Application of the fractional complex transform and variational iteration methods for studying the solution for nonlinear fractional Klein-Gordon equation

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Abstract

In this paper, we implement the fractional complex transform method to convert the nonlinear fractional Klein-Gordon equation (FKGE) to ordinary differential equation. We use the variational iteration method (VIM) to solve the resulting ODE. Some numerical examples are presented to validate the proposed techniques. Finally, a comparison with the numerical solution using Runge-Kutta of order four is given.

Keywords: Nonlinear fractional Klein-Gordon equation; Riemann-Liouville derivative; Fractional complex transform method; Variational iteration method.

1. Introduction

Fractional differential equations (FDEs) have recently been applied in various areas of engineering, science, finance, applied mathematics, bio-engineering and others. However, many researchers remain unaware of this field. FDEs have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering [18]. Consequently, considerable attention has been given to the solutions of FDEs of physical interest. Most FDEs do not have exact solutions, so approximate and numerical techniques ([9]-[13], [24]), must be used. Recently, several numerical methods to solve fractional differential equations have been given, such as variational iteration method [6], homotopy perturbation method [3], Adomian decomposition method [2], homotopy analysis method [5] and collocation method [23].

The Klein-Gordon equation plays a significant role in mathematical physics and many scientific applications, such as solid-state physics, nonlinear optics, and quantum field theory [25]. The equation has attracted much attention in studying solitons ([21], [22]) and condensed matter physics, in investigating the interaction of solitons in a collisionless plasma, the recurrence of initial states, and in examining the nonlinear wave equations [2]. Wazwaz has obtained the various exact travelling wave solutions such as, compactons, solitons and periodic solutions by using the tanh method [25]. The study of numerical solutions of the Klein-Gordon equation has been investigated considerably in the last few years. In the previous studies, the most papers have carried out different spatial discretization of the equation ([4], [26]). Where, the numerical solution using radial basis functions is given in [1], collocation and finite difference-collocation methods for the solution of proposed problem is introduced in [14], finally, the tension spline approach for the numerical solution of nonlinear Klein-Gordon equation is implemented in [19]. We describe some necessary definitions and mathematical preliminaries of the fractional calculus theory which will be used further in this work.

Definition 1.

The Caputo fractional derivative operator D^α of order α is defined in the following form

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt, \quad \alpha > 0,$$

where $m-1 < \alpha \leq m,\ m \in \mathbb{N},\ x > 0$.
Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation

$$D^\alpha \left(\lambda\, f(x) + \mu\, g(x)\right) = \lambda\, D^\alpha\, f(x) + \mu\, D^\alpha\, g(x),$$

where λ and μ are constants. For the Caputo's derivative we have $D^\alpha\, C = 0$, if C is a constant [18] and

$$D^\alpha\, x^n = \begin{cases} 0, & \text{for} \quad n \in \mathbb{N}_0 \text{ and } n < \lceil \alpha \rceil; \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, & \text{for} \quad n \in \mathbb{N}_0 \text{ and } n \geq \lceil \alpha \rceil. \end{cases} \tag{1}$$

We use the ceiling function $\lceil \alpha \rceil$ to denote the smallest integer greater than or equal to α and $\mathbb{N}_0 = \{0, 1, ...\}$. Recall that for $\alpha \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order. For more details on fractional derivatives definitions and theirs properties see ([17], [18]).

2. Chain rule for fractional calculus

In previous papers ([8], [15], [16]), the authors used the following chain rule

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial u}{\partial s} \frac{\partial^\alpha s}{\partial t^\alpha},$$

to convert a fractional differential equation with Jumarie's modification of Riemann-Liouville derivative into its classical differential partner. In [7], the authors showed that this chain rule is invalid by giving a counter example as follows:

Assume, $s = t^\alpha,\ 0 < \alpha < 1$ and $u = s^m$, i.e., $u = t^{m\alpha}$, then

$$\frac{\partial^\alpha u}{\partial t^\alpha} = D_t^\alpha u = D_t^\alpha t^{m\alpha} = \frac{\Gamma(1+m\alpha)t^{m\alpha-\alpha}}{\Gamma(1+m\alpha-\alpha)}.$$

Now we calculate $\frac{\partial u}{\partial s}\frac{\partial^\alpha s}{\partial t^\alpha}$.

