Scattering relations and extinction cross-section for the conductive Helmholtz problem

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Abstract

In the present study we consider the scattering of time-harmonic acoustic waves from a penetrable obstacle on which is imposed conductive transmission conditions. For this problem the far-field pattern is derived and general scattering theorems for both plane and spherical waves are proved. For the spherical waves has been used a far-field generator in order to formulate the general scattering theorem in plane-wave form. The extinction cross-section is evaluated through optical theorem which is derived as corollary of the general scattering theorem.

Keywords: conductive conditions; extinction cross-section; optical theorem; far field pattern.
AMS: 35P25, 35J08, 76Q05.

1 Introduction

Conservation of energy for electromagnetic or acoustic waves, or conservation of probability in quantum mechanics, leads to the optical theorem. The optical theorem correlates the power extinguished from an incident wave on an object to the far-field pattern [8],[16]. The extinction cross-section is the power lost due to scattering and absorption of the incident field (total extinguished power) normalized by the power per unit area incident on the scatterer. The optical theorem in essence implies that the total power extinguished from the incident field is removed from the incident field by the interference between the incident field and the forward scattered field. For details on the connection of the extinction cross-section and the optical theorem we refer to the books [8] and [11].

In this paper we study general scattering and optical theorems for a scatterer which is covered by a thin layer with very high conductivity. From a physical viewpoint, scatterers covered by a thin layer of high conductivity can be important in the modeling of electromagnetic induction in the Earth [1]. The physical meaning of conductive conditions have been previously investigated, for example by Senior [13] and [14]. Angell, Kleinman and Hettlich in [1] have proved the well-posedness of the corresponding direct scattering problem. Bondarenko and Liu in [3] apply the factorization method to solve an inverse scattering problem for an obstacle with conductive boundary conditions. Generalized optical theorems and extinction of power for scalar fields are proved in [5]. General scattering and optical theorems for both spherical acoustic and electromagnetic waves by a layered scatterer with a resistive or conductive core are included in [15]. In [17] an extended optical theorem for a non-diffracting beam is used to examine the exctintion cross-section of a sphere centered on the axis of the beam. In [18] a generalizated optical theorem which relates the extincion cross-section to the angular function of the incident field is derived. A formula for the acoustic integral extinction is derived in [12].

We consider time-harmonic acoustic plane and spherical waves. The conditions on the surface of the scatterer describe a conductive boundary transmission problem which includes the classical acoustic transmission problem [6], [8]. Relations between the solutions of the scattering problems due to two distinct waves incident on the same scatterer are proved. The paper is organized as follows: In section 2 we formulate the conductive scattering problem and calculate the corresponding far-field pattern. In section 3 we consider plane incidence, prove a general scattering theorem and as corollary we derive the optical theorem. In section 4 we have defined the plane and spherical far-field generators in order to be used for the formulation of scattering theorems. Spherical waves are considered in section 5 and a general scattering theorem is proved. Using these results we prove an optical theorem for spherical waves. Finally, in section 6 we discus some remarks and special problems which can be obtained as degenerate cases of the conductive problem.

2 Formulation

Let D be a bounded region in $\mathbb{R}^3$ with a C^2- boundary S. The complement of D, that is $\mathbb{R}^3 \backslash \bar{D}$, is assumed to be simply connected and will be denoted by D_0. The set D will be referred to as the scatterer. The exterior D_0 of the scatterer is an infinite homogeneous isotropic lossless acoustic medium with mass density ρ_0 and mean compressibility γ_0. The interior of D within lies the origin is a homogeneous, isotropic, lossy medium with mass density ρ, mean compressibility γ and compressional viscosity δ. At each point $\mathbf{r}$ on the surface S there is a normal unit vector $\hat{\nu}(\mathbf{r})$ pointing into D_0.

