Carleman type estimates for some non linear parabolic problems

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Abstract

We consider a more general nonlinear parabolic partial differential equations system with non homogeneous Neumann boundary conditions. Under some growing conditions on the non linear term we establish some Carleman type estimates for the solution of the considered system.

Key words : *Global Carleman inequalities, Nonlinear parabolic equations, null controllability.*

1 Introduction

Carleman estimates, introduced in 1939 [1] as exponnential weighted energy inequalities for proving existence and uniqueness of solutions of linear elliptic PDEs, have since been used with or without succes to some controllabilty of PDEs systems. Particularly, it is well known that they play an important role for establishing null controllability for parabolic and hyperbolic PDEs. That is why, a larger number of papers is devoted to Carleman estimates and their applications. See [4], [5], [3], [7], [10] and the references therein. In addition, it can also be noted that Carleman estimates have particularly been proved to be useful for the resolution of inverse problems of some PDEs [6], [11], [12].
Carleman estimates, specifically for problems with inner control depending whether of boundary condition type, have been proposed in the literature [5], [8] and [9]. It should be noted however that Carleman type estimates are not easy to obtain. Difficulties can arise when considering PDEs with nonlinear terms or for problems associated with non homogeneous boundary conditions.
In this paper, we are then concerned with a problem of establishing Carleman estimates for a nonlinear parabolic equation with non homogeneous Neumann boundary condition. To this end, we first establish some preliminaries results in section 3, then we prove in section 4 some useful results concerning Carleman estimates.

2 Problem formulation

Let Ω be a bounded domain of $\mathbb{R}^d$, ($d = 2$ or 3) whose boundary is denoted by Γ. For given time parameter $T > 0$, we consider a non linear parabolic partial differential equation

$$\partial_t \psi - \nabla.\mathcal{A}\nabla\psi + F(\psi) = 0 \;\text{ in } Q_T = (0,\ T) \times \Omega \tag{2.1}$$

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with following Neumann boundary condition

$$\mathcal{A}\nabla\psi.n = g \text{ on } \Sigma_T = (0,\ T) \times \Gamma \tag{2.2}$$

and initial condition

$$\psi(0) = \psi_0 \text{ in } \Omega. \tag{2.3}$$

Here, $\mathcal{A}$ denotes a linear operator from $\mathbb{R}^d$ to $\mathbb{R}^d$ that satisfies elliptic property:

$$\exists C > 0 \text{ such that } \xi^t \mathcal{A}\xi \geq C|\xi|^2 \quad \forall \xi \in \mathbb{R}^d. \tag{2.4}$$

Furthermore, we shall assume that the operator $\mathcal{A}$ is symmetric, that is

$$\xi^t \mathcal{A}\eta = \eta^t \mathcal{A}\xi \quad \forall\ \eta, \xi \in \mathbb{R}^d. \tag{2.5}$$

F is a given non linear scalar function of class $\mathcal{C}^1$ while g and ψ_0 are given functions with some regularities properties. We shall also assume that

$$(u - v)(F(u) - F(v)) \geq 0 \quad \forall u, v \in \mathbb{R} \tag{2.6}$$

and

$$F(0) = 0. \tag{2.7}$$

In this paper, we would like to establish some Carleman type inequalities that will be useful to the following controllability problem: find φ solution of the following system equations

$$-\partial_t\varphi + \nabla.\mathcal{A}\nabla\varphi + F(\varphi) = v\chi_\omega \quad \text{in } Q_T = (0,\ T) \times \Omega \tag{2.8}$$

$$\mathcal{A}\nabla\varphi.n = g \text{ on } \Sigma_T = (0,\ T) \times \Gamma \tag{2.9}$$

$$\varphi(T) = \varphi_T \text{ in } \Omega \tag{2.10}$$

such that

$$\varphi(0) = 0 \text{ in } \Omega. \tag{2.11}$$

For a given non empty $\omega \subset \Omega$, we would like to establish an observability inequality. In fact, difficulties in establishing Carleman type estimates lies in the presence of the nonlinear term $F(\varphi)$ so that additional conditions to this term is required. Also, it should be noticed that the observability inequality is well known to be very helpfully for establishing the controllability of the problem (2.8)-(2.11).