Since $\frac{\partial u}{\partial s} = ms^{m-1} = mt^{\alpha m-\alpha}$ and

$$\frac{\partial^\alpha s}{\partial t^\alpha} = D_t^\alpha s = D_t^\alpha t^\alpha = \frac{\Gamma(1+\alpha)t^{\alpha-\alpha}}{\Gamma(1+\alpha-\alpha)} = \frac{\Gamma(1+\alpha)t^0}{\Gamma(1)} = \Gamma(1+\alpha).$$

Then,

$$\frac{\partial u}{\partial s}\frac{\partial^\alpha s}{\partial t^\alpha} = mt^{m\alpha-\alpha}\Gamma(1+\alpha) = m\Gamma(1+\alpha)t^{m\alpha-\alpha}.$$

This shows that, $\frac{\partial^\alpha u}{\partial t^\alpha} \neq \frac{\partial u}{\partial s}\frac{\partial^\alpha s}{\partial t^\alpha}$. In [7] the authors show that

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \sigma_t\frac{\partial u}{\partial s}\frac{\partial^\alpha s}{\partial t^\alpha},$$

where σ_t denotes the sigma index. From the above example we can see that $\sigma_t = \frac{\Gamma(1+m\alpha)}{m\Gamma(1+m\alpha-\alpha)}$. For more details, see [20].

3. Reducing the nonlinear FKGE to ordinary differential equation

In this section, to demonstrate the effectiveness of our approach, we will apply the complex transformation of Li and He to construct an approximate solution for the nonlinear fractional Klein-Gordon equation. Consider the following general form of FKGE

$$D_t^{2\alpha}u(x,t) + aD_x^{2\beta}u(x,t) + bu(x,t) + u^\gamma(x,t) = 0, \quad x \in (0,1), \quad t > 0, \tag{2}$$

where D_t^α denotes the fractional derivative of order α with respect to t, D_x^α denotes the fractional derivative of order β with respect to x, $u(x,t)$ is unknown function, and a, b, c and γ are known constants with $\gamma \in R, \gamma \neq \pm 1$. We also assume the following initial conditions

$$u(x,0) = g_1(x), \quad u_t(x,0) = g_2(x), \quad x \in (0,1), \tag{3}$$

and the following boundary conditions $u(0,t) = u(1,t) = 0$.

Li and He proposed a fractional complex transform for converting fractional differential equations into ordinary differential equations, so that all analytical methods for advanced calculus can be easily applied to fractional calculus. Now, take the following fractional complex transform

$$u(x,t) = U(\xi), \qquad \xi = \frac{Kt^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{Lx^{2\beta}}{\Gamma(1+2\beta)}, \tag{4}$$

where K and L are constants. By using the fractional chain rule

$$D_t^{2\alpha}u = \sigma_t\frac{d^2u}{d\xi^2}D_t^{2\alpha}\xi = \sigma_t KU'',$$

$$D_x^{2\beta}u = \sigma_x\frac{d^2u}{d\xi^2}D_x^{2\beta}\xi = \sigma_x LU''.$$

Without loss of generality we can take $\sigma_t = \sigma_x = \ell$ where ℓ is a constant. By using the definition of Caputo derivative and the above modified chain rule, Eq.(2) converts to the following ordinary differential equation

$$(K\ell + aL\ell)U'' + bU + cU^{\gamma} = 0,$$

we can write the above ODE in the form

$$U''(\xi) + \chi_1 U(\xi) + \chi_2 U^{\gamma}(\xi) = 0, \tag{5}$$

where $\chi_1 = \frac{b}{(K\ell + aL\ell)}$ and $\chi_2 = \frac{c}{(K\ell + aL\ell)}$.

4. Procedure of solution with VIM

In this section, we implement VIM for solving nonlinear ODE (5) with suitable boundary conditions. According to VIM, we construct the following recurrence formula

$$U_{n+1}(\xi) = U_n(\xi) + \int_0^{\xi} \lambda(\tau)\Big[U_n^{''}(\tau) + \chi_1 \tilde{U}_n(\tau) + \chi_2 \tilde{U}_n^{\gamma}(\tau)\Big]d\tau, \tag{6}$$

where λ is a general Lagrange multiplier. Making the above correction functional stationary

$$\begin{aligned} \delta U_{n+1}(\xi) &= \delta U_n(\xi) + \delta \int_0^{\xi} \lambda(\tau)[U_n^{''}(\tau)]d\tau \\ &= \delta U_n(\xi) + \int_0^{\xi} [\lambda(\tau)\delta U_n^{''}(\tau)]d\tau \\ &= \delta U_n(\xi) + [\delta\, U_n^{'}\lambda - \delta\, U_n \lambda^{'}]_{\tau=\xi} + \int_0^{\xi} [\delta\, U_n(\tau)\lambda^{''}(\tau)]d\tau = 0, \end{aligned}$$

where $\delta\tilde{U}_n$, is considered as restricted variation, i.e., $\delta\tilde{U}_n = 0$, yields the following stationary conditions (by comparison the two sides in the above equation)

$$\lambda^{''}(\tau) = 0, \qquad 1 - \lambda^{'}(\tau)|_{\tau=\xi} = 0, \qquad \lambda(\tau)|_{\tau=\xi} = 0. \tag{7}$$

