Numerical solutions of integro-differential equation with purely integral condition by using Laplace transform method

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Abstract

The aims of this paper are to prove the existence, uniqueness and continuous dependence upon the data of solution of following intogro-differential hyperbolique equation with purely integral conditions.

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a\dfrac{\partial^2 u}{\partial x^2} - c\dfrac{\partial u}{\partial x} + \alpha u(x,t) = \displaystyle\int_0^t \vartheta(t-s)u(x,s)\,ds, \quad 0 < x < \ell, \quad 0 < t \leq T \\ u(x,0) = \phi(x), \quad 0 < x < \ell \\ \dfrac{\partial u}{\partial t}(x,0) = \chi(x), \quad 0 < x < \ell, \\ \displaystyle\int_0^\ell u(x,t)\,dx = E(t), \quad 0 < t \leq T, \\ \displaystyle\int_0^\ell x u(x,t)\,dx = G(t), \quad 0 < t \leq T, \end{cases}$$

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The proofs are based on a priori estimates and Laplace transform method. We present a numerical approximate solution to integro-differential equation with integral conditions. A Laplace transform method is described for the solution of considered equation. Following Laplace transform of the original problem, an appropriate method of solving differential equations is used to solve the resultat time-independent modified equation and solution is inverted numerically back into the time domain. Numerical results are provided to show the accuracy of the proposed method.

1 Introduction

In this paper, we deal with a class of hyperbolic integro-differential equation with purely non-local conditions. The precise statement of the problem is a follows : let $\ell > 0,\ T > 0$, and $\Omega = \{(x,t) \in \mathbb{R}^2 \ : \ 0 < x < \ell,\ 0 < t < T\}$. We shall determine a solution u, in Ω of the differential equation

$$\frac{\partial^2 u}{\partial t^2} - a\frac{\partial^2 u}{\partial x^2} - c\frac{\partial u}{\partial x} + \alpha u(x,t) = \int_0^t \vartheta(t-s)u(x,s)\,ds, \quad 0 < x < \ell, \quad 0 < t \leq, T \tag{1.1}$$

satisfying the initial conditions

$$u(x,0) = \varphi(x), \quad 0 < x < \ell, \tag{1.2}$$

$$\frac{\partial u}{\partial t}(x,0) = \chi(x), \quad 0 < x < \ell, \tag{1.3}$$

and the integral conditions

$$\int_0^\ell u(x,t)\,dx = E(t), \quad 0 < t \leq T, \tag{1.4}$$

$$\int_0^\ell xu(x,t)\,dx = G(t), \quad 0 < t \leq T, \tag{1.5}$$

where $\vartheta,\ \xi,\ \varphi, E,\ G$ are known functions, $\ell,\ T$ and a are pasitive constants, and α, c are the reals.

Assumption 1.1
For all $x,t) \in \overline{\Omega}$, *we assume that*

$$\vartheta_0 \leq \vartheta(x,t) \leq \vartheta_1,\ \vartheta_0 > 0,\ \vartheta_1. > 0$$

$$\varphi \textit{is continuously derivate on}(0,\ \ell).$$

The notion of nonlocal condition has been introduced to extend the study of the classical initial value problems and it is more precise for describing nature phenomena than the classical condition since more information is taken into account, thereby decreasing the negative effects incurred by a possibly erroneous single measurement taken at the initial value. The inportance of nonlocal conditions in many applications is discussed in [10], [16] .

Mathematical modelling by evolution problems with a nonlocal constraint of the form $\frac{1}{1-\ell}\int_{\ell}^{1} u(x,t)\,dx =$ is encountered in heat transmission theory, thermoelasticity, chemical engineering, underground water flow, and plasma physic.