3 Preliminary results

Let us consider a function η^0 of class $\mathcal{C}^2(\Omega)$ that satisfies

$$\left\{ \begin{array}{rl} \eta^0 & > 0 \quad \text{in } \Omega \\ |\nabla\eta^0(x)| & \neq 0 \quad \forall x \in \overline{\Omega \setminus \omega'} \\ \eta^0 & = 0 \quad \text{ on } \Gamma \end{array} \right. \tag{3.1}$$

where $\omega' \subset \Omega$. One can refer to [7, 2] for the existence of η^0. In addition, we consider following weight functions:

$$\alpha(t, x) = (t(T - t))^{-1} \left(e^{2\lambda\|\eta^0\|_\infty} - e^{\lambda(\|\eta^0\|_\infty + \eta^0(x))} \right) \tag{3.2}$$

and

$$\xi(t,x) = (t(T-t))^{-1}\, e^{\lambda(\|\eta^0\|_\infty + \eta^0(x))} \tag{3.3}$$

with $\lambda > 0$ and where $\|.\|_\infty$ denotes a uniform norm. Also, for the sake of simplicity, we shall sometimes set

$$\Delta'(.) = \nabla.\mathcal{A}\nabla(.). \tag{3.4}$$

We easily derive following formulas:

$$\begin{array}{rcl} \nabla\xi &=& \lambda\xi\nabla(\eta^0),\ \nabla\alpha = -\lambda\xi\nabla(\eta^0),\quad \mathcal{A}\nabla\alpha = -\lambda\xi\mathcal{A}\nabla(\eta^0), \\ \Delta'(\alpha) &=& -\lambda^2\xi.\nabla(\eta^0).\mathcal{A}\nabla(\eta^0) - \lambda\xi\Delta'(\eta^0), \\ \dfrac{\partial\xi}{\partial t} &=& 2\,(T-t)^{-1}\,\xi - T\,(t(T-t))^{-1}\,\xi, \\ \dfrac{\partial\alpha}{\partial t} &=& 2t\,(t(T-t))^{-2}\, e^{2\lambda\|\eta^0\|_\infty} - T\,(t(T-t))^{-2}\, e^{2\lambda\|\eta^0\|_\infty} \\ && -2\,((T-t))^{-1}\,\xi + T\,(t(T-t))^{-1}\,\xi, \\ \dfrac{\partial^2\xi}{\partial t^2} &=& 2\,((t(T-t)) + (2t-T)^2)\,(t(T-t))^{-3}\, e^{\lambda(\|\eta^0\|_\infty + \eta^0)}, \\ \dfrac{\partial(\xi^2)}{\partial t} &=& 4t\,(T-t)^{-1}\,\xi^2 - 2T\,(t(T-t))^{-1}\,\xi^2. \end{array} \tag{3.5}$$

From which, it is not difficult to see that there exists a positive constant C (C may denote different constant) such that for $t \in (0,\ T)$ and for sufficiently larger value of λ, we have

$$\left|\frac{\partial\alpha}{\partial t}\right| \le C\xi^2,\ \left|\frac{\partial\xi}{\partial t}\right| \le C\xi^2,\ \left|\frac{\partial^2\xi}{\partial t^2}\right| \le C\xi^2, \left|\frac{\partial^2\alpha}{\partial t^2}\right| \le C\xi^3 \text{ and } |\Delta'(\alpha)| \le C\lambda^2\xi \tag{3.6}$$

We now define a sufficiently regular function that is

$$\widetilde{\psi}(t,x) = e^{-s\alpha(t,x)}\psi(t,x) \tag{3.7}$$

for some non negative real s to be chosen later. Such a function obviously satisfies

$$\lim_{t\to 0}\widetilde{\psi}(t,x) = \lim_{t\to T}\widetilde{\psi}(t,x) = 0. \tag{3.8}$$

Then, after some calculations, one can see that ψ is solution of (2.1) - (2.3) if and only if $\widetilde{\psi}$ is solution of the following equations system:

$$\begin{array}{rcl} s\dfrac{\partial\alpha}{\partial t}\widetilde{\psi} + \dfrac{\partial\widetilde{\psi}}{\partial t} + e^{-s\alpha}F\left(e^{s\alpha}\widetilde{\psi}\right) &=& s^2\widetilde{\psi}\nabla(\alpha).\mathcal{A}\nabla(\alpha) + 2s\nabla\left(\widetilde{\psi}\right).\mathcal{A}\nabla(\alpha) \quad \text{in } (0,T)\times\Omega \\ && +s\widetilde{\psi}\Delta'(\alpha) + \Delta'\left(\widetilde{\psi}\right) \end{array} \tag{3.9}$$

