**Control chart limits for monitoring process mean based on Downton’s estimator**

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**Abstract**

This paper proposes control limits for monitoring process mean using the Downton estimator and the necessary tables of factors for computing the control limits for sample size $n\leq 10$. The derived control limits for process mean were compared with control limits based on range statistic as a measure of controlling the variability. The average run length performance of the proposed control charts was also investigated. The obtained results showed that the Shewhart $\overbar{X}$control chart using the Downton statistic for monitoring the process mean performed better than the traditional $\overbar{X}$control chart using the range statistic especially for detection of small shift in the process mean.

**Keywords:** Average Run Length, process mean, process shift, Downton’s estimator,

**1.0 Introduction**

Control charts are statistical process control tools that are used for the control and monitoring of repetitve processes.The Shewhart $\overbar{X}$ chart is one of the most widely used SPC techniques developed to monitor and control the process average while the range (R) and standard deviation (S) are used to monitor the process variability. The charts called $\overbar{X}$ and R or $\overbar{X}$ and S chart are combined to evaluate the stability of a process. In the $\overbar{X}$ chart if the average of a subgroup exceed the control limits, it is an indication that the process averages are out of control and in R or S chart, if the standard deviation of subgroup exceed the contol limits, it is an indication that the process variability is out of control (Montgomery, 2005). These charts are based on the fundamental assumption of normality for the quality variables. However in practice, the normality assumption is often violated by real-life process data. When the normality assumption is violated, one approach is to transform the data to normality using the Box-Cox transformation method which often lead to loss of information that may affect the conclusion drawn on such data. Several authors have worked on control charts to handle non-normality in process data. Abu-Shawiesh(2008) derived the control limits for the standard deviation control chart based on the MAD from the sample median. However, the derived control limits by Abu-Shawiesh(2008) was for situation when a standard value of sigma is assumed known or specified by the management (Adekeye et al., 2012). The MAD control chart provides an alternative to the Shewhart S control chart for non-normal process. Adekeye et al.(2012) extended the work of Abu-Shawiesh(2008) deriving the control limits for process mean using MAD when quality characteristic under study was non-normal. He concluded that derived control limits is an improvement on the control limits of the Shewhart chart based on the standard as a measure of controlling process variability.Recently Abbasi and Miller (2011) proposed a control chart to monitor process dispersion of a non-normal population based on the Downton’s estimator. It is shown that the D chart is superior to R and close competition to the S chart under normality assumption but more superior to both the R and S charts in detecting process variability when assumption of normality is violated. In this paper, the concept of Abbasi and Miller(2011) is extended to develop the control limits for the corresponding $\overbar{X}$ chart for monitoring the process mean when the standard deviation is unknown and we are interested in monitoring past data (Phase I analysis) .

**2.0 Control Chart Based on Downton’s Estimator**

The control chart based on Downton’s statistics hereby called D- control chart is a dispersion control chart proposed by Abbasi and Miller (2011) for monitoring the variability of process. The D control chart is a chart of subgroup standard deviations in which the control limits are set using an unbiased estimator based on the Downton’s statistics. This estimator is simple, easy to compute and used in the design of the D control chart. In the design of the D-chart the relationship between D and $σ$ is defined by a random variable Z where $Z=^{D}/\_{σ}$ (Abbasi and Miller, 2011). Let $X\_{1}, X\_{2}, ……….,X\_{n}$ be a random sample from a normal distribution with mean, $μ$ and variance $σ^{2}$. If the observations of the sample is re-arrange in ascending order so that $X\_{\left(1\right)}\leq X\_{\left(2\right)}……..\leq X\_{\left(n\right)}$, Downton(1966) proposed the estimator D given by

$$D=\sqrt{π}\sum\_{i=1}^{n}\frac{\left(2i-n-1\right)X\_{\left(i\right)}}{n\left(n-1\right)}$$

 = $\frac{2\sqrt{π}}{n\left(n-1\right)}\sum\_{i=1}^{n}\left[i-\frac{1}{2}\left(n+1\right)\right]X\_{\left(i\right)}$ (1)

as an unbiased estimator of sigma($σ$). (Barnett, Mullen, and Saw; 1967). It should be noted that the Downton’s estimator, D, is not much affected by non-normality.

