}] & -[\mathbf{B}]\\ [\mathbf{C}] & [0]\end{vmatrix}=const\,, \qquad (20)$$

the degree of the polynomial $\psi(p)$ is zero ($\mu = 0$). Availability unimodal Rosenbrock matrix indicates the absence decoupled zeros; the system has no uncontrolled and unobservable subsystems. Equation (19) finally determines the class of systems (1) for which can be found by the expression (15).

4. Conclusions

When placing the roots of the characteristic equation of the system transient response significantly depends on the actual poles and zeros of a plurality of individual transfer functions. The proposed algorithm for finding a feedback matrix completely inhibits the action of individual zeros of the system, pulling them to the point of the complex plane. In calculating the coefficients of the matrix controller SF there is no need to form a system of nonlinear equations or the transition from the original basis for a new, more acceptable to the calculation of the matrix of feedback (for example, the canonical form of control), as in the classical approach of calculation. The proposed algorithm is applicable if the following conditions. If a set of trailing zeros is empty, then the spectrum of individual zeros of transfer functions and the spectrum of the system poles can be arbitrarily set using matrix feedback. This fact can be used as the basis of the compensation algorithm for calculating the matrix state feedback.

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