Many methods were used to investigate the existence and uniqueness of the solution of mixed problems which combine classical and integral conditions. J. R. CANNON [5] used the potentiel method, combining a Dirichlet and an intégral condition for a parabolic equation. L. A. MOURAVEY and V. PHILINOVOSKI [9] used the maximum principle, combining a Neumann and an integral condition for heat equation. M.Z DJIBIBE and K. TCHARIE [13], IONKIN [6] and L. BOUGOFFA[4] used the Fourier method for same purpose.
Recently, mixed problems with integral conditions for generalization of equation (1.1) have been treated using the energy-integral method. See M. Z DJIBIBE and K. TCHARIE [11] , M.Z DJIBIBE and K. TCHARIE [12], M. Z. DJIBIBE el al. [14],[15], N. I. YURCHUK[16],[17], M. MESLOUB, A. BOUZIANI and N. KECHKAR[8]. Differently to these works, in the present paper we combine a priori estimate and Fourier's method to prove existence and uniqueness solution of the problem (1.1)- (1.5).
The results obtained in this paper generalize the results of [1], and constitute a new contribution to this emerging field of research

It can be a part in the contribution of the development of a priori estimates an Laplace methods for solving such problems.

The questions related to these problems are so miscellaneous that the elaboration of a general theory is still premature. Therefore, the investigationof these problems requires at every time a separate study.

The remainder of the paper is organized as follows. After this introduction, in section 2, we present some preliminaries and basic lemmas. Then in Section 3, we establish a priori estimate. Finally, in section 4, we prove existence solution.

2 Prelimimaries

We transform the problem (1.1)-(1.5) with nonhomogeneous boundary conditions (1.4) and (1.5) into a problem with homogeneous boundary conditions. For this, we introduce a new unknow function defined by $u(x,y) = v(x,t) + w(x,t)$, where

$$w(x,t) = \frac{2\ell - 4x}{\ell^4}E(t) + \frac{4x - 2\ell}{\ell^4}G(t) = \frac{4x - 2\ell}{\ell^4}(G(t) - E(t)).$$

Then, problem becomes :

$$\frac{\partial^2 v}{\partial t^2} - a\frac{\partial^2 v}{\partial x^2} - c\frac{\partial v}{\partial x} + \alpha v(x,t) = f(x,t), \quad 0 < x < \ell, \quad 0 < t \leq T, \tag{2.1}$$

$$v(x,0) = \varphi(x), \quad 0 < x < \ell \tag{2.2}$$

$$\frac{\partial v}{\partial t}(x,0) = \psi(x), \quad 0 < x < \ell, \tag{2.3}$$

$$\int_0^\ell v(x,t)\,dx = 0, \quad 0 < t \leq T, \tag{2.4}$$

$$\int_0^\ell xv(x,t)\,dx = 0, \quad 0 < t \leq T, \tag{2.5}$$

where

$$\varphi(x) = \phi(x) - \frac{4x - 2\ell}{\ell^4}(G(0) - E(0))$$

$$\psi(x) = \chi(x) - \frac{4x - 2\ell}{\ell^4}(G'(0) - E'(0))$$

$$f(x,t) = \int_0^t \vartheta(t-s)u(x,s)\,ds - \frac{2(2x-\ell)}{\ell^4}(G''(t) - E''(t)) - \frac{2(2\alpha x - \alpha\ell - 2c}{\ell^4}(G(t) - E(t))$$

Instead of searching for the function u, we search for the function v. So the solution of problem (1.1), (1.2), (1.3), (1.4) and (1.5) will be given by $u(x,t) = v(x,t) + w(x,t)$.

Lemma 2.1 (Gronwall)
Let $x(t) \geq 0, \quad h(t), \quad y(t)$ *the integrables fonctions on* $[a;b]$. *If*

$$y(t) \leq h(t) + \int_a^t x(\tau)y(\tau)\,d\tau, \quad \forall t \in [a;b]$$

then

$$y(t) \leq h(t) + \int_a^t h(\tau)x(\tau)\exp(\int_\tau^t x(s)\,ds)\,d\tau, \quad \forall t \in [a;b].$$

In particular, if $x(t) \equiv c$ *is invariable fonction and* $h(t)$ *is an increasing fonction, then*

$$y(t) \leq h(t)e^{c(t-a)}, \forall t \in [a;b].$$

3 Uniqueness and Continuous Dependence of the solution

Theorem 3.1
If $v(x,t)$ *is a solution of problem* (1.1), (1.2), (1.3), (1.4) *and* (1.5), *then we have a priori estimates*

$$\sup_{0\le t\le T}\int_0^\ell \left(J_x^2\left(\frac{\partial u(x,\tau)}{\partial t}\right)+J_x^2u\,dx+u^2\right)\,dx \le A\left(\int_0^\ell \varphi^2(x)\,dx+\int_0^\ell J_x^2\psi(x)\,dx+\int_0^\ell J_x^2\varphi(x)\,dx\right) \tag{3.1}$$