$$s\widetilde{\psi}\mathcal{A}\nabla(\alpha)\,\mathbf{n} + \mathcal{A}\nabla\left(\widetilde{\psi}\right)\mathbf{n} = e^{-s\alpha}g \quad \text{on } \Sigma_T \tag{3.10}$$

$$\widetilde{\psi}(0) = \widetilde{\psi}(T) = 0. \tag{3.11}$$

We rewrite the equation (3.9) as follows

$$L(\widetilde{\psi}) + G(\widetilde{\psi}) = 0 \text{ in } (0,T)\times\Omega \tag{3.12}$$

by setting

$$L(\widetilde{\psi}) := s\frac{\partial\alpha}{\partial t}\widetilde{\psi} - s\widetilde{\psi}\Delta'(\alpha) - s^2\widetilde{\psi}\nabla(\alpha).\mathcal{A}\nabla(\alpha) \tag{3.13}$$

and

$$G(\widetilde{\psi}) := \frac{\partial\widetilde{\psi}}{\partial t} + e^{-s\alpha}F\left(e^{s\alpha}\widetilde{\psi}\right) - 2s\nabla\left(\widetilde{\psi}\right).\mathcal{A}\nabla(\alpha) - \Delta'\left(\widetilde{\psi}\right). \tag{3.14}$$

Next, we shall set $\langle f,g\rangle = \displaystyle\int_0^T\int_\Omega fg.$

We now establish some preliminary results. Our first result is:

Proposition 3.1. *Assume that $0 < \lambda < \lambda_0$ and $0 < s < s_0$ for given λ_0 and s_0. Then, under hypotheses (2.5) and (2.6) there exists a positive constant C such that for every $\psi \in \mathcal{C}\left([0,\ T]; L^2(\Omega)\right) \bigcap L^2\left(0,T; H^1(\Omega)\right)$ and $g \in L^2\left(\Sigma_T\right)$ we have*

$$\begin{array}{rcl}\langle L(\widetilde{\psi}), G(\widetilde{\psi})\rangle & \geq & -Cs\lambda^2 \int_0^T \int_\Omega \xi e^{-2s\alpha} \left|\nabla\psi\right|^2 - Cs^2\lambda^3 \left(\int_0^T \int_\Omega \xi^2 e^{-2s\alpha}\psi^2 + \int_0^T \int_\Omega \xi^2 e^{-2s\alpha} \left|\nabla\psi\right|^2\right) \\ & & -Cs^3\lambda \left(\int_0^T \int_\Omega \xi^3 e^{-2s\alpha}\psi^2 + \int_0^T \int_\Omega \xi^3 e^{-2s\alpha} \left|\nabla\psi\right|^2\right) \\ & & -Cs^4\lambda^5 \int_0^T \int_\Omega \xi^4 e^{-2s\alpha}\psi^2 - Cs^5\lambda^3 \int_0^T \int_\Omega \xi^5 e^{-2s\alpha}\psi^2 \\ & & -Cs^2\lambda^2 \int_0^T \int_\Omega \xi^2 e^{-2s\alpha}\psi F\left(\psi\right) - Cs\lambda^2 \int_0^T \int_\Omega \xi e^{-2s\alpha}\psi F\left(\psi\right) \\ & & -Cs^2\lambda^2 \int_{\Sigma_T} \xi^2 e^{-2s\alpha} \left|\psi\right| \left|g\right| d\sigma dt \\ & & -Cs^3\lambda^3 \int_{\Sigma_T} \xi^3 e^{-2s\alpha}\psi^2 d\sigma dt. \end{array} \tag{3.15}$$

To prove the proposition 3.1 we need following lemmas.

Lemma 3.1. *Under hypotheses of the proposition 3.1, there exists a positive constant C such that*

$$\langle L(\widetilde{\psi}), \frac{\partial\widetilde{\psi}}{\partial t}\rangle \;\; \geq \;\; -C\left(s\lambda + s\lambda^2\right) \int_0^T \int_\Omega \xi^2 \widetilde{\psi}^2 - C(s + s^2\lambda^2) \int_0^T \int_\Omega \widetilde{\psi}^2 \xi^3. \tag{3.16}$$

