Combinatorial Interpretation of the Identity $\det (h_{\lambda(I)-I+j}) = \det (e_{\lambda'(I)-I+j})$

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Abstract

In this note, we consider the n-tuples of lattice paths as the combinatorial objects for the right hand side of the identity $\det (h_{\lambda(I)-I+j}) = \det (e_{\lambda'(I)-I+j})$ but under some restrictions. The proof of the identity then follows by showing that the generating function for the set **D-A** is equal to zero.

1 Introduction

Let λ be a partition and λ' be its conjugate partition. Let $h_k(x_1, x_2, \ldots x_n)$ be the k^{th} homogeneous symmetric functions of $x_1, x_2, \ldots x_n$ defined by

$$h_k = \sum x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$$

where the summation is over all $a_1, a_2, \ldots a_n \geq 0$ such that $a_1 + a_2 + \ldots + a_n = k$.

Let $e_k(x_1, x_2, \ldots x_n)$ be the k^{th} elementary symmetric function defined by

$$E_k = \sum_{a_1 < a_2 < \ldots < a_k} x_{a_1} x_{a_2} \ldots x_{a_k}$$

Where $a_1, a_2, \ldots a_k$ is any choice of k numbers from 1 to n and n satisfies the Equation $\lambda_1 + \lambda'_1 = n+1$.

If we define $h_k = 0$ for $k<0$ and $e_k = 0$ for $k<0$, it is easy to see that $| h_{\lambda(1)-I+j} |$ is a $\lambda'_1 \times \lambda'_1$ determinant and $| e_{\lambda'(1)-I+j} |$ is a $\lambda_1 \times \lambda_1$ determinant.

Macdonald [1] has given a proof of the identity based on the fact $\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0$ which provides that each minor of H = $(h_{i-j})_{nxn}$ is equal to the complementary cofactor of E', where E $=(e_{i-j})_{nxn}$. Considering the minor of H with row indices $\lambda_I - \lambda_1' - I$ $(1 \leq I \leq \lambda_1')$ and the column indices $\lambda_1' - I$ $(1 \leq I \leq \lambda_1)$ and according to another fact that the m-n numbers $\lambda_I + n - I$ $(1 \leq I \leq n)$, $n-1+j-\lambda'_j$ $(1 \leq j \leq m)$ are a permutation of $\{0, 1, \ldots n+m-1\}$, where m and n are two integers such that $m \geq \lambda_1$ and $n \geq \lambda'_1$. The identity then is straightforward.

Gessel and Viennot [2] have given a combinatorial proof that the left hand side of the identity is equal to the generating function for column- strict plane partition with shape $(\lambda_1, \lambda_2, \ldots \lambda_n)$ by considering the n-tuples of lattice paths and column – strict plane partition as the combinatorial objects and apply the involutionary method.

In this article, we also consider the n-tuples of lattice paths as the combinatorial Objects for the right hand side of the identity but under some restrictions. We define a set **D** consisting of all those paths and a weight for each element D of **D** such that the generating function for **D** with respect to this weight function is exactly equal to the expansion of the right hand side of the identity. We then find a subset **A** of **D** such that there is a one to one correspondence between the Generating function for the set **A** and the generating function for column- strict plane partition with the shape $(\lambda_1,\lambda_2, \dots \lambda_n)$. The proof of the identity then follows by showing that the generating function for the set **D-A** is equal to zero, which can be proved by finding an involution on the set **D-A.**

II N-Tuples of lattice paths and Column-Strict plane partition.

Consider n-tuples of paths, the ith path being from the point $P_j = (n-j+1, n)$ to a point $Q_I = (\lambda'_I+n-I+1, 1)$ with three restrictions:

1) Each path consists of horizontal steps to the right and vertical steps downward.
2) Each path of the n-tuple can have at most one unit at any given height.
3) No two paths start at the same point.