Let u^i be an incident on D acoustic wave which may be either plane or spherical. We note that the incident plane waves are defined in $\mathbb{R}^3$, whereas the spherical incident waves are defined in $\mathbb{R}^3 \backslash \{\mathbf{a}\}$, where $\mathbf{a}$ is the position vector of point source. If u^s is the corresponding scattered field then the total exterior field u^t is given by

$$u^t = u^i + u^s \tag{1}$$

and satisfies the Helmholtz equation

$$\Delta u^t + k^2 u^t = 0 \text{ in } D_0, \tag{2}$$

where $k = \omega\sqrt{\gamma_0\rho_0}$ is the wave number and ω is the angular frequency.

The scatterer field u^s is a solution of (2) in $\mathbb{R}^3 \backslash \bar{D}$ since u^i solves the equation (2). Also u^s satisfies the Sommerfeld radiation condition

$$\mathbf{r} \cdot \nabla u^s - iku^s = o(\frac{1}{r}), \quad r = |\mathbf{r}| \to \infty, \tag{3}$$

uniformly in all directions $\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} \in S^2$, where S^2 is the unit sphere. In the sequel for a vector $\mathbf{w}$ we shall denote $w = |\mathbf{w}|$ for the measure and $\hat{\mathbf{w}} = \frac{\mathbf{w}}{w}$ for the corresponding unit vector. The total field u in D solves the Helmholtz equation

$$\Delta u + \eta^2 k^2 u = 0 \text{ in } D, \tag{4}$$

where η is the relative index of diffraction and it is given by $\eta = \frac{\sqrt{\gamma\rho}}{\sqrt{\gamma_0\rho_0(1-i\omega\gamma\delta)}}$.

The surface S is covered by a thin layer with very high conductivity. This physical consideration is described by the general conductive transmission conditions [1]

$$u^t = u \text{ on } S, \tag{5}$$

$$\frac{\partial u^t}{\partial v} = \lambda \frac{\partial u}{\partial \nu} - \mu u \text{ on } S, \tag{6}$$

where $\lambda \in \mathbb{C}, \lambda \neq 0$ and $\mu \in L^\infty(S)$ with $Re(\mu) \leq 0$. If $\mu = 0$ and $\lambda = \frac{\rho_0}{\rho}(1 - i\omega\gamma\delta)$ then relations (5),(6) become the classical transmission conditions [8].
The scattered field has the following asymptotic form [8]

$$u^s(\mathbf{r}) = h(kr)u^\infty(\hat{\mathbf{r}}) \; + \; O\left(\frac{1}{r^2}\right), \quad r \to \infty, \tag{7}$$

where $h(x) = \frac{e^{ix}}{ix}$ is the spherical Hankel function of the first kind and zero order. The function $u^\infty(\hat{\mathbf{r}})$ is the far-field pattern and is given by

$$u^\infty(\hat{\mathbf{r}}) = -\frac{ik}{4\pi} \int_S \left[\frac{\partial u^s(\mathbf{r}')}{\partial \nu} + ik(\hat{\mathbf{r}} \cdot \hat{\nu}) u^s(\mathbf{r}') \right] e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'} ds(\mathbf{r}') \; . \tag{8}$$

In particular, applying the conductive transmission conditions and using the Green's first theorem we have the far-field pattern for the problem (1)-(6)

$$u^\infty(\hat{\mathbf{r}}) = \frac{ik^3\eta^2(\lambda - 1)}{4\pi} \int_D u(\mathbf{r}') e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'} dv(\mathbf{r}') \; + \; \frac{ik}{4\pi} \int_S \mu(\mathbf{r}') u(\mathbf{r}') e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'} ds(\mathbf{r}') \tag{9}$$

In what follows we shall make use of Twersky's notation [16]

$$\{u, \nu\}_S = \int_S \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) ds \tag{10}$$

and we shall denote with $\bar{z}$ the complex conjugation of z.