Proof
Appying J_x to (1.1), Multiplying with $J_x\dfrac{\partial u}{\partial t}$ and integrating the results obtained over $\Omega_\tau=(0,\ell)\times(0,\tau)$. Observe that

$$\int_0^\ell\int_0^\tau J_x\left(\frac{\partial^2 v}{\partial t^2}\right)J_x\left(\frac{\partial v}{\partial t}\right)\,dt\,dx-a\int_0^\ell\int_0^\tau J_x\left(\frac{\partial^2 v}{\partial x^2}\right)J_x\left(\frac{\partial v}{\partial t}\right)\,dt\,dx$$

$$-c\int_0^\ell\int_0^\tau J_x\left(\frac{\partial v}{\partial x}\right)J_x\left(\frac{\partial v}{\partial t}\right)\,dt\,dx+\alpha\int_0^\ell\int_0^\tau J_xv(x,t)J_x\left(\frac{\partial v}{\partial t}\right)\,dt\,dx$$
$$=\int_0^\ell\int_0^\tau J_x\left(\int_0^t \vartheta(t-s)u(x,s)\,ds\right)J_x\left(\frac{\partial v}{\partial t}\right)\,dt\,dx, \tag{3.2}$$

where $J_x(u(x,t)=\int_0^x u(\xi,t)\,d\xi$. Successive intrgration by parts of integrals on the left-hand of (3.2) are straight-foward but somewhat tedious. We give only their results.

$$\frac{1}{2}\int_0^\ell J_x^2\left(\frac{\partial u(x,\tau)}{\partial t}\right)\,dx-\frac{1}{2}\int_0^\ell J_x^2\psi(x)\,dx+\frac{\alpha}{2}\int_0^\ell J_x^2u\,dx-\frac{\alpha}{2}\int_0^\ell J_x^2\varphi(x)\,dx+\frac{a}{2}\int_0^\ell u^2\,dx$$
$$-\frac{a}{2}\int_0^\ell \varphi^2(x)\,dx=\int_0^\ell\int_0^\tau J_x\left(\int_0^t \vartheta(t-s)u(x,s)\,ds\right)J_x\left(\frac{\partial v}{\partial t}\right)\,dt\,dx+a\int_{\Omega_\tau}u(\ell,t)\frac{\partial u}{\partial t}\,dxdt$$
$$+c\int_{\Omega_\tau}uJ_x\left(\frac{\partial u}{\partial t}\right)\,dxdt-\int_{\Omega_\tau}\left(a\frac{\partial u}{\partial x}(0,t)+u(0,t)\right)J_x\left(\frac{\partial u}{\partial t}\right)\,dxdt. \tag{3.3}$$

By the Cauchy inequality,

$$\int_0^\ell\int_0^\tau J_x\left(\int_0^t \vartheta(t-s)u(x,s)\,ds\right)J_x\left(\frac{\partial v}{\partial t}\right)\,dt\,dx\le\frac{T\vartheta_0}{2}\int_{\Omega_T}J_x^2u\,dx\,dt$$
$$+\frac{\vartheta_0 T}{2}\int_0^T\int_0^\ell J_x^2\left(\frac{\partial u}{\partial t}\right)\,dx\,d\tau, \tag{3.4}$$

$$\int_{\Omega_\tau}u(\ell,t)\frac{\partial u}{\partial t}\,dxdt\le\frac{1}{2}\int_{\Omega_T}u^2\,dxdt+\frac{1}{2}\int_{\Omega_T}J_x^2\left(\frac{\partial u}{\partial t}\right)\,dxdt, \tag{3.5}$$

$$\int_{\Omega_\tau}uJ_x\left(\frac{\partial u}{\partial t}\right)\,dxdt\le\frac{1}{2}\int_{\Omega_T}u^2\,dxdt+\frac{1}{2}\int_{\Omega_T}J_x^2\left(\frac{\partial u}{\partial t}\right)\,dxdt. \tag{3.6}$$