Proof. Replacing $L(\widetilde{\psi})$ by its expression (3.13) one writes

$$\langle L(\widetilde{\psi}), \frac{\partial\widetilde{\psi}}{\partial t}\rangle \;\; = \;\; s\int_0^T \int_\Omega \frac{\partial\alpha}{\partial t}\widetilde{\psi}\frac{\partial\widetilde{\psi}}{\partial t} - s\int_0^T \int_\Omega \widetilde{\psi}\frac{\partial\widetilde{\psi}}{\partial t}\Delta'\left(\alpha\right) - s^2 \int_0^T \int_\Omega \widetilde{\psi}\frac{\partial\widetilde{\psi}}{\partial t}\nabla\left(\alpha\right).\mathcal{A}\nabla\left(\alpha\right).$$

Knowing that $\lim_{t\to 0}\widetilde{\psi}(t,x) = \lim_{t\to T}\widetilde{\psi}(t,x) = 0$ and integrating by parts over $(0,\ T)$ one obtains

$$\langle L(\widetilde{\psi}), \frac{\partial\widetilde{\psi}}{\partial t}\rangle \;\; = \;\; -\frac{s}{2}\int_0^T \int_\Omega \widetilde{\psi}^2\frac{\partial^2\alpha}{\partial t^2} + \frac{s}{2}\int_0^T \int_\Omega \widetilde{\psi}^2\frac{\partial}{\partial t}\Delta'\left(\alpha\right) + \frac{s^2}{2}\int_0^T \int_\Omega \widetilde{\psi}^2\frac{\partial}{\partial t}\left(\nabla\left(\alpha\right).\mathcal{A}\nabla\left(\alpha\right)\right).$$

We replace $\Delta'(\alpha)$ and $\nabla(\alpha)$ by theirs expressions in formulas (3.5) to obtain

$$\begin{array}{rcl}\langle L(\widetilde{\psi}), \frac{\partial\widetilde{\psi}}{\partial t}\rangle & = & -\frac{s}{2}\int_0^T \int_\Omega \frac{\partial^2\alpha}{\partial t^2}\widetilde{\psi}^2 - \frac{s\lambda^2}{2}\int_0^T \int_\Omega \frac{\partial\xi}{\partial t}\widetilde{\psi}^2\nabla(\eta^0).\mathcal{A}\nabla(\eta^0) \\ & & -\frac{s\lambda}{2}\int_0^T \int_\Omega \frac{\partial\xi}{\partial t}\widetilde{\psi}^2\Delta'\left(\eta^0\right) + \frac{s^2\lambda^2}{2}\int_0^T \int_\Omega \frac{\partial\xi^2}{\partial t}\widetilde{\psi}^2\left(\nabla\eta^0\right).\mathcal{A}\nabla\left(\eta^0\right).\end{array}$$

Using inequalities in (3.6) yealds that there exists a positive constant C such that

$$\langle L(\widetilde{\psi}), \frac{\partial\widetilde{\psi}}{\partial t}\rangle \;\; \geq \;\; -Cs\int_0^T \int_\Omega \xi^3\widetilde{\psi}^2 - C\left(s + s\lambda + s\lambda^2\right)\int_0^T \int_\Omega \xi^2\widetilde{\psi}^2 + \frac{s^2\lambda^2}{2}\int_0^T \int_\Omega \widetilde{\psi}^2\frac{\partial\xi^2}{\partial t}\left(\nabla\left(\eta^0\right).\mathcal{A}\nabla\left(\eta^0\right)\right).$$

We remark that

$$\frac{s^2\lambda^2}{2}\int_0^T \int_\Omega \widetilde{\psi}^2\frac{\partial\xi^2}{\partial t}\left(\nabla\left(\eta^0\right).\mathcal{A}\nabla\left(\eta^0\right)\right) = s^2\lambda^2\int_0^T \int_\Omega \widetilde{\psi}^2\xi\frac{\partial\xi}{\partial t}\left(\nabla\left(\eta^0\right).\mathcal{A}\nabla\left(\eta^0\right)\right)$$

and using again inequalities in (3.6), we obtain:

$$\left|s^2\lambda^2\int_0^T \int_\Omega \widetilde{\psi}^2\xi\frac{\partial\xi}{\partial t}\left(\nabla\left(\eta^0\right).\mathcal{A}\nabla\left(\eta^0\right)\right)\right| \;\; \leq \;\; Cs^2\lambda^2\int_0^T \int_\Omega \widetilde{\psi}^2\xi^3.$$