3 Plane waves

We consider the scattering of time-harmonic plane acoustic waves by the conductive scatterer D. Let the incident field be

$$u^i(\mathbf{r};\hat{\mathbf{p}}) = e^{ik\hat{\mathbf{p}}\cdot\mathbf{r}}, \tag{11}$$

where $\hat{\mathbf{p}}$ describes the direction of propagation. We shall indicate the dependence of the scattered field, the far-field pattern and the total fields in D_0 and D on the incident direction $\hat{\mathbf{p}}$ by writing $u^s(\mathbf{r};\hat{\mathbf{p}}), u^\infty(\hat{\mathbf{r}};\hat{\mathbf{p}}), u^t(\mathbf{r};\hat{\mathbf{p}})$ and $u(\mathbf{r};\hat{\mathbf{p}})$ respectively.

We prove a general scattering theorem for plane acoustic waves, [8], from which the optical theorem is derived as a simply consequence.

Theorem 3.1. *Theorem Let $u^i(\mathbf{r};\hat{\mathbf{p}})$ and $u^i(\mathbf{r};\hat{\mathbf{q}})$ be two plane waves incident upon the conductive scatterer D. Then for the corresponding far-field patterns we have*

$$\overline{u^\infty(\hat{\mathbf{q}};\hat{\mathbf{p}})} + u^\infty(\hat{\mathbf{p}};\hat{\mathbf{q}}) + \frac{1}{2\pi}\int_{S^2} \overline{u^\infty(\hat{\mathbf{r}};\hat{\mathbf{p}})} u^\infty(\hat{\mathbf{r}};\hat{\mathbf{q}}) ds(\hat{\mathbf{r}}) = \mathcal{A}(\hat{\mathbf{p}};\hat{\mathbf{q}}), \tag{12}$$

where

$$\mathcal{A}(\hat{\mathbf{p}};\hat{\mathbf{q}}) = -\frac{k^3 Im(\lambda\eta^2)}{2\pi}\int_D \overline{u(\mathbf{r};\hat{\mathbf{p}})} u(\mathbf{r};\hat{\mathbf{q}}) dv(\mathbf{r}) + \frac{k Im(\lambda)}{2\pi}\int_D \nabla\overline{u(\mathbf{r};\hat{\mathbf{p}})}\cdot\nabla u(\mathbf{r};\hat{\mathbf{q}}) dv(\mathbf{r})$$

$$-\frac{k}{2\pi}\int_S Im(\mu(\mathbf{r}))\overline{u(\mathbf{r};\hat{\mathbf{p}})} u(\mathbf{r};\hat{\mathbf{q}}) ds(\mathbf{r}) \tag{13}$$

Proof. In view of (1) and the bilinearity of the form (10) we take

$$\begin{aligned}\{\overline{u^t(\cdot;\hat{\mathbf{p}})}, u^t(\cdot;\hat{\mathbf{q}})\}_S &= \{\overline{u^i(\cdot;\hat{\mathbf{p}})}, u^i(\cdot;\hat{\mathbf{q}})\}_S + \{\overline{u^i(\cdot;\hat{\mathbf{p}})}, u^s(\cdot;\hat{\mathbf{q}})\}_S \\ &\quad + \{\overline{u^s(\cdot;\hat{\mathbf{p}})}, u^i(\cdot;\hat{\mathbf{q}})\}_S + \{\overline{u^s(\cdot;\hat{\mathbf{p}})}, u^s(\cdot;\hat{\mathbf{q}})\}_S\end{aligned} \tag{14}$$