Define a weight for the element D of **D** as follows:

$$\mathrm{WT}\ (D) = \mathrm{Sgn}\ (\sigma)\ X_1{}^{a_1}\ X_2{}^{a_2} \dots X_n{}^{a_n}\ {}^{a_2}$$

Where, σ is the permutation of $(1,2, \dots n)$ which indicates the n paths contained in n-tuple D are from point $P_{\sigma(I)}$ to Q_I, and a_I is the number of unit horizontal segment on the level $y = I$ contained in the n-tuple of paths D.

For any subset **S** of **D**, denote the generating function for **S** with respect to the weight by Φ (**S**), therefore

$$\Phi\ (\mathbf{S}) = \sum_{D\in S} WT(D)$$

Proposition 2.1

For any partition $\lambda' = (\lambda'_1, \lambda'_2, \dots \lambda'_n)$, we have

$$\Phi\ (\mathbf{D}) = \sum_{\sigma\in S_n} Sgn(\sigma) \prod_{i=1}^{n} e_{\lambda'_i - i + \sigma(i)} = \det\left| e_{\lambda'_i - i + j} \right|$$

Proof: Let **S** denote the subset of **D** such that all elements of **S** correspond to a fixed permutation σ (σ indicates a mapping from $P_{\sigma(I)}$ to Q_i.). Now, the path from $P_{\sigma(I)}$ to Q_I of all elements of **S** will contain $(\lambda'_I+n-I+1) - (n-\sigma(I)+1) = \lambda'_I - I + \sigma(I)$ horizontal unit segments and all these segments, by the restriction 2), are on different levels. In other words, these $\lambda'_I - I + \sigma(I)$ unit segments are chosen from the n different levels. Combining the definition of weight function, the sum of the weights of all possible paths from $P_{\sigma(I)}$ to Q_I is

$$\sum_{\alpha_1<\alpha_2<\ldots<\alpha_{\ell(i)}} X_{\alpha_1} X_{\alpha_2} \ldots X_{\alpha_{\ell(i)}}$$

where $\ell(i) = \alpha'_i - i + \sigma(i)$. This is exactly the definition of the elementary symmetric function $e_{\lambda'_i - i + \sigma(i)}$. Because each n-tuple contains n paths, the generating function for subset **S** is

$$\Phi\ (S) = \mathrm{Sgn}\ (\sigma) \prod_{i=1}^{n} e_{\lambda'_i - i + \sigma(i)}$$

Hence
$$\Phi\ (D) = \prod_{\sigma \in S_n} Sgn(\sigma) \prod_{i=1}^{n} e_{\lambda'_i - i + \sigma(i)}\ .$$

Now, let ***A*** be the subset of **D** satisfying the following restrictions:

a) All elements of ***A*** correspond to the permutation $\sigma(I) = I$
b) For each I and j, the j^{th} highest horizontal segment of the path P_I to Q_I does not exceed the j^{th} highest horizontal segment of the path P_{I-1} to Q_{I-1}.

For example, for $\lambda'_I = (2, 1, 1, 0)$, the n-tuple of paths D_1 (Figure 1(a)) belongs to ***A*** but the n-tuple of paths D_2 (Figure 1(b)) does not belong to ***A***.