Substuting (3.6) into (3.5), yields

$$\frac{1}{2}\int_0^\ell J_x^2\left(\frac{\partial u(x,\tau)}{\partial t}\right)dx+\frac{\alpha}{2}\int_0^\ell J_x^2 u\,dx+\frac{a}{2}\int_0^\ell u^2\,dx=\frac{a}{2}\int_0^\ell \varphi^2(x)\,dx+\frac{1}{2}\int_0^\ell J_x^2\psi(x)\,dx$$
$$+\frac{\alpha}{2}\int_0^\ell J_x^2\varphi(x)\,dx+\frac{\vartheta_0 T+|c|+1}{2}\int_{\Omega_T} J_x^2\left(\frac{\partial u}{\partial t}\right)dx\,d\tau+\frac{T\vartheta_0}{2}\int_{\Omega_T} J_x^2 u\,dxdt$$
$$+\frac{|a|+T\vartheta_0+|c|}{2}\int_{\Omega_T} u^2\,dxdt. \tag{3.7}$$

The right-hand side of (3.7) is independent of τ, hence, replacing the left-hand side by the upper bound with respect to τ and by Gronwall Lemma, we get

$$\sup_{0\le\tau\le T}\int_0^\ell\left(J_x^2\left(\frac{\partial u(x,\tau)}{\partial t}\right)+J_x^2u\,dx+u^2\right)dx\le e^{\lambda T}\left(\int_0^\ell\varphi^2(x)\,dx+\int_0^\ell J_x^2\psi(x)\,dx+\int_0^\ell J_x^2\varphi(x)\,dx\right), \tag{3.8}$$

where

$$\lambda=\frac{\max\left(\frac{1}{2},\frac{\alpha}{2},\frac{T\vartheta_0+|c|+1}{2},\frac{T\vartheta_0+|a|+|c|}{2}\right)}{\min\left(\frac{1}{2},\frac{\alpha}{2},\frac{a}{2}\right)}.$$

From (3.8), we obtain the priori estimates (3.1). This complete the proof of Theoreme (3.1), with $A=e^{\lambda T}$

Consequence 3.1
If problem (1.1), (1.2), (1.3), (1.4) *and* (1.5) *has a solution, then this solution is unique and depends continuously on* $(\varphi,\ \psi)$.

4 Existence of the Solution

Laplace transform is widely used in the area of engineering technology and mathematical science. There are many problems whose solution may be found in terms of the Laplace. In fact, it is an efficient method for solving many diffential equations and partial differential equation. The mains difficult of the method of Laplace domain into the real domain.
Hence in this section, we apply the technique of the Laplace transform to find solutions of the problem (1.1)-(1.5).

Suppose that $v(x,t)$ is défined and is of the exponential order for $t\ge 0$, there exists $\lambda>0,\ \beta>0$ and $t_0>0$ such that $|v(x,t)|\le\lambda e^{\beta t}$ for $t\ge t_0$. Then the Laplace transform $V(x,s)$ includind the function $v(x,t)$ is defined by

$$V(x,s)=\int_0^{+\infty}v(x,t)e^{-st}\,dt, \tag{4.1}$$

where s is know as a Laplace variable and V is a function in the Laplace domain.

Applying the Laplace transform on both sides of (3.4), we obtain

$$\int_0^{+\infty} \frac{\partial^2 v}{\partial t^2} e^{-st}\, dt - a \int_0^{+\infty} \frac{\partial^2 v}{\partial x^2} e^{-st}\, dt - c \int_0^{+\infty} \frac{\partial v}{\partial x} e^{-st}\, dt + \\ +\alpha \int_0^{+\infty} v(x,t) e^{-st}\, dt = \int_0^{+\infty} \int_0^{t} \vartheta(t-s) v(x,s)\, ds\, dt. \tag{4.2}$$

The standard integration by parts of the terms on the left-hand of (4.2), leads

$$\int_0^{+\infty} \frac{\partial^2 v}{\partial t^2} e^{-st}\, dt = -\psi(x) - s\varphi(x) + s^2 V(x,s), \tag{4.3}$$

$$\int_0^{+\infty} \frac{\partial^2 v}{\partial x^2} e^{-st}\, dt = \frac{d^2 V(x,s)}{dx^2}, \tag{4.4}$$

$$\int_0^{+\infty} \frac{\partial v}{\partial x} e^{-st}\, dt = \frac{dV(x,s)}{dx}. \tag{4.5}$$