If $X\_{1}, X\_{2}, ……….,X\_{n}$ are normally distributed, then the unbiased estimator for $σ$ and $σ\_{D}$ based on Downton’s estimator is $\hat{σ}=\overbar{D}$ and $σ\_{D}=z\_{3}σ$ respectively where $\overbar{D}$ = 

is computed from an appropriate number of random samples obtained from a process during normal operating conditions.

 $z\_{3}=\frac{1}{\sqrt{n\left(n-1\right)}}\sqrt{n\left(\frac{1}{3}π+2\sqrt{3}-4\right)+\left(6-4\sqrt{3}+\frac{1}{3}π\right)}$ (2)

The 3-sigma control limits for the estimate of the standard deviation based on the Downton’s as derived by Abbasi and Miller (2011) are:

 $LCL=max\left(0,\overbar{D}-3z\_{3}\overbar{D}\right)= Z\_{3}\overbar{D}$

 $CL=\overbar{D}$ (3)

 $UCL=\overbar{D}+3z\_{3}\overbar{D}=Z\_{4}\overbar{D}$

 where,  and 

The values of $z\_{3}$, $Z\_{3}$ and $Z\_{4}$ computed by the authors from equations (2) and (3) for different values of $n\leq 10$ are presented in Table 1

 **Table 1: Control limits factors for constructing D chart**

|  |  |
| --- | --- |
|  **n** | **Factors for control chart limits** |
| **z3** | **Z3** | **Z4** |
| 2 | 0.756 | 0 | 3.268 |
| 3 | 0.674 | 0 | 3.022 |
| 4 | 0.624 | 0 | 2.872 |
| 5 | 0.598 | 0 | 2.794 |
| 6 | 0.581 | 0 | 2.743 |
| 7 | 0.570 | 0 | 2.710 |
| 8 | 0.562 | 0 | 2.686 |
| 9 | 0.256 | 0.809 | 1.191 |
| 10 | 0.241 | 0.813 | 1.187 |

For the range and standard deviation control chart, the corresponding control limits for the $\overbar{X}$ control chart had been derived for monitoring process mean (Montgomery, 2005). In this study, the concept of Abbasi and Miller(2011) is extended to develop the control limits for the corresponding $\overbar{X}$ chart control limits for monitoring the process mean based on the Downton’s estimator. Also, Downton’s statistic for sample sizes $n\leq 10$ is given in Table 2