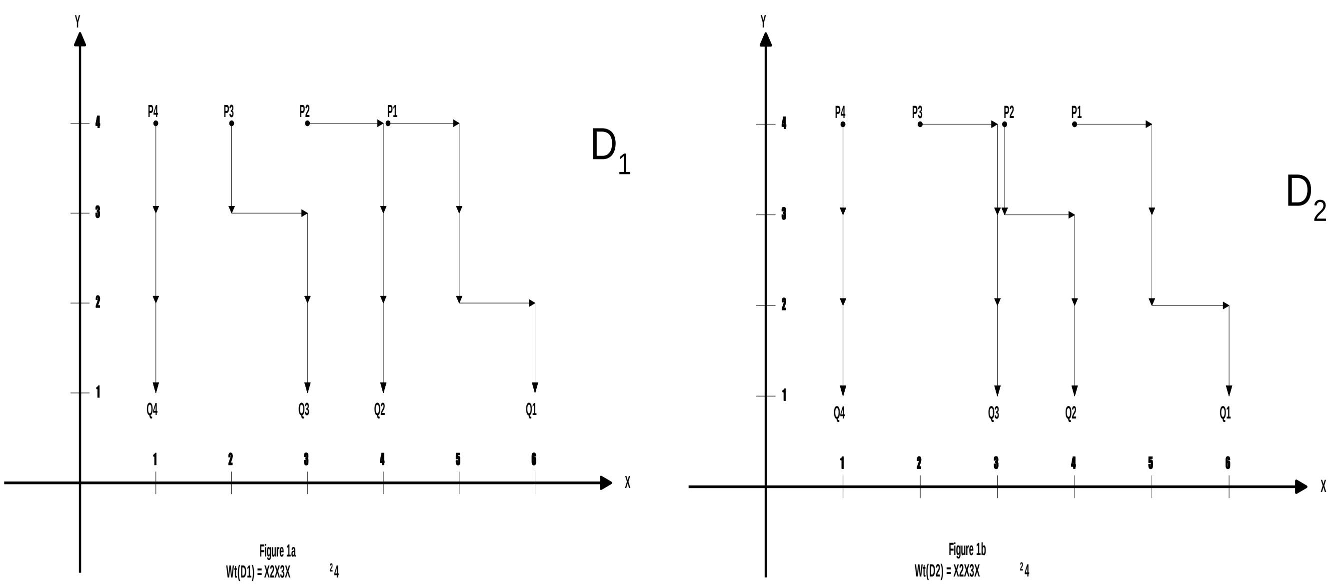

Figure 1a
$Wt(D_1) = X_2X_3X_4^2$

Figure 1b
$Wt(D_2) = X_2X_3X_4^2$

Proposition 2.2

$$\Phi\,(\boldsymbol{A}) = \sum_{\sigma \in S_n} Sgn\,(\sigma) \prod_{i=1}^{n} h_{\lambda_i - i + \sigma(i)} = \det(h_{\lambda_i - i + j})$$

Proof: Gessel and Viennot have proved that det (h $_{\lambda_i - i + j}$) is equal to the generating function for column-strict plane partition with the shape (λ_1,λ_2, …λ_n). The weight of a column-strict plane partition with the shape (λ_1,λ_2, …λ_n) is defined as follows:

1) The largest integer can be appeared in the plane partition is n and the smallest integer can be appeared is 1.
2) The weight of a column-strict plane partition P is defined as:

$$\text{WT (P)} = X_1^{\alpha_1} X_2^{\alpha_2} \ldots X_n^{\alpha_n}$$

Where α_i indicates the number I appears α_i times in the plane-strict plane partition P.

Gessel and Viennot proved that the generating function for the set of all column-strict partition is

$$\Phi\,(\boldsymbol{P}) = \sum_{\sigma \in S_n} Sgn(\sigma) \prod_{i=1}^{n} h_{\lambda_i - i + \sigma(i)} = \det(h_{\lambda_i - i + j})\,.$$

Now, we prove that there exists a one to one weight preserving correspondence between the column-strict plane partition and the elements of ***A***. Let D be any element of ***A***, the i^{th} path of D from point P_I to Q_I) contains λ'_I horizontal unit segments. We now draw n columns of squares. From right to left, the first column contains λ'_n squares; the second column contains λ'_{n-1} squares and so on, then the last column contains λ'_1 squares. In each column of squares we put in the numbers decreasingly from high to low such that those numbers indicate the horizontal levels of the corresponding path.