Eqs.(7) are called Lagrange-Euler equation and its natural boundary conditions, the Lagrange multiplier, therefore

$$\lambda(\tau) = \tau - \xi. \tag{8}$$

Now, by substituting from (8) in (6), the following variational iteration formula can be obtained

$$U_{n+1}(\xi) = U_n(\xi) + \int_0^{\xi} (\tau - \xi)\Big[U_n^{''}(\tau) + \chi_1 U_n(\tau) + \chi_2 U_n^{\gamma}(\tau)\Big]d\tau. \tag{9}$$

Now, we start with initial approximation

$$U_0(\xi) = U(0) + \frac{U'(0)}{1!}\xi = A + B\xi,$$

for some constants $A = U(0)$ and $B = U'(0)$. By using the above iteration formula (9), we can directly obtain the first components of the solution of (5) as follows

$U_0(\xi) = A + B\xi,$

$U_1(\xi) = A+Bx+2.1761Ax^2+2.1761A^3x^2+0.7254Bx^3+2.1761A^2Bx^3+1.0881AB^2x^4+0.21763B^3x^5, ...,$

and so on. The unknown variables A and B are computed if we satisfy the boundary conditions.

5. Numerical simulation

In this section, we solve numerically the nonlinear fractional Klein-Gordon equation where we use the complex transformation method to reduce it as ODE, then we solve the resulting ODE using VIM. Some numerical examples are presented to validate the solution scheme. To achieve this propose we consider the following three cases.

Case study 1:

In this case, we take the values of the constants as follows

$$a = -1, \quad b = 1, \quad c = 1, \quad \gamma = 3, \quad K = 0.25, \quad L = 0.50, \quad \ell = 1,$$

with different values of α and β ($\alpha = 0.5, 0.7, 1.0$ and $\beta = 0.5, 0.7, 1.0$). In this case, the values of A and B are $A = 0.0,\ \ B = 0.05$.

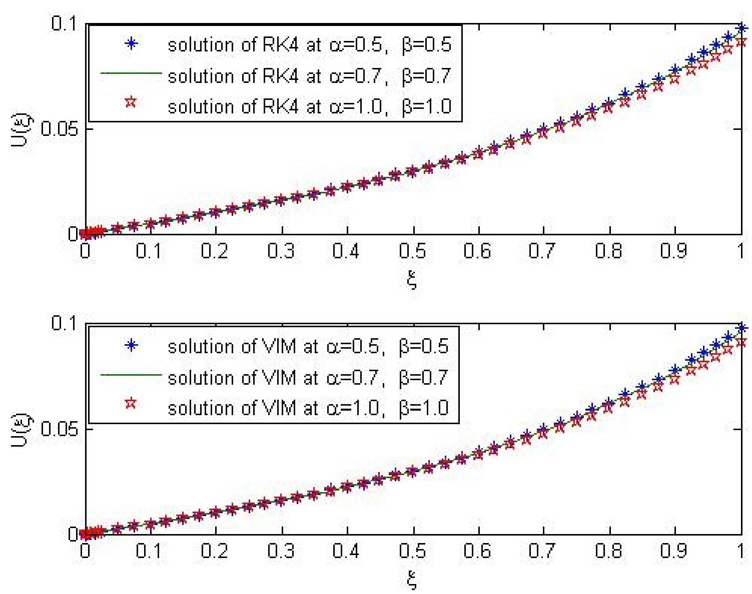

Figure 1. The behavior of the approximate solution using RK4 (Top) and VIM (Bottom) with different values of α and β.

The obtained numerical results by means of the proposed methods are shown figure 1. In this figure, we presented a comparison between the numerical solution using Runge-Kutta of order four (RK4) and the approximate solution using the proposed method, VIM with $n = 5$.

Case study 2:

In this case, we take the values of the constants as follows

$$\alpha = 0.9, \quad \beta = 0.9, \quad \gamma = 3, \quad K = 0.25, \quad L = 0.50, \quad \ell = 1,$$

with different values of a, b and c ($a = -1, 1, -1,\ \ b = 1, -1, 1$ and $c = 1, -1, -1$). In this case, the values of A and B are $A = 0.0,\ \ B = 0.05$.