 **Table 2 Downton’s statistics for sample sizes (**$n\leq 10$**)**

|  |  |
| --- | --- |
| **Sample size (n)** | **Downton’s statistics** |
| 2 | $$D=\frac{√π}{2}\left(X\_{\left(2\right)}-X\_{\left(1\right)}\right)$$ |
| 3 | $$D=\frac{√π}{3}\left(X\_{\left(3\right)}-X\_{\left(1\right)}\right)$$ |
| 4 | $$D=\frac{√π}{6}\left(\frac{3}{2}X\_{\left(4\right)}+\frac{1}{2}X\_{\left(3\right)}-\frac{1}{2}X\_{\left(2\right)}-\frac{3}{2}X\_{\left(1\right)}\right)$$ |
| 5 | $$D=\frac{√π}{10}\left(2X\_{\left(5\right)}+X\_{\left(4\right)}-X\_{\left(2\right)}-2X\_{\left(1\right)}\right)$$ |
| 6 | $$D=\frac{√π}{15}\left(\frac{5}{2}X\_{\left(6\right)}+\frac{3}{2}X\_{\left(5\right)}+\frac{1}{2}X\_{\left(4\right)}-\frac{1}{2}X\_{\left(3\right)}-\frac{3}{2}X\_{\left(2\right)}-\frac{5}{2}X\_{\left(1\right)}\right)$$ |
| 7 | $$D=\frac{√π}{21}\left(3X\_{\left(7\right)}+2X\_{\left(6\right)}+X\_{\left(5\right)}-X\_{\left(3\right)}-2X\_{\left(2\right)}-3X\_{\left(1\right)}\right)$$ |
| 8 | $$D=\frac{√π}{28}\left(\frac{7}{2}X\_{\left(8\right)}+\frac{5}{2}X\_{\left(7\right)}+\frac{3}{2}X\_{\left(6\right)}+\frac{1}{2}X\_{\left(5\right)}-\frac{1}{2}X\_{\left(4\right)}-\frac{3}{2}X\_{\left(2\right)}-\frac{5}{2}X\_{\left(2\right)}-\frac{7}{2}X\_{\left(1\right)}\right)$$ |
| 9 | $$D=\frac{√π}{36}\left(4X\_{\left(9\right)}+3X\_{\left(8\right)}+2X\_{\left(7\right)}+X\_{\left(6\right)}-X\_{\left(4\right)}-2X\_{\left(3\right)}-3X\_{\left(2\right)}-4X\_{\left(1\right)}\right)$$ |
| 10 | $$D=\frac{√π}{45}\left(\frac{9}{2}X\_{\left(10\right)}+\frac{7}{2}X\_{\left(9\right)}+\frac{5}{2}X\_{\left(8\right)}+\frac{3}{2}X\_{\left(7\right)}+\frac{1}{2}X\_{\left(6\right)}-\frac{1}{2}X\_{\left(5\right)}-\frac{3}{2}X\_{\left(4\right)}-\frac{5}{2}X\_{\left(3\right)}-\frac{7}{2}X\_{\left(2\right)}-\frac{9}{2}X\_{\left(1\right)}\right)$$ |

 **2.1 Derivation of The Control Limits for** $\overbar{X}$ **Chart Based on Downton’s Statistic**

The proposed control chart, that is, the $\overbar{X}\_{D}$ control chart is constructed based on the sample mean to estimate the process mean, $μ$ and the Downton estimator as an alternative for the sample standard deviation, S, to estimate the process standard deviation, $σ$. Thus, let $X\_{i1}, X\_{i2}, X\_{i3}, …………..,X\_{in}$ be a random sample of independent observations of size n taken over m subgroup, *i = 1,2,…, m.* The sample are assumed to be equal and taken from identical distributions functions. To derive the control limits for the corresponding $\overbar{X}\_{D}$ control chart, the process mean, $μ$ is estimated by the average of the *m* subgroup sample means and the process standard deviation is estimated using the average of the subgroup Downton estimator values as an estimate of variability,that is, $\hat{σ}=\overbar{D}$ , where $̿=\frac{1}{m}\sum\_{j=1}^{m}\overbar{X}\_{j}$. Then, the 3-sigma lower and upper control limits, LCL and UCL,respectively and the central line for the mean chart are constructed using the *m* subgroups consisting of *n* observations obtained from the in-control process. The derivation of the control limits using the 3-sigma approach for $\overbar{X}$ chart is

LCL = $\overbar{X}-3σ\_{\overbar{X}}$

CL = $\overbar{X}$ (4)

UCL = $\overbar{X}+3σ\_{\overbar{X}}$

 But  and , Therefore, substituting these information into Eq (4) will give

LCL = 

CL = $\overbar{X}$ (5)

UCL = 

 where 

To decide the stability of the process mean, values of  are plotted on the chart for each sample period, if value of the , j = 1, 2, .....,*m* falls outside the control limits in Equation (5), the process is considered to be out of control. The values of A which is a function of *n* are obtained from Montgomery (2005). The summary of the control limits for the charts are given in Table 3.