For the first integral in (14) using the conductive transmission conditions (5), (6), applying the Green's first theorem in D and taking into account that $\overline{u(\mathbf{r};\hat{\mathbf{p}})}, u(\mathbf{r};\hat{\mathbf{q}})$ are regular solutions of Helmholtz equation in D we take

$$\{\overline{u^t(\cdot;\hat{\mathbf{p}})}, u^t(\cdot;\hat{\mathbf{q}})\}_S = -2ik^2 Im(\lambda\eta^2)\int_D \overline{u(\mathbf{r};\hat{\mathbf{p}})} u(\mathbf{r};\hat{\mathbf{q}}) dv(\mathbf{r})$$

$$+ 2i Im(\lambda)\int_D \nabla\overline{u(\mathbf{r};\hat{\mathbf{p}})}\cdot\nabla u(\mathbf{r};\hat{\mathbf{q}}) dv(\mathbf{r}) - 2i\int_S Im(\mu(\mathbf{r}))\overline{u(\mathbf{r};\hat{\mathbf{p}})} u(\mathbf{r};\hat{\mathbf{q}}) ds(\mathbf{r}) \tag{15}$$

The incident waves $\overline{u^i(\cdot;\hat{\mathbf{p}})}, u^i(\cdot;\hat{\mathbf{q}})$ are regular solutions of Helmholtz equation in $\mathbb{R}^3$ and an application of the second Green's theorem in D gives

$$\{\overline{u^i(\cdot;\hat{\mathbf{p}})}, u^i(\cdot;\hat{\mathbf{q}})\}_S = 0. \tag{16}$$

From (8) we have

$$\{\overline{u^i(\cdot;\hat{\mathbf{p}})}, u^s(\cdot;\hat{\mathbf{q}})\}_S = -\frac{4\pi}{ik} u^\infty(\hat{\mathbf{p}};\hat{\mathbf{q}}), \tag{17}$$

$$\{\overline{u^s(\cdot;\hat{\mathbf{p}})}, u^i(\cdot;\hat{\mathbf{q}})\}_S = -\frac{4\pi}{ik} \overline{u^\infty(\hat{\mathbf{p}};\hat{\mathbf{q}})}. \tag{18}$$

For the last integral of (14) we consider a sphere S_R, centered on the origin with radius R surrounding the scatterer. Let D_R be the region exterior to S and interior to S_R. Applying Green's second theorem on the functions $\overline{u^s(\mathbf{r};\hat{\mathbf{p}})}$ and $u^s(\mathbf{r};\hat{\mathbf{q}})$ in D_R, where they are regular solutions of the Helmholtz equation, we take

$$\{\overline{u^s(\cdot;\hat{\mathbf{p}})}, u^s(\cdot;\hat{\mathbf{q}})\}_S = \{\overline{u^s(\cdot;\hat{\mathbf{p}})}, u^s(\cdot;\hat{\mathbf{q}})\}_{S_R}. \tag{19}$$

Then, letting $R \to \infty$ we pass to radiation zone and thus we can use the asymptotic form (7) giving

$$\{\overline{u^s(\cdot;\hat{\mathbf{p}})}, u^s(\cdot;\hat{\mathbf{q}})\}_S = \frac{2i}{k} \int_{S^2} \overline{u^\infty(\hat{\mathbf{r}};\hat{\mathbf{p}})} u^\infty(\hat{\mathbf{r}};\hat{\mathbf{q}}) ds(\hat{\mathbf{r}}). \tag{20}$$

Substituting (15)-(20) into (14) the theorem is proved. □

The energy that a scatterer extracts from the incident wave is described by scattering cross-section and by absorption cross-section. The scattering cross-section is a measure of the disturbance caused by the obstacle to the incident wave and is defined by

$$\sigma^s(\hat{\mathbf{p}}) = \frac{1}{k^2} \int_{S^2} |u(\hat{\mathbf{r}};\hat{\mathbf{p}})|^2 ds(\hat{\mathbf{r}}). \tag{21}$$

The absorption cross-section is defined by

$$\sigma^\alpha(\hat{\mathbf{p}}) = \frac{1}{k} \int_S u^t(\mathbf{r};\hat{\mathbf{p}}) \overline{\frac{\partial u^t(\mathbf{r};\hat{\mathbf{p}})}{\partial \nu}} ds(\mathbf{r}), \tag{22}$$

and the extinction cross-section is defined by

$$\sigma^e(\hat{\mathbf{p}}) = \sigma^s(\hat{\mathbf{p}}) + \sigma^\alpha(\hat{\mathbf{p}}). \tag{23}$$

The cross-section and the optical theorem connect the far-field pattern and the extinction. For a conductive scatterer the optical theorem is the following.