Substuting (4.3), (4.4) and (4.5) into (4.2), we get

$$-a \frac{d^2 V(x,s)}{dx^2} - c \frac{dV(x,s)}{dx} + (\alpha + s^2) V(x,s) = f(x,s) + \psi(x) + s\varphi(x), \tag{4.6}$$

where $f(x,s) = \int_0^{+\infty} \int_0^{t} \vartheta(t-s) v(x,s)\, ds\, dt.$
Similarly, we have

$$\int_0^{\ell} V(x,s)\, dx = 0, \tag{4.7}$$

$$\int_0^{\ell} x V(x,s)\, dx = 0. \tag{4.8}$$

Now, we distinguish the following cas :

- Case 1 : If $c^2 + 4a\alpha + 4as^2 = 0.$
- Case 2 : If $c^2 + 4a\alpha + 4as^2 > 0.$
- Case 3 : If $c^2 + 4a\alpha + 4as^2 < 0.$

In this article, we only deal Case 1 and Case 2

For Case 1, that is $c^2 + 4a\alpha + 4as^2 = 0$, the solution general of (4.8) is given by

$$V(x,s) = (C_1(s)x + C_2(s)) e^{-\frac{c}{2a}x} - \frac{1}{a} \int_0^{x} (x-\tau)(f(\tau,s) + \psi(\tau) + s\varphi(\tau)) e^{\frac{c}{2a}\tau}\, d\tau. \tag{4.9}$$

Putting the intagral conditions (4.7) and (4.8) in (4.9), we get

$$\left(2-\left(1+\frac{c\ell}{2a}e^{\frac{-c\ell}{2a}}\right)\right)C_1(s)+\frac{c}{2a}\left(1-e^{-\frac{c}{2a}}\right)C_2(s)$$
$$=2\int_0^\ell\int_0^x(x-y)(f(\tau,s)+\psi(\tau)+s\varphi(\tau))e^{\frac{c}{2a}\tau}\,d\tau\,dx,$$

$$4a^2+(c^2\ell^2-4ac\ell-4a^1))C_1(s)+(2a-(2a+\ell c))C_2(s)$$
$$=4a^2\int_0^\ell\int_0^x x(x-\tau)(f(\tau,s)+\psi(\tau)+s\varphi(\tau))e^{\frac{c}{2a}\tau}\,d\tau\,dx$$

For Case 1, that is $c^2+4a\alpha+4as^2>0$, Using the method of variation of parameter, to solve (4.8), we have the general solution as

$$V(x,s)=C_1(s)exp\left(-\frac{c-\sqrt{c^2+4a(\alpha+s^2)}}{2a}x\right)+C_2(s)exp\left(-\frac{c+\sqrt{c^2+4a(\alpha+s^2)}}{2a}x\right)$$
$$-\frac{2}{\sqrt{c^2+4a(\alpha+s^2)}}\int_0^x(f(\tau,s)+\psi(\tau)+s\varphi(\tau)]\,e^{-\frac{c(x-\tau)}{2a}}\sinh\left(\frac{\sqrt{c^2+4a(\alpha+s^2)}}{2a}(x-\tau)\right)d\tau, \tag{4.10}$$

where C_1 et C_2 are arbitrary functions of s.

Substuting (4.10) into (4.7) and (4.8), we have

$$C_1(s)\int_0^\ell exp\left(-\frac{c-\sqrt{c^2+4a(\alpha+s^2)}}{2a}x\right)dx+C_2(s)\int_0^\ell exp\left(-\frac{c+\sqrt{c^2+4a(\alpha+s^2)}}{2a}x\right)dx$$
$$=\frac{2}{\sqrt{c^2+4a(\alpha+s^2)}}\int_0^\ell\int_0^x(f(\tau,s)+\psi(\tau)+s\varphi(\tau)]\,e^{-\frac{c(x-\tau)}{2a}}\sinh\left(\frac{\sqrt{c^2+4a(\alpha+s^2)}}{2a}(x-\tau)\right)d\tau\,dx, \tag{4.11}$$