Consequently, it follows

$$\langle L(\widetilde{\psi}), \frac{\partial \widetilde{\psi}}{\partial t}\rangle \geq -C\left(s\lambda + s\lambda^2\right)\int_0^T\int_\Omega \xi^2\widetilde{\psi}^2 - C(s+s^2\lambda^2)\int_0^T\int_\Omega \widetilde{\psi}^2\xi^3$$

for some positive constant C. The lemma is then proved. □

Lemma 3.2. *Under hypotheses of the proposition 3.1, there exists a positive constant C such that*

$$\begin{aligned}\langle L(\widetilde{\psi}), e^{-s\alpha}F\left(e^{s\alpha}\widetilde{\psi}\right)\rangle \geq\ & -C(s+s^2\lambda^2)\int_0^T\int_\Omega \xi^2 e^{-s\alpha}\widetilde{\psi}F\left(e^{s\alpha}\widetilde{\psi}\right) \\ & -Cs\lambda^2\int_0^T\int_\Omega \xi e^{-s\alpha}\widetilde{\psi}F\left(e^{s\alpha}\widetilde{\psi}\right).\end{aligned} \tag{3.17}$$

Proof. We have

$$\begin{aligned}\langle L(\widetilde{\psi}), e^{-s\alpha}F\left(e^{s\alpha}\widetilde{\psi}\right)\rangle =\ & s\int_0^T\int_\Omega \frac{\partial\alpha}{\partial t}\widetilde{\psi}e^{-s\alpha}F\left(e^{s\alpha}\widetilde{\psi}\right) \\ & -s\int_0^T\int_\Omega \widetilde{\psi}e^{-s\alpha}F\left(e^{s\alpha}\widetilde{\psi}\right)\Delta'(\alpha) \\ & -s^2\int_0^T\int_\Omega \widetilde{\psi}e^{-s\alpha}F\left(e^{s\alpha}\widetilde{\psi}\right)\nabla(\alpha).\mathcal{A}\nabla(\alpha).\end{aligned}$$

From hypotheses (2.6) and (2.7) one can see that $e^{s\alpha}\widetilde{\psi}F\left(e^{s\alpha}\widetilde{\psi}\right) \geq 0$ and, thanks to (3.6), we obtain the following estimation:

$$\left|\int_0^T\int_\Omega \frac{\partial\alpha}{\partial t}\widetilde{\psi}e^{-s\alpha}F\left(e^{s\alpha}\widetilde{\psi}\right)\right| \leq C\int_0^T\int_\Omega \xi^2\widetilde{\psi}e^{-s\alpha}F\left(e^{s\alpha}\widetilde{\psi}\right).$$

Thus, we deduce

$$s\int_0^T\int_\Omega \frac{\partial\alpha}{\partial t}\widetilde{\psi}e^{-s\alpha}F\left(e^{s\alpha}\widetilde{\psi}\right) \geq -Cs\int_0^T\int_\Omega \xi^2\widetilde{\psi}e^{-s\alpha}F\left(e^{s\alpha}\widetilde{\psi}\right)$$

Consequently, using (3.5) and (3.6) one finally obtains

$$\begin{aligned}\langle L(\widetilde{\psi}), e^{-s\alpha}F\left(e^{s\alpha}\widetilde{\psi}\right)\rangle \geq\ & -C(s+s^2\lambda^2)\int_0^T\int_\Omega \xi^2\widetilde{\psi}e^{-s\alpha}F\left(e^{s\alpha}\widetilde{\psi}\right) \\ & -Cs\lambda^2\int_0^T\int_\Omega \xi\widetilde{\psi}e^{-s\alpha}F\left(e^{s\alpha}\widetilde{\psi}\right).\end{aligned}$$

Thus, we deduce the inequality (3.17). □

Lemma 3.3. *Under hypotheses of the proposition 3.1, there exists a positive constant C such that*

$$\begin{aligned}-2s\langle L(\widetilde{\psi}), \nabla\left(\widetilde{\psi}\right).\mathcal{A}\nabla(\alpha)\rangle \geq\ & -C\left(s^2\lambda^3+s^2\lambda^2\right)\left(\int_0^T\int_\Omega \xi^2\widetilde{\psi}^2 + \int_0^T\int_\Omega \xi^2\left|\nabla\left(\widetilde{\psi}\right)\right|^2\right) \\ & -C\left(s^2\lambda+s^3\lambda^3\right)\left(\int_0^T\int_\Omega \xi^3\widetilde{\psi}^2 + \int_0^T\int_\Omega \xi^3\left|\nabla\left(\widetilde{\psi}\right)\right|^2\right).\end{aligned} \tag{3.18}$$