Theorem 3.2. *Theorem Let $u^i(\mathbf{r};\hat{\mathbf{p}})$ be a plane wave incident on the conductive scatterer D and $\sigma^e(\hat{\mathbf{p}})$ the corresponding extinction cross-section. Then we have*

$$\sigma^e(\hat{\mathbf{p}}) = -\frac{4\pi}{k^2} Re[u^\infty(\hat{\mathbf{p}};\hat{\mathbf{p}})]. \tag{24}$$

Proof. If we put $\hat{\mathbf{p}} = \hat{\mathbf{q}}$ in (12) we obtain

$$2Re[u^{\infty}(\hat{\mathbf{p}};\hat{\mathbf{p}})] = -\frac{1}{2\pi}\int_{S^2} |u(\hat{\mathbf{r}};\hat{\mathbf{p}})|^2 ds(\hat{\mathbf{r}}) + \mathcal{A}(\hat{\mathbf{p}};\hat{\mathbf{p}}). \tag{25}$$

Applying the conductive transmission conditions in (22) and using the Green's first theorem we find

$$\sigma^{\alpha}(\hat{\mathbf{p}}) = -\frac{2\pi}{k^2}\mathcal{A}(\hat{\mathbf{p}};\hat{\mathbf{p}}). \tag{26}$$

From (25), (26) we get (24). □

4 Far-field generators

We consider an incident spherical acoustic wave due to a source located at a point with position vector $\mathbf{a}$, given by

$$u_a^i(\mathbf{r}) = ae^{ika}\frac{e^{ik|\mathbf{r}-\mathbf{a}|}}{|\mathbf{r}-\mathbf{a}|}, \mathbf{r} \neq \mathbf{a}, \tag{27}$$

where $a = |\mathbf{a}|$. The corresponding scattered, total exterior, total interior and far-field pattern are denoted by u_a^s, u_a^t, u_a and u_a^{∞} respectively.

The scattering theorems for spherical waves are more complicated than the corresponding theorems for plane waves. This due to the fact that the spherical acoustic waves are singular solutions of the Helmholtz equation and the Green's theorem does not apply in spaces where there are point sources. In order to formulate the general scattering theorem in the plane-wave form we introduce the spherical and the plane far-field generators [2].

For two point sources $\mathbf{a}$ and $\mathbf{b}$ we define the spherical far-field pattern generator by

$$U_b(\mathbf{a}) = ikae^{ika}\left[u_b^s(\mathbf{a}) - \frac{1}{2\pi}\int_{S^2} u_b^{\infty}(\hat{\mathbf{r}})e^{ik\mathbf{a}\cdot\hat{\mathbf{r}}}ds(\hat{\mathbf{r}})\right] \tag{28}$$

and the plane far-field pattern generator by

$$U(\mathbf{a};-\hat{\mathbf{b}}) = ikae^{ika}\left[u^s(\mathbf{a};-\hat{\mathbf{b}}) - \frac{1}{2\pi}\int_{S^2} u^{\infty}(\hat{\mathbf{r}};-\hat{\mathbf{b}})e^{ik\mathbf{a}\cdot\hat{\mathbf{r}}}ds(\hat{\mathbf{r}})\right]. \tag{29}$$

When the observation point $\mathbf{a}$ recedes to infinite the generators become far-field patterns. When both observation point $\mathbf{a}$ and the point - source $\mathbf{b}$ recede to infinity $U_b(\mathbf{a})$ reduces to the plane far-field pattern defined on $-\hat{\mathbf{a}}$ and is derived from plane incidence in the direction $-\hat{\mathbf{b}}$. In particular, in [2] the following theorem is proved.