$$C_1(s)\int_0^\ell xexp\left(-\frac{c-\sqrt{c^2+4a(\alpha+s^2)}}{2a}x\right)dx+C_2(s)\int_0^\ell xexp\left(-\frac{c+\sqrt{c^2+4a(\alpha+s^2)}}{2a}x\right)dx$$
$$=\frac{2}{\sqrt{c^2+4a(\alpha+s^2)}}\int_0^\ell\int_0^x x\,(f(\tau,s)+\psi(\tau)+s\varphi(\tau)]\,e^{-\frac{c(x-\tau)}{2a}}\sinh\left(\frac{\sqrt{c^2+4a(\alpha+s^2)}}{2a}(x-\tau)\right)d\tau\,dx, \tag{4.12}$$

where

$$\begin{pmatrix}C_1(s)\\C_2(s)\end{pmatrix}=\begin{pmatrix}a_{11}(s)&a_{12}(s)\\a_{21}(s)&a_{22}(s)\end{pmatrix}^{-1}\times\begin{pmatrix}b_1(s)\\b_2(s)\end{pmatrix}, \tag{4.13}$$

and

$$
\begin{cases}
a_{11}(s) &= \int_0^{\ell} \exp\left(-\frac{c-\sqrt{c^2+4a(\alpha+s^2)}}{2a}x\right)\,dx \\[2ex]
a_{12}(s) &= \int_0^{\ell} \exp\left(-\frac{c+\sqrt{c^2+4a(\alpha+s^2)}}{2a}x\right)\,dx \\[2ex]
a_{21}(s) &= \int_0^{\ell} x\exp\left(-\frac{c-\sqrt{c^2+4a(\alpha+s^2)}}{2a}x\right)\,dx \\[2ex]
a_{22}(s) &= \int_0^{\ell} x\exp\left(-\frac{c+\sqrt{c^2+4a(\alpha+s^2)}}{2a}x\right)\,dx \\[2ex]
b_1(s) = - & \frac{a}{\sqrt{c^2+4a(\alpha+s^2)}}\int_0^{\ell}\int_0^{x} (f(\tau,s)+\psi(\tau)+s\varphi(\tau)]\, e^{-\frac{c(x-\tau)}{2a}} \\
& \times \sinh\left(\frac{\sqrt{c^2+4a(\alpha+s^2)}}{2a}(x-\tau)\right)\,d\tau\,dx, \\[2ex]
b_2(s) &= -\frac{a}{\sqrt{c^2+4a(\alpha+s^2)}}\int_0^{\ell}\int_0^{x} x\,(f(\tau,s)+\psi(\tau)+s\varphi(\tau)]\, e^{-\frac{c(x-\tau)}{2a}} \\
& \times \sinh\left(\frac{\sqrt{c^2+4a(\alpha+s^2)}}{2a}(x-\tau)\right)\,d\tau\,dx.
\end{cases}
\tag{4.14}
$$

If it is not possible to calculate the integral directly, then we calculate tem numerically. If the Laplace inversion is possibly computed directly for (4.10) and (4.14), we obtain our solution explicitly. Otherwise, we use the suitable approximate method, then we use the numerical inversion of the Laplace transform. We have