 **Table 3 Summary of control limits for the proposed** $\overbar{X}$ **and D and** $\overbar{X}$ **and R charts**

|  |  |
| --- | --- |
| $\overbar{X}$ **and D** | $\overbar{X}$ **and R** |
| $\overbar{X}\_{D}$chart$$LCL=\overbar{X}\_{D}-A\overbar{D}$$$$CL=\overbar{X}\_{D}$$$$UCL=\overbar{X}\_{D}+A\overbar{D}$$D chart$$LCL= Z\_{3}\overbar{D}$$$$CL=\overbar{D}$$$$UCL=Z\_{4}\overbar{D}$$ | $\overbar{X}$ chart$$LCL=\overbar{X}-A\_{2}\overbar{R}$$$$CL=\overbar{X}$$$$UCL=\overbar{X}-A\_{2}\overbar{R}$$R chart$$LCL= D\_{3}\overbar{R}$$$$CL=\overbar{R}$$$$UCL=D\_{4}\overbar{R}$$ |

**3.0 Example**

To illustrate the proposed control chart, data set taken from Montgomery(2005, pg 200) and a randomly generated data from the gamma distribution representing the normal and non-normal case are used.

**3.1 Data on Hard-Brake Process**

The first data set is taken from a hard-brake process used in conjuction with photolithography in semiconductor manufacturing from Montgomery(2005, pg 200). This data is applied to demostrate the application of the $\overbar{X}\_{D}$ chart based on the Downton’s estimator. Twenty five samples each of size 5 is taken to monitor the process that is assumed to come from a normal distribution. Therefore, m = 25 and n = 5. Table 4 show the average ($\overbar{X}$), range (R) and Downton (D) values for the 25 subgroups

 **Table 4:** $\overbar{X}$, **R and Downton values for dataset in Montgomery(2005)**

|  |  |  |  |
| --- | --- | --- | --- |
| Sample number | $$\overbar{X}$$ | D | R |
| 1 | 1.5119 | 0.1768 | 0.3679 |
| 2 | 1.4951 | 0.1204 | 0.2517 |
| 3 | 1.4817 | 0.0600 | 0.1390 |
| 4 | 1.4712 | 0.1543 | 0.3521 |
| 5 | 1.4882 | 0.1533 | 0.3706 |
| 6 | 1.4492 | 0.1280 | 0.2674 |
| 7 | 1.5805 | 0.1699 | 0.4189 |
| 8 | 1.5343 | 0.1180 | 0.2447 |
| 9 | 1.5076 | 0.1533 | 0.3589 |
| 10 | 1.5134 | 0.1047 | 0.2658 |
| 11 | 1.5242 | 0.1553 | 0.3509 |
| 12 | 1.5284 | 0.1829 | 0.4204 |
| 13 | 1.3947 | 0.1866 | 0.4470 |
| 14 | 1.5261 | 0.1025 | 0.2422 |
| 15 | 1.4083 | 0.1702 | 0.3499 |
| 16 | 1.5344 | 0.2520 | 0.6823 |
| 17 | 1.4874 | 0.1459 | 0.3589 |
| 18 | 1.4573 | 0.1420 | 0.3153 |
| 19 | 1.5777 | 0.1338 | 0.3062 |
| 20 | 1.5060 | 0.2439 | 0.5240 |
| 21 | 1.4691 | 0.0894 | 0.2185 |
| 22 | 1.5390 | 0.0873 | 0.1863 |
| 23 | 1.5592 | 0.1008 | 0.2533 |
| 24 | 1.5688 | 0.0441 | 0.1156 |
| 25 | 1.5264 | 0.1587 | 0.3224 |

Using the information in Table 3, the following were obtained: . Inserting these estimates into Equation (5), the control limits for the Downton-based and that of the range- based  control chart were computed. The control limits are presented in Table 5

 **Table 5: Control Limits for** $\overbar{X}$R  **and** $\overbar{X}\_{D}$ **charts for Normal case.**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Control charts** | **Location and Scale estimator** | **LCL** | **CL** | **UCL** |
|  $μ$ |  $\hat{σ}$ |
| $\overbar{X}$R chart |  |  R | 1.3180 | 1.5056 | 1.6933 |
| $\overbar{X}\_{D}$ chart |  |  D | 1.3158 | 1.5056 | 1.6954 |