Proof. We have

$$\begin{aligned}-2s\langle L(\widetilde{\psi}), \nabla\left(\widetilde{\psi}\right).\mathcal{A}\nabla(\alpha)\rangle =\ & -2s^2\int_0^T\int_\Omega \frac{\partial\alpha}{\partial t}\widetilde{\psi}\nabla\left(\widetilde{\psi}\right).\mathcal{A}\nabla(\alpha) \\ & +2s^2\int_0^T\int_\Omega \widetilde{\psi}\Delta'(\alpha)\nabla\left(\widetilde{\psi}\right).\mathcal{A}\nabla(\alpha) \\ & +2s^3\int_0^T\int_\Omega \widetilde{\psi}\nabla(\alpha).\mathcal{A}\nabla(\alpha)\nabla\left(\widetilde{\psi}\right).\mathcal{A}\nabla(\alpha).\end{aligned}$$

Thanks to (3.5) and (3.6), one obtains the following estimation:

$$\begin{aligned}\left|\int_0^T\int_\Omega \frac{\partial\alpha}{\partial t}\widetilde{\psi}\nabla\left(\widetilde{\psi}\right).\mathcal{A}\nabla\left(\alpha\right)\right| &\leq \int_0^T\int_\Omega\left|\frac{\partial\alpha}{\partial t}\right|\left|\widetilde{\psi}\right|\left|\nabla\left(\widetilde{\psi}\right)\right|\left|\mathcal{A}\nabla\left(\alpha\right)\right| \\ &\leq C\lambda\int_0^T\int_\Omega\xi^3\left|\widetilde{\psi}\right|\left|\nabla\left(\widetilde{\psi}\right)\right|.\end{aligned}$$

Applying Young's inequality yealds

$$-2s^2\int_0^T\int_\Omega\frac{\partial\alpha}{\partial t}\widetilde{\psi}\nabla\left(\widetilde{\psi}\right).\mathcal{A}\nabla\left(\alpha\right) \geq -Cs^2\lambda\left(\int_0^T\int_\Omega\xi^3\left|\widetilde{\psi}\right|^2+\int_0^T\int_\Omega\xi^3\left|\nabla\left(\widetilde{\psi}\right)\right|^2\right) \quad (i)$$

Also thanks to formulas (3.5), we can write

$$\int_0^T\int_\Omega\widetilde{\psi}\nabla\left(\alpha\right).\mathcal{A}\nabla\left(\alpha\right)\nabla\left(\widetilde{\psi}\right).\mathcal{A}\nabla\left(\alpha\right) = -\lambda^3\int_0^T\int_\Omega\xi^3\widetilde{\psi}\nabla\left(\eta^0\right).\mathcal{A}\nabla\left(\eta^0\right)\nabla\left(\widetilde{\psi}\right).\mathcal{A}\nabla\left(\eta^0\right).$$

Applying Young's inequality and formulas (3.6) one obtains

$$2s^3\int_0^T\int_\Omega\widetilde{\psi}\nabla\left(\alpha\right).\mathcal{A}\nabla\left(\alpha\right)\nabla\left(\widetilde{\psi}\right).\mathcal{A}\nabla\left(\alpha\right) \geq -Cs^3\lambda^3\left(\int_0^T\int_\Omega\xi^3\widetilde{\psi}^2+\int_0^T\int_\Omega\xi^3\left|\nabla\left(\widetilde{\psi}\right)\right|^2\right). \quad (ii)$$

Otherwise,

$$\begin{aligned}2s^2\int_0^T\int_\Omega\widetilde{\psi}\Delta'\left(\alpha\right)\nabla\left(\widetilde{\psi}\right).\mathcal{A}\nabla\left(\alpha\right) &= 2s^2\lambda^3\int_0^T\int_\Omega\widetilde{\psi}\xi^2.\nabla\left(\widetilde{\psi}\right).\mathcal{A}\nabla\left(\eta^0\right)\left(\nabla(\eta^0).\mathcal{A}\nabla(\eta^0)\right) \\ &+2s^2\lambda^2\int_0^T\int_\Omega\widetilde{\psi}\xi^2\nabla\left(\widetilde{\psi}\right).\mathcal{A}\nabla\left(\eta^0\right)\Delta'(\eta^0).\end{aligned}$$

