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Catalan's constant is an Irrational Number

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Abstract

In this paper, I focus on the Catalan's constant, being the main contribution the proof about this number is irrational. The key idea is to get to a contradiction using a series of properties of summaries and Leibniz's test for alternating series.

Introduction

The Catalan's Constant, which is denoted by the letter G and named by Eugene Charles Catalan, is a number defined by the summation. Being its numerical value is approximately $G = 0.915965594177219\ldots$, that is a number between zero and one.

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \cdots$$

Before starting we must know some properties about the Catalan's constant that help in the proof to determine the irrationality of this number. First, when the summatory starts with an even number then the result is positive, this is the case of G, otherwise if it is an odd number the number will be negative, this happens because the term $a_n = \frac{1}{(2n+1)^2}$ of the alternative series **decreases monotonically**.

$$\because |a_{n+1}| \leq |a_n| \Rightarrow \left|\frac{1}{(2n+3)^2}\right| \leq \left|\frac{1}{(2n+1)^2}\right|$$

That is true for all $n \in \mathbb{N}$

Proof

To prove the Catalan's constant is Irrational number I am going to use a reduction to the absurd assuming that G is a rational number which one can be written with a fraction of two Integers numbers $\{a, b\} \in \mathbb{Z}$. Following that, I am going to separate the sum in two parts being the start of these expressions $2b + 1$ and $2b + 2$, and odd number and even number respectively, at last multiplying the whole equation by $[(4b + 3)!]^2$.

$$\frac{a}{b} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$$

$$\frac{a}{b} = \sum_{n=0}^{2b+1} \frac{(-1)^n}{(2n+1)^2} + \sum_{n=2b+2}^{\infty} \frac{(-1)^n}{(2n+1)^2}$$

$$\frac{a}{b} - \sum_{n=0}^{2b+1} \frac{(-1)^n}{(2n+1)^2} = \sum_{n=2b+2}^{\infty} \frac{(-1)^n}{(2n+1)^2}$$

$$[(4b+3)!]^2 \cdot \frac{a}{b} - \sum_{n=0}^{2b+1} \frac{[(4b+3)!]^2}{(2n+1)^2} (-1)^n = [(4b+3)!]^2 \sum_{n=2b+2}^{\infty} \frac{(-1)^n}{(2n+1)^2}$$

After multiplying the entire equation by $[(4b + 3)!]^2 \in \mathbb{Z}$ we are going to see that the left part of the equation divided into two parts is an Integer expression.

$$[(4b+3)!]^2 \cdot \frac{a}{b} = (4b+3)^2 \cdot (4b+2)^2 \cdot (4b+1)^2 \cdot (4b)^2 \cdot (4b-1)^2 \cdot \dots \cdot 3^2 \cdot 1^2 \cdot \frac{a}{b}$$

$$[(4b+3)!]^2 \cdot \frac{a}{b} = (4b+3)^2 \cdot (4b+2)^2 \cdot (4b+1)^2 \cdot 16b \cdot (4b-1)^2 \cdot \dots \cdot 3^2 \cdot 1^2 \cdot a$$

$$[(4b+3)!]^2 \cdot \frac{a}{b} = (4b+3)^2 \cdot (4b+2)^2 \cdot (4b+1)^2 \cdot 16b \cdot [(4b-1)!]^2 \cdot a$$

$$\Rightarrow \left\{[(4b+3)!]^2 \cdot \frac{a}{b}\right\} \in \mathbb{Z}$$

$$\sum_{n=0}^{2b+1} \frac{[(4b+3)!]^2}{(2n+1)^2} (-1)^n = \frac{[(4b+3)!]^2}{1^2} - \frac{[(4b+3)!]^2}{3^2} + \frac{[(4b+3)!]^2}{5^2} - \dots - \frac{[(4b+3)!]^2}{(4b+3)^2}$$

$$= [(4b+3)!]^2 - [(4b+3)^2 \cdot \dots \cdot 7^2 \cdot 5^2 \cdot 1^2] + [(4b+3)^2 \cdot \dots \cdot 7^2 \cdot 3^2 \cdot 1^2] - \dots - [(4b+2)^2 \cdot \dots \cdot 5^2 \cdot 3^2 \cdot 1^2]$$

$$\Rightarrow \left\{\sum_{n=0}^{2b+1} \frac{[(4b+3)!]^2}{(2n+1)^2} (-1)^n\right\} \in \mathbb{Z}$$

Already proving that the left part of the equation is Integer then by consequence the right part **must** be also an Integer expression thanks to the equality. To continue with this problem, we need to know some characteristics about our summation to infinity, for example, it is correct to ensure that sum is positive due to starts with an even number that is $2b + 2$ it is important to notice that due to is key to know that the summation to the infinity is equal to its absolute value.

To continue with the proof, we are going to do the Leibniz's test and for that we have to know two things about the summation, the first condition is that $|a_n|$ decreases monotonically, that is true as we saw on the introduction of this paper, the second condition is that the limit to the infinity of a_n must be zero.

$$\lim_{n\to\infty} \frac{1}{(2n+3)^2} = \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty} (2n+3)^2} = \frac{1}{\infty} = 0$$

$$\therefore \lim_{n\to\infty} a_n \text{ is equal to zero.}$$

Leibniz's test

The Leibniz's test states that when an alternative series is convergent then the absolute value of the summation k minus the whole sum is lesser (or equal) than the absolute value of the summation k minus the sum $k + 1$, this last one being equal to the succession $k + 1$.

$$|S_k - L| \le |S_k - S_{k+1}| = a_{k+1}$$

Let as denote as the summation finishes with a term k as S_k and the complete sum as L satisfying the Leibniz's test $|S_k - L| \le a_{k+1}$, so we are going to choose k equal to $2b + 1$.

$$|S_{2b+1} - L| \le a_{2b+2}$$

$$\left|\sum_{n=0}^{2b+1} a_n(-1)^n - \sum_{n=0}^{\infty} a_n(-1)^n\right| \le \frac{1}{(2(2b+2)+1)^2}$$

$$\left|-\sum_{n=2b+2}^{\infty} a_n(-1)^n\right| \le \frac{1}{(4b+5)^2}$$

$$\left|\sum_{n=2b+2}^{\infty} \frac{(-1)^n}{(2n+1)^2}\right| \le \frac{1}{(4b+5)^2}$$

We are going to equal $\sum_{n=2b+2}^{\infty}\frac{(-1)^n}{(2n+1)^2}$ with its absolute value because this series is positive due to starts with an even number, and how b is an Integer number then the expression $\frac{1}{(4b+5)^2} \in \mathbb{Q}$ and this is between zero and one thanks to the numerator equal to one.

$$0 < \sum_{n=2b+2}^{\infty}\frac{(-1)^n}{(2n+1)^2} \leq \frac{1}{(4b+5)^2} < 1$$

$$\therefore \sum_{n=2b+2}^{\infty}\frac{(-1)^n}{(2n+1)^2} \notin \mathbb{Z}$$

And this inequation makes impossible that our summation to be an Integer number, reaching a contradiction due to the left part of the equation **is not** equal to the right part of the equation, this happens to assume that G can be written by a rational number. Which proves the Catalan's constant is Irrational number.

$$\therefore G \in Irrational\blacksquare$$

Conclusion

In this paper, I have presented one of the infinities solutions of this problem using the Leibniz's test, as well as, used to proof that e is irrational. Also, this demonstration permits to know one important characteristic about G and how to work with alternating series like this.

References

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