The obtained numerical results by means of the proposed methods are shown figure 2. In this figure, we presented a comparison between the numerical solution using Runge-Kutta of order four (RK4) and the approximate solution using the proposed method, VIM with $n = 5$.

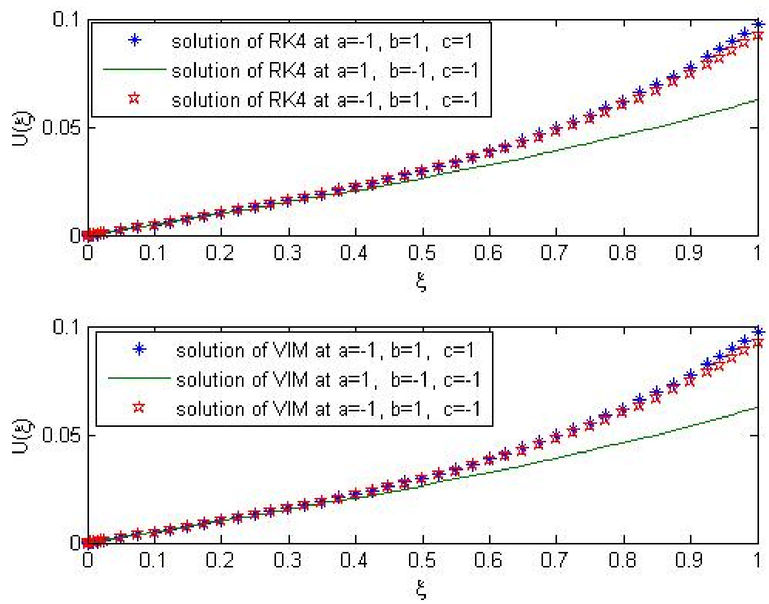

Figure 2. The behavior of the approximate solution using RK4 (Top) and VIM (Bottom) with different values of a, b and c.

Case study 3:

In this case, we take the values of the constants as follows

$$a = -1, \quad b = 1, \quad c = 1, \quad \alpha = 0.75, \quad \beta = 0.75, \quad \gamma = 3, \quad \ell = 1,$$

with different values of K and L ($K = 0.25, 0.50, 0.25,$ and $L = 0.50, 0.25, 0.75$). In this case, the values of A and B are $A = 0.0,\ \ B = 0.05$.

The obtained numerical results by means of the proposed methods are shown figure 3. In this figure, we presented a comparison between the numerical solution using Runge-Kutta of order four (RK4) and the approximate solution using the proposed method, VIM with $n = 5$.

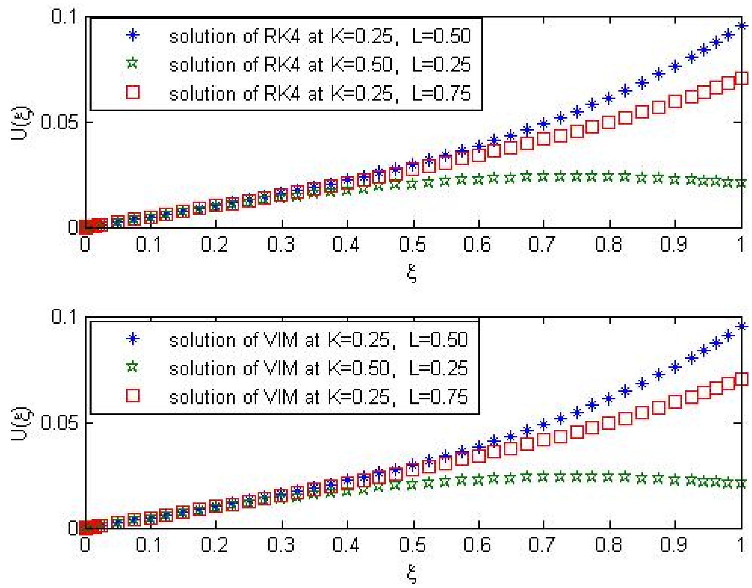

Figure 3. The behavior of the approximate solution using RK4 (Top) and VIM (Bottom) with different values of K and L.

6. Conclusion and remarks

In this article, the properties of the fractional complex transform method are used to reduce the nonlinear fractional Klein-Gordon equation to the solution of ordinary differential equation. The resulting ODE is solved by using variational iteration method. The obtained approximate solution using the suggested methods is in excellent agreement with the numerical solution using the forth order Runge-Kutta method and show that these approaches can be solved the problem effectively and illustrates the validity and the great potential of the proposed technique. All computations in this paper are done using Matlab 8.0.

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