$$
\begin{aligned}
a_{11}(s) &= \int_0^{\ell} exp\left(-\frac{c-\sqrt{c^2+4a(\alpha+s^2)}}{2a}x\right)\,dx \\
&= \frac{1}{2}\sum_{i=1}^{n} k_i exp\left(-\frac{c-\sqrt{c^2+4a(\alpha+s^2)}}{2a}(x_i+1)\right) \\
\\
a_{12}(s) &= \int_0^{\ell} exp\left(-\frac{c+\sqrt{c^2+4a(\alpha+s^2)}}{2a}x\right)\,dx \\
&= \frac{1}{2}\sum_{i=1}^{n} k_i exp\left(-\frac{c+\sqrt{c^2+4a(\alpha+s^2)}}{2a}(x_i+1)\right) \\
\\
a_{21}(s) &= \int_0^{\ell} x exp\left(-\frac{c-\sqrt{c^2+4a(\alpha+s^2)}}{2a}x\right)\,dx \\
&= \frac{1}{4}\sum_{i=1}^{n} k_i(x_i+1) exp\left(-\frac{c-\sqrt{c^2+4a(\alpha+s^2)}}{2a}(x_i+1)\right) \\
\\
a_{22}(s) &= \int_0^{\ell} x exp\left(-\frac{c+\sqrt{c^2+4a(\alpha+s^2)}}{2a}x\right)\,dx \\
&= \frac{1}{4}\sum_{i=1}^{n} k_i(x_i+1) exp\left(-\frac{c+\sqrt{c^2+4a(\alpha+s^2)}}{2a}(x_i+1)\right) \\
\\
b_1(s) &= -\frac{a}{\sqrt{c^2+4a(\alpha+s^2)}}\int_0^{\ell}\int_0^{x} \left(f(\tau,s)+\psi(\tau)+s\varphi(\tau)\right] e^{-\frac{c(x-\tau)}{2a}} \\
&\quad \times \sinh\left(\frac{\sqrt{c^2+4a(\alpha+s^2)}}{2a}(x-\tau)\right)\,d\tau\,dx, \\
&= -\frac{a}{4\sqrt{c^2+4a(\alpha+s^2)}}\sum_{i=1}^{n} k_i\left[f\left(\frac{1}{2}(x_i+1),s\right)+\psi\left(\frac{1}{2}(x_i+1\right)+s\varphi\left(\frac{1}{2}(x_i+1\right)\right] \\
&\quad \times\left(1-(\frac{1}{2}(x_i+1)\right) exp\left(-\frac{c}{2a}\left[\left(1-\frac{1}{2}(x_i+1)\right)x_j+\left(1+\frac{1}{2}(x_i+1)\right)\right]\right)\times \\
&\quad \sum_{j=1}^{n}\sinh\left(\frac{\sqrt{c^2+4a(\alpha+s^2)}}{2a}\left[\frac{1}{2}\left[\left(1-\frac{1}{2}(x_i+1)\right)x_j+\left(1+\frac{1}{2}(x_i+1)\right)\right]-\frac{1}{2}(x_i+1)\right]\right) \\
\\
b_2(s) &= -\frac{a}{\sqrt{c^2+4a(\alpha+s^2)}}\int_0^{\ell}\int_0^{x} x\left(f(\tau,s)+\psi(\tau)+s\varphi(\tau)\right] e^{-\frac{c(x-\tau)}{2a}} \\
&\quad \times \sinh\left(\frac{\sqrt{c^2+4a(\alpha+s^2)}}{2a}(x-\tau)\right)\,d\tau\,dx
\end{aligned}
\tag{4.15}
$$

$$b_2(s) = -\frac{a}{4\sqrt{c^2+4a(\alpha+s^2)}}\sum_{i=1}^{n} k_i\left[f\left(\frac{1}{2}(x_i+1),s\right)+\psi\left(\frac{1}{2}(x_i+1\right)+s\varphi\left(\frac{1}{2}(x_i+1\right)\right]$$
$$\times\left(1-(\frac{1}{2}(x_i+1)\right)\left(\frac{1}{2}\left[\left(1-\frac{1}{2}(x_i+1)\right)x_j+\left(1+\frac{1}{2}(x_i+1)\right)\right]\right)$$
$$\times exp\left(-\frac{c}{2a}\left[\left(1-\frac{1}{2}(x_i+1)\right)x_j+\left(1+\frac{1}{2}(x_i+1)\right)\right]\right)\times$$
$$\sum_{j=1}^{n}\sinh\left(\frac{\sqrt{c^2+4a(\alpha+s^2)}}{2a}\left[\frac{1}{2}\left[\left(1-\frac{1}{2}(x_i+1)\right)x_j+\left(1+\frac{1}{2}(x_i+1)\right)\right]-\frac{1}{2}(x_i+1)\right]\right) \tag{4.16}$$

where x_i and k_i are the abscissa and weights, defined as

$$x_i \ : \ i^{th} \quad \text{zero of} \quad P_n(x), \quad k_i = \frac{2}{(1-x_i^2)(P'(x))^2}$$

Thier tabulated values can be found in [7] for different values of N.

Using Stehfest's algorithm, the time domain solution is approximated as

$$u(x,t) \simeq \frac{\ln 2}{t}\sum_{k=1}^{n}\sum_{i=\left[\frac{k+1}{2}\right]}^{\min(k,n)}\frac{i^n(2i)!(-1)^{k+n}}{(n-i)!i!(i-1)!(k-i)!(2i-k)!} \tag{4.17}$$

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