We apply again Young's inequality and formulas (3.6) to obtain

$$\begin{aligned}2s^2\int_0^T\int_\Omega\widetilde{\psi}\Delta'\left(\alpha\right)\nabla\left(\widetilde{\psi}\right).\mathcal{A}\nabla\left(\alpha\right) &\geq -Cs^2\lambda^3\left(\int_0^T\int_\Omega\xi^2\widetilde{\psi}^2+\int_0^T\int_\Omega\xi^2\left|\nabla\left(\widetilde{\psi}\right)\right|^2\right) \\ &-Cs^2\lambda^2\left(\int_0^T\int_\Omega\xi^2\widetilde{\psi}^2+\int_0^T\int_\Omega\xi^2\left|\nabla\left(\widetilde{\psi}\right)\right|^2\right).\end{aligned} \quad (iii)$$

Combining (i), (ii), and (iii), we finally obtain

$$\begin{aligned}-2s\langle L(\widetilde{\psi}),\nabla\left(\widetilde{\psi}\right).\mathcal{A}\nabla\left(\alpha\right)\rangle &\geq -C\left(s^2\lambda^3+s^2\lambda^2\right)\left(\int_0^T\int_\Omega\xi^2\widetilde{\psi}^2+\int_0^T\int_\Omega\xi^2\left|\nabla\left(\widetilde{\psi}\right)\right|^2\right) \\ &-C\left(s^2\lambda+s^3\lambda^3\right)\left(\int_0^T\int_\Omega\xi^3\widetilde{\psi}^2+\int_0^T\int_\Omega\xi^3\left|\nabla\left(\widetilde{\psi}\right)\right|^2\right).\end{aligned}$$

This achieves the proof. □

Lemma 3.4. *Under hypotheses of the proposition 3.1, we have*

$$\begin{aligned}-\langle L(\widetilde{\psi}),\Delta'\left(\widetilde{\psi}\right)\rangle &\geq -Cs\lambda^2\int_0^T\int_\Omega\xi\left|\nabla\widetilde{\psi}\right|^2-C(s+s^2\lambda^2)\int_0^T\int_\Omega\xi^2|\nabla\widetilde{\psi}|^2 \\ &-C(s\lambda+s\lambda^2+s\lambda^3+s^2\lambda^3)\left(\int_0^T\int_\Omega\xi^2\widetilde{\psi}^2+\int_0^T\int_\Omega\xi^2\left|\nabla\widetilde{\psi}\right|^2\right) \\ &-Cs\lambda^2\int_{\Sigma_T}\xi\left|\widetilde{\psi}\right|e^{-s\alpha}|g|d\sigma dt-C(s+s^2\lambda^2)\int_{\Sigma_T}\xi^2\left|\widetilde{\psi}\right|e^{-s\alpha}|g|d\sigma dt \\ &-Cs^2\lambda^3\int_{\Sigma_T}\xi\widetilde{\psi}^2d\sigma dt-Cs^3\lambda^3\int_{\Sigma_T}\xi^2\widetilde{\psi}^2d\sigma dt.\end{aligned} \quad (3.19)$$

Proof. We have

$$-\langle L(\widetilde{\psi}), \Delta'\left(\widetilde{\psi}\right)\rangle = -s\int_0^T\int_\Omega \frac{\partial\alpha}{\partial t}\widetilde{\psi}\Delta'\left(\widetilde{\psi}\right) + s\int_0^T\int_\Omega \widetilde{\psi}\Delta'\left(\alpha\right)\Delta'\left(\widetilde{\psi}\right) + s^2\int_0^T\int_\Omega \widetilde{\psi}\Delta'\left(\widetilde{\psi}\right)\nabla\left(\alpha\right).\mathcal{A}\nabla\left(\alpha\right).$$