For an example, for λ'_I=(2, 1, 1,0) and the n-tuple of path E (Figure 2(a)), since λ'_4= 0, we omit it, the λ'_3= 1, so the first column has only one square; λ'_2 = 1, so the second column has also one square; λ'_1 =2, so the third column contains two squares. Since the path from P_3 to Q_3 contains one horizontal unit segment on the second level, we put the number 2 in the first square of the first column, the path from P_2 to Q_2 has one horizontal unit segment on the third level, so we put 3 in the square; the path from P_1 to Q_1 one horizontal unit segment on the fourth level and one unit segment on the third level, so we put 4 and 3 decreasingly in the last column squares. (Figure 2(b))

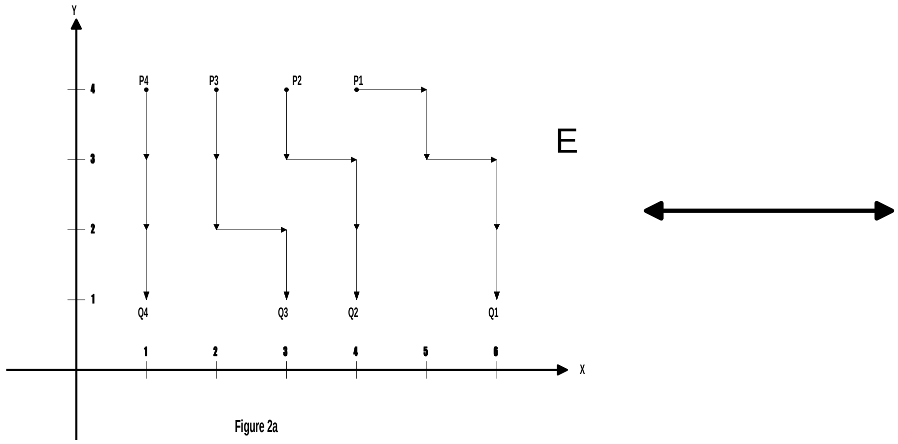

Figure 2a

4	3	2
3		

Figure 2b

By the restrictions of set ***A***, each elements of ***A*** corresponds to a column-strict plane partition as in figure 2(b) in which the numbers decrease from left to right and decrease strictly from top to bottom. It is easy to see that for each column strict partition of shape $(\lambda_1,\lambda_2, \ldots\lambda_n)$ there also exists a unique n-tuple of path which belongs to ***A*** and the corresponding n-tuple of path and the column-strict plane partition have the same weight. For an example, the n-tuple of paths E and the plane-strict partition E' have weight:

$$WT (E) = WT (E') = X_2X_3X_4^2$$

This proves that $\qquad \Phi(\boldsymbol{P}) = \Phi(\boldsymbol{A})$,

Hence $\qquad \Phi(\boldsymbol{A}) = \sum_{\sigma\in S_n} Sgn(\sigma)\prod_{i=1}^{n} h_{\lambda_i - i+\sigma(i)} = \det\left|h_{\lambda_i - i+j}\right|$

Proposition 2.3

$$\boldsymbol{\Phi(\mathbf{D}\text{-}\, A) = 0}$$

The proof of this proposition will be presented through the following two lemmas:

Lemma 1. Any n-tuple of **D-*A*** has at least two intersecting paths.

Proof: Case (a) If a n-tuple D in **D-*A*** corresponds to a non-identity permutation σ, that is : $\sigma(I) \neq I$, for some i. Assume I is the largest number of 1, 2, … n such that $\sigma(I) \neq I$, say $\sigma(I)=j$, where $I>j$. Clearly, there must exist a member k such that $\sigma(k) = I$, than the path from P_j to Q_I and the path from P_I to Q_k are obviously intersecting.

Case (b) If a n-tuple D in **D-*A*** corresponds to the identity permutation. By the restriction (b) of the definition of set ***A***, there exist a pair of I and j such that the jth highest horizontal segment of the path P_I to Q_I is higher than the jth highest horizontal segment of the path P_{I-1} to Q_{I-1}. [Without loss of generality, suppose from first to (j-1)th highest horizontal segments of path P_I to Q_I do not exceed the corresponding higher horizontal segments of path P_{I-1} to Q_{I-1}]. Clearly, the right end point R of the jth highest horizontal unit segment of path P_I to Q_I is also on the path P_{I-1} to Q_{I-1}. Therefore these two paths are intersecting.