Theorem 4.1. *Theorem For two incident spherical waves u_a^i and u_b^i we have*

$$\lim_{a\to\infty} U_b(\mathbf{a}) = u_b^\infty(-\hat{\mathbf{a}}), \tag{30}$$

$$\lim_{a\to\infty} U^\infty(\mathbf{a}; -\hat{\mathbf{b}}) = u^\infty(-\hat{\mathbf{a}}; -\hat{\mathbf{b}}), \tag{31}$$

$$\lim_{a,b\to\infty} U_b(\mathbf{a}) = u^\infty(-\hat{\mathbf{a}}; -\hat{\mathbf{b}}). \tag{32}$$

Now we can formulate the general scattering theorem for spherical acoustic waves.

5 Spherical waves

From the normalization of the spherical wave (27), which first introduced in [18], it follows that as the location of the point $\mathbf{a}$ recedes to infinite, the point source field degenerates to a plane wave with direction of propagation $-\hat{\mathbf{a}}$, that is

$$\lim_{a\to\infty} u_a^i(\mathbf{r}) = e^{-ik\hat{\mathbf{a}}\cdot\mathbf{r}} = u^i(\mathbf{r}; -\hat{\mathbf{a}}). \tag{33}$$

Furthermore the normal energy fluxes for the spherical and plane waves are the same [7].
We consider the incident spherical acoustic wave u_a^i due to a source located at a point with position vector $\mathbf{a}$ given by (27). In the exterior domain $D_0\backslash\{\mathbf{a}\}$ we have again the superposition

$$u_a^t = u_a^i + u_a^s \text{ in } D_0\backslash\{\mathbf{a}\}. \tag{34}$$

The scattered and the incident fields u_a^s and u_a^i satisfy the Sommerfeld radiation condition (3).
A general scattering theorem is also valid for the spherical waves. We consider two point-source positions $\mathbf{a}$ and $\mathbf{b}$ and using the notation (28) the general scattering theorem for acoustic spherical waves is formulated as follows.

Theorem 5.1. *Theorem Let $u_a^i(\mathbf{r})$ and $u_b^i(\mathbf{r})$ be two spherical incident waves on the conductive scatterer D. Then we have*

$$\overline{U_a(\boldsymbol{b})} + U_b(\mathbf{a}) + \frac{1}{2\pi}\int_{S^2} \overline{u_a^\infty(\hat{\mathbf{r}})} u_b^\infty(\hat{\mathbf{r}}) ds(\hat{\mathbf{r}}) = \mathcal{B}_{a,b}, \tag{35}$$

where

$$\mathcal{B}_{a,b} = -\frac{k^3}{2\pi} Im(\lambda\eta^2)\int_D \overline{u_a(\mathbf{r})} u_b(\mathbf{r}) dv(\mathbf{r}) + \frac{k}{2\pi} Re(\lambda)\int_D \nabla\overline{u_a(\mathbf{r})}\cdot\nabla u_b(\mathbf{r}) dv(\mathbf{r})$$

$$-\frac{k}{2\pi}\int_S Im(\mu(\mathbf{r}))\overline{u_a(\mathbf{r})} u_b(\mathbf{r}) ds(\mathbf{r}). \tag{36}$$

Proof. As in the plane case, in view of (34) and the bilinearity of the form (10) we take

$$\{\overline{u_a^t}, u_b^t\}_S = \{\overline{u_a^i}, u_b^i\}_S + \{\overline{u_a^i}, u_b^s\}_S + \{\overline{u_a^s}, u_b^i\}_S + \{\overline{u_a^s}, u_b^s\}_S. \tag{37}$$