**3.2 Generated Data From Gamma(0.5, 2)**

The data for the non-normal case was generated from Gamma(0.5, 2) distribution, consisting of 20 subgroups of size n= 5 observations each using MATLAB. The generated data, the average, range, and Downton values are presented in Table 6

**Table 6. Simulated data from Gamma(0.5, 2) distribution**

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Sample** | **X1** | **X2** | **X3** | **X4** | **X5** | $$\overbar{X}$$ | **D** | **R** |
| 1 | 2.6661 | 0.8141 | 1.3687 | 0.6665 | 1.4557 | 1.3942 | 0.8223 | 1.9996 |
| 2 | 1.5229 | 1.3077 | 0.6857 | 0.4067 | 0.0527 | .7951 | 0.6807 | 1.4702 |
| 3 | 0.2422 | 1.7740 | 0.2846 | 1.2061 | 1.0677 | .9149 | 0.7062 | 1.5318 |
| 4 | 1.0746 | 0.8843 | 0.9029 | 0.8551 | 1.1602 | .7754 | 0.1418 | 0.3051 |
| 5 | 4.7285 | 0.3014 | 3.8643 | 1.7936 | 0.9576 | 2.3291 | 2.0840 | 4.4271 |
| 6 | 0.7933 | 2.2658 | 0.9602 | 0.6548 | 1.5228 | 1.2394 | 0.7002 | 1.6110 |
| 7 | 0.7079 | 1.0968 | 0.1047 | 0.6754 | 1.1629 | .7495 | 0.4497 | 1.0582 |
| 8 | 2.2122 | 1.7827 | 0.1288 | 0.6622 | 0.6353 | 1.0842 | 0.9417 | 2.0834 |
| 9 | 2.1237 | 0.8545 | 0.3815 | 0.3476 | 1.7001 | 1.0815 | 0.8631 | 1.7761 |
| 10 | 0.2717 | 1.7648 | 0.5289 | 0.5342 | 1.9407 | 1.0081 | 0.8105 | 1.6690 |
| 11 | 2.3907 | 0.8900 | 0.5028 | 1.6236 | 0.7903 | 1.2395 | 0.8167 | 1.8879 |
| 12 | 1.6955 | 0.4401 | 1.4075 | 0.8205 | 0.7006 | 1.0128 | 0.5702 | 1.2554 |
| 13 | 0.6524 | 3.4576 | 0.7142 | 0.4170 | 0.8483 | 1.2179 | 1.1123 | 3.0406 |
| 14 | 0.4214 | 1.4155 | 0.4004 | 0.7503 | 1.4884 | .8952 | 0.5617 | 1.0880 |
| 15 | 0.2905 | 1.5486 | 0.3152 | 2.0454 | 1.1725 | 1.0744 | 0.8405 | 1.7549 |
| 16 | 0.4124 | 0.2163 | 0.9160 | 0.5080 | 1.3055 | .6716 | 0.4753 | 1.0892 |
| 17 | 2.1493 | 1.1899 | 0.2715 | 0.3971 | 0.3297 | .8675 | 0.8179 | 1.8778 |
| 18 | 0.4348 | 0.7131 | 1.4876 | 3.7585 | 1.0477 | 1.4883 | 1.3152 | 3.3237 |
| 19 | 0.6870 | 1.0361 | 0.3199 | 1.0480 | 1.2121 | .8606 | 0.3802 | 0.8922 |
| 20 | 1.0520 | 2.3357 | 0.6695 | 0.3902 | 0.3614 | .9618 | 0.8169 | 1.9743 |

Using the information in Table 6, the following parameters were obtained: . Inserting these estimates into Equation(5), the control limits for the Downton-based and that of the range- based  control chart were computed. The control limits are presented in Table 7