Lemma 2. For any n-tuple D in **D-*A***, there exists a corresponding n-tuple D' also in **D-*A*** such that WT (D) = -WT (D').

Proof: Case I. If a n-tuple D in **D-*A*** has positive weight.

(a) Suppose D corresponds to permutation σ, where σ is not identity permutation. As we proved in lemma 1, case (a) the path from P_j to Q_I and path from P_I to Q_k are intersecting. Now we fix all other paths of D. Because point P_j is to the right of P_I and the point Q_k is to the right of Q_I, there exists at least one vertical unit segment which belongs to both of these two paths. Let point R be the highest end point of this vertical unit segment. Then we change two paths P_j to Q_I and P_I to Q_k to be P_j to R then R to Q_k (the second half path originally belongs to path P_I to Q_k) and from P_I to R then from R to Q_I (the second half path originally belongs to the path P_j to Q_I). The new n-tuple D' corresponds a negative sign permutation and WT (D) = - WT (D')

(b) Suppose D corresponds to the identity permutation. By the proof of lemma 1 case (b), we can find the right end point of jth highest horizontal segment of path P_I to Q_I which is also on the path P_{I-1} to Q_{I-1} and clearly, both of these two paths contain the vertical segment from R to the point one unit down from R. Now, we fix all other paths of D but change path P_I to Q_I to be P_I to R then R to Q_{I-1} and change path P_{I-1} to Q_{I-1} to be P_{I-1} to R then R to $Q_{i.}$ Clearly. The new n-tuple D' contains same number of horizontal segments on each level as the n-tuple D but D' corresponds a negative permutation. Therefore WT(D) = -WT(D').

Case II If a n-tuple D in **D-*A*** has negative weight. The proof is exactly the same as the proof of lemma 2 case I (a). (Since the negative weight n-tuple always corresponds to a non-identity permutation) if we can prove that the new n-tuple corresponds to this negative weight n-tuple D will not belong to the set ***A*.** This is equivalent to proving that any negative weight n-tuple which has the same horizontal and vertical segments at exactly same places as an n-tuple of ***A*** will not satisfy the restrictions of set **D.**

By the restrictions of the set ***A***, if for each I and j the jth highest horizontal segment of tha path P_I to Q_I is lower than the jth highest horizontal segment of the path P_{I-1} to $Q_{I-1,}$ then this n-tuple of paths are non-intersecting. Obviously, there is no corresponding negative weight n-tuple for this n-tuple in ***A***. If some jth highest horizontal segment of path P_I to Q_I has same height as the jth highest horizontal segment of the path P_{I-1} to Q_{I-1}, then these two paths are intersecting but any permutation of the end points will create a path containing more than one horizontal unit segment on a level, so that n-tuple does not belong to the set **D**. That proves that any negative weight n-tuple can not correspond to an n-tuple of paths in ***A***.

Proposition 2.4 $$\Phi(\boldsymbol{A}) = \sum_{\sigma\in S_n} Sgn(\sigma)\prod_{i=1}^{n} e_{\lambda_i - i+\sigma(i)} = \det\left|e_{\lambda'_i - i+j}\right|$$

Proof: $\Phi(\mathbf{D}) = \Phi[A+(\boldsymbol{D}\text{-}\boldsymbol{A})] = \Phi(\boldsymbol{A}) + \Phi(\mathbf{D}\text{-}\boldsymbol{A})$. Therefore, by the propositions 2.1 and 2.3, we have $\Phi(\boldsymbol{A}) = \Phi(\mathbf{D}) = \det\left|e_{\lambda'_i - i+j}\right|$.

Proposition 2.5 $\det\left|h_{\lambda_i - i + j}\right| = \det\left|e_{\lambda'_i - i + j}\right|$

Proof: The formula above is straight forward from propositions **2.2** and **2.4**.

References

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