The three integrals $\{\overline{u_a^t}, u_b^t\}_S$, $\{\overline{u_a^i}, u_b^i\}_S$, $\{\overline{u_a^s}, u_b^s\}_S$ are evaluated as in Theorem 1 provided the fact that the point sources are located outside the area of calculation. In particular we have

$$\{\overline{u_a^t}, u_b^t\}_S = -2ik^2 Im(\lambda\eta^2) \int_D \overline{u_a(\mathbf{r})} u_b(\mathbf{r}) dv(\mathbf{r}) + 2iIm(\lambda) \int_D \nabla\overline{u_a(\mathbf{r})} \cdot \nabla u_b(\mathbf{r}) dv(\mathbf{r})$$

$$-2i \int_S Im(\mu(\mathbf{r}))\overline{u_a(\mathbf{r})} u_b(\mathbf{r}) ds(\mathbf{r}). \tag{38}$$

As in the plane case, we have

$$\{\overline{u_a^i}, u_b^i\}_S = 0, \tag{39}$$

$$\{\overline{u_a^s}, u_b^s\}_S = \frac{2i}{k} \int_{S^2} \overline{u_a^\infty(\hat{\mathbf{r}})} u_b^\infty(\hat{\mathbf{r}}) ds(\hat{\mathbf{r}}). \tag{40}$$

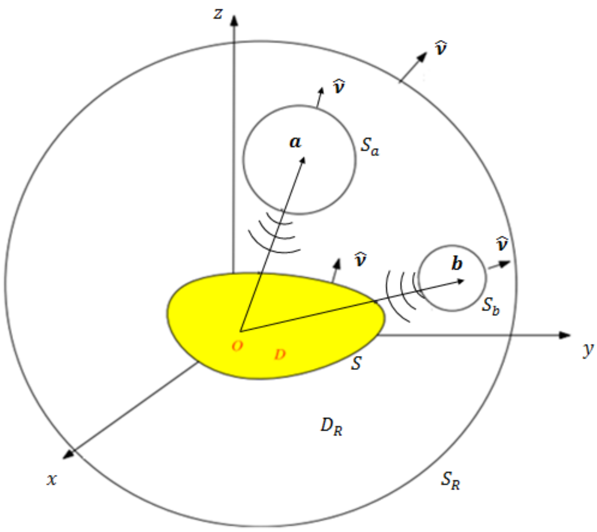

Figure 1: The scatterer D and the point sources with vector positions **a** and **b**.

For the two other integrals of (37) we consider two small spheres of surfaces S_a and S_b with **a** and **b** their centers respectively and a large sphere S_R surrounding S_a, S_b and the scatterer. We apply the second Green's theorem in the region between the surfaces S_a, S_b, S_R and S, where the spherical incident waves are regular solutions of the Helmholtz equation and we take

$$\{\overline{u_a^i}, u_b^s\}_S = -4\pi a e^{ika} u_b^s(\mathbf{a}) + 2ae^{ika}\int_{S^2} u_b^\infty(\hat{\mathbf{r}}) e^{ik\hat{\mathbf{r}}\cdot\mathbf{a}} ds(\hat{\mathbf{r}}) = -\frac{4\pi}{ik} U_b(\mathbf{a}), \tag{41}$$

$$\{\overline{u_a^s}, u_b^i\}_S = 4\pi b e^{-ikb}\overline{u_a^s(\mathbf{b})} - 2be^{ikb}\int_{S^2} \overline{u_a^\infty(\hat{\mathbf{r}})} e^{-ik\hat{\mathbf{r}}\cdot\mathbf{b}} ds(\hat{\mathbf{r}}) = -\frac{4\pi}{ik}\overline{U_a(\mathbf{b})}. \tag{42}$$

Substituting (38)-(42) to (36) the theorem is proved.