 **Table 7:** **Control Limits for** $\overbar{X}$R  **and** $\overbar{X}\_{D}$ **charts for Non-normal case**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Control charts** | **Location and Scale estimator** | **LCL** | **CL** | **UCL** |
| $$μ$$ | $$\hat{σ}$$ |
| $\overbar{X}$R chart | $$\overbar{X}$$ | R | 0.0511 | 1.0930 | 2.1347 |
| $\overbar{X}\_{D}$ chart | $$\overbar{X}$$ | D | 0 | 1.0930 | 2.1604 |

**4.0 Discussion**

The results of the derived control limits for the $\overbar{X}\_{D}$ and D charts when used on dataset which is assumed to be normal in Table 5, gave the same results as $\overbar{X}$ and R chart which is in conformity with the conclusion of Montgomery(2005). Moreover, there were no point outside the control limits on the Downton-based, $\overbar{X}\_{D}$ chart while a point is outside the limits on the range-based control chart. Thus, we say that the process is in statistical control at the stated levels and can be adopted for use in phase II. In Table 7 the control limit interval is higher when Downton is used to compute control limits than when R chart was used. That is, when R is used there is the chance that some points will be outside the limits whereas when D is used to compute the limits they will fall into the control limits for normal and non-normal case. Thus, the Downton-based control limits accomodates more sample points than range-based control limits. The use of range-based control limits might lead to frequent false alarm. It was also observed that the Downton-based control limits detect changes in variability faster than R based control limits.

**5.0 The Performance of The** $\overbar{X}$ **Chart Based on Downton’s Estimator**

The Average Run Length (ARL) of the $\overbar{X}$ based on the Downton’s estimator is studied and compared with the traditional $\overbar{X}$ and R charts. The ARL represent the average number of points until a chart signals where $ARL= \frac{1}{P\_{0}}$, $P\_{0}$ is the probability that one points plot out-of-control. Here, the in-control average run length ARL0 is equal to 370 when the control limits are based on 3$σ$ from the mean and the out-of-control ARL1 for the two types of charts are computed using MATLAB version 7.10. Sample sizes of n=5 is considered and the process mean of observations is assume to have shifted from $μ\_{0}$ to $μ\_{1}$ where $μ\_{1}=μ\_{0}+kσ$ and $k\in \left\{1.00, 1.25, 1.50,……,2.00, 2.25,…….,3.00\right\}$. The corresponding out-of-control ARL for the different possible shifts are displayed in Table 8.

The results in Table 8 indicates that the out-of-control signal is quickly detected for small shifts when compared with the range-based statistic for sample size n = 5. When the magnitude of shift is large in the process observations, the out-of-control ARL performance of both charts are the same. The $\overbar{X}$ control limits based on the Downton’s estimator performs well for small shift and can be used instead of the traditional $\overbar{X}$ chart based on range statistic when normality assumption is violated and the desire shift is small.(k less or equal to 2).

 **Table 8 : ARL for** $\overbar{X}$ **based on Downton and Range Estimates For Varying shift values**

|  |  |  |
| --- | --- | --- |
| **Shift (k)** | $\overbar{X}$ **ARL Based on Range**  | $\overbar{X}$**ARL Based on Downton**  |
| 1.00 | 4.533 | 4.333 |
| 1.25 | 2.399 | 2.376 |
| 1.50 | 1.570 | 1.560 |
| 1.75 | 1.222 | 1.218 |
| 2.00 | 1.076 | 1.075 |
| 2.25 | 1.022 | 1.021 |
| 2.50 | 1.005 | 1.005 |
| 2.75 | 1.000 | 1.000 |
| 3.00 | 1.000 | 1.000 |

**5.0 Conclusion**

The derived $\overbar{X}$ chart limits based on the Downton’s estimator has been proved to be more effective than the $\overbar{X}$ chart limits based on the range statistic for monitoring process data when normality assumptions is violated. Therefore, it is recommended that the proposed $\overbar{X}$ chart control limits based on Downton statistic be used as an attractive alternative by quality control practitioners when the assumption of normality is violated for small shifts.

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