□

The scattering and the absorption cross-sections σ_a^s and σ_a^α respectively are defined as in the plane case and they are given by

$$\sigma_a^s = \frac{1}{k^2}\int_{S^2} |u_a^\infty(\hat{\mathbf{r}})|^2 ds(\hat{\mathbf{r}}), \tag{43}$$

$$\sigma_a^\alpha = \frac{1}{k}\int_S u_a^t(\mathbf{r}) \overline{\frac{\partial u_a^t(\mathbf{r})}{\partial \nu}} ds(\mathbf{r}). \tag{44}$$

The extinction cross-section σ_a^e is given by

$$\sigma_a^e = \sigma_a^s + \sigma_a^\alpha. \tag{45}$$

The optical theorem for spherical acoustic waves can be formulated in the same form as the plane waves where the far-field pattern has been replaced by its generator.

Theorem 5.2. *Theorem Let $u_a^i(\mathbf{r})$ be a spherical wave incident on the conductive scatterer D and σ_a^e be the corresponding extinction cross-section, then we have*

$$\sigma_a^e = -\frac{4\pi}{k^2} Re[U_a(\mathbf{a})]. \tag{46}$$

Proof. If we put $\mathbf{a} = \mathbf{b}$ in (35) we obtain

$$2Re[U_a(\mathbf{a})] = -\frac{1}{2\pi}\int_{S^2} |u_a^\infty(\hat{\mathbf{r}})|^2 ds(\hat{\mathbf{r}}) + \mathcal{B}_{a,a}. \tag{47}$$

Taking into account the conductive transmission conditions the absorption cross-section from (44) is given by

$$\sigma_a^\alpha = -\frac{2\pi}{k^2}\mathcal{B}_{a,a}. \tag{48}$$

From (47) and (48) we obtain (46). □

6 Discussion and Summary

The absorption cross-section provides a measure of total energy taken from the incident wave. We remark that in the case of the conductive scatterer both the plane and spherical absorption cross-sections are expressed as a sum of three integrals (13), (26) for plane and (36), (48) for spherical waves. The first two integrals give the energy that is absorbed by the lossy medium occupying D and the third integral gives the energy absorbed by the surface of the scatterer and it depends on the conductive parameter μ. From (24) we see that for plane incident waves the extinction cross-section $\sigma^e(\hat{\mathbf{p}})$ is related to the far-field pattern in the forward direction $\hat{\mathbf{p}}$. From (46) and (36) the extinction cross-section due to a point source at $\mathbf{a}$ is related to the scattered field at $\mathbf{a}$ and a Herglotz wave function with kernel u^∞. The formula (9) expresses the far-field pattern for both plane and spherical waves for the conductive scatterer.

Taking into account the formula (9) and the optical theorem we conclude:

- If $\lambda = 1$ and $\mu \neq 0$, then the far-field pattern is depended on the surface of the scatterer only.
- If $\lambda \neq 1$ and $\mu = 0$, then we have a sample transmission problem. Especially if $\lambda = \frac{\rho_0}{\rho}(1 - i\omega\delta\gamma)$ then we have the classical acoustic penetrable problem [8].
- If $\lambda = 1$ and $\mu = 0$ then $u^\infty = 0$. It is known that [6] if the far-field pattern is zero then the scattered field is identically zero and therefore no scattering occurs.
- If $Im(\lambda) = Im(\mu(\mathbf{r})) = 0$ and the material in D is lossless then the absorption cross-section is zero.

In the study of inverse scattering problem we use mixed scattering relations [2] which connect plane and spherical waves. If we consider that $b \to \infty$ in Theorem 2 we take the mixed general scattering theorem for the conductive problem

$$\overline{u_a^\infty(-\hat{\mathbf{b}})} + U(\mathbf{a}; -\hat{\mathbf{b}}) + \frac{1}{2\pi}\int_{S^2} \overline{u_a^\infty(\hat{\mathbf{r}})} u^\infty(\hat{\mathbf{r}}; -\hat{\mathbf{b}}) ds(\hat{\mathbf{r}}) = \mathcal{B}_a(-\hat{\mathbf{b}}), \tag{49}$$

where

$$\mathcal{B}_a(-\hat{\mathbf{b}}) = \lim_{b\to\infty} \mathcal{B}_{a,b}. \tag{50}$$

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