The use of Green's formula gives

$$\begin{aligned}
-\langle L(\widetilde{\psi}), \Delta'\left(\widetilde{\psi}\right)\rangle = & s\int_0^T\int_\Omega \frac{\partial\alpha}{\partial t}\nabla\widetilde{\psi}.\mathcal{A}\nabla\left(\widetilde{\psi}\right) - s\lambda\int_0^T\int_\Omega \frac{\partial\xi}{\partial t}\widetilde{\psi}.\mathcal{A}\nabla\left(\widetilde{\psi}\right).\nabla(\eta^0)\\
& -s\int_{\Sigma_T}\frac{\partial\alpha}{\partial t}\widetilde{\psi}\mathcal{A}\nabla\left(\widetilde{\psi}\right)nds - s\int_0^T\int_\Omega \Delta'\left(\alpha\right)\nabla\widetilde{\psi}.\mathcal{A}\nabla\left(\widetilde{\psi}\right)\\
& -s\int_0^T\int_\Omega \widetilde{\psi}\mathcal{A}\nabla\left(\widetilde{\psi}\right).\nabla\left(\Delta'\left(\alpha\right)\right) + s\int_{\Sigma_T}\Delta'\left(\alpha\right)\widetilde{\psi}\mathcal{A}\nabla\left(\widetilde{\psi}\right).nd\sigma dt\\
& -s^2\int_0^T\int_\Omega \nabla\widetilde{\psi}.\mathcal{A}\nabla\left(\widetilde{\psi}\right)\nabla\left(\alpha\right).\mathcal{A}\nabla\left(\alpha\right)\\
& -s^2\int_0^T\int_\Omega \widetilde{\psi}\mathcal{A}\nabla\left(\widetilde{\psi}\right).\nabla\left(\nabla\left(\alpha\right).\mathcal{A}\nabla\left(\alpha\right)\right)\\
& +s^2\int_{\Sigma_T}\nabla\left(\alpha\right).\mathcal{A}\nabla\left(\alpha\right)\widetilde{\psi}\mathcal{A}\nabla\left(\widetilde{\psi}\right).nd\sigma dt.
\end{aligned}$$

We assume that this equality can be written in the form $-\langle L(\widetilde{\psi}), \Delta'\left(\widetilde{\psi}\right)\rangle = (a)+(b)+(c)+(d)$ where

$$\begin{aligned}
(a) &= s\int_0^T\int_\Omega \frac{\partial\alpha}{\partial t}\nabla\widetilde{\psi}.\mathcal{A}\nabla\left(\widetilde{\psi}\right) - s\int_0^T\int_\Omega \Delta'\left(\alpha\right)\nabla\widetilde{\psi}.\mathcal{A}\nabla\left(\widetilde{\psi}\right) - s^2\int_0^T\int_\Omega \nabla\widetilde{\psi}.\mathcal{A}\nabla\left(\widetilde{\psi}\right)\nabla\left(\alpha\right).\mathcal{A}\nabla\left(\alpha\right)\\
(b) &= -s^2\int_0^T\int_\Omega \widetilde{\psi}\mathcal{A}\nabla\left(\widetilde{\psi}\right).\nabla\left(\nabla\left(\alpha\right).\mathcal{A}\nabla\left(\alpha\right)\right) - s\lambda\int_0^T\int_\Omega \frac{\partial\xi}{\partial t}\widetilde{\psi}.\mathcal{A}\nabla\left(\widetilde{\psi}\right).\nabla(\eta^0)\\
(c) &= -s\int_0^T\int_\Omega \widetilde{\psi}\mathcal{A}\nabla\left(\widetilde{\psi}\right).\nabla\left(\Delta'\left(\alpha\right)\right)\\
(d) &= -s\int_{\Sigma_T}\frac{\partial\alpha}{\partial t}\widetilde{\psi}\mathcal{A}\nabla\left(\widetilde{\psi}\right)nd\sigma dt + s\int_{\Sigma_T}\Delta'\left(\alpha\right)\widetilde{\psi}\mathcal{A}\nabla\left(\widetilde{\psi}\right).nd\sigma dt + s^2\int_{\Sigma_T}\nabla\left(\alpha\right).\mathcal{A}\nabla\left(\alpha\right)\widetilde{\psi}\mathcal{A}\nabla\left(\widetilde{\psi}\right).nd\sigma
\end{aligned}$$

And, we estimate each term of the above expression. From inequalities in formulas (3.6) and knowing that the operator $\mathcal{A}$ is positive, we successively deduce

$$\left|\int_0^T\int_\Omega \frac{\partial\alpha}{\partial t}\nabla\widetilde{\psi}.\mathcal{A}\nabla\left(\widetilde{\psi}\right)\right| \leq C\int_0^T\int_\Omega \xi^2|\nabla\widetilde{\psi}|^2,$$

$$\left|\int_0^T\int_\Omega \Delta'\left(\alpha\right)\nabla\widetilde{\psi}.\mathcal{A}\nabla\left(\widetilde{\psi}\right)\right| \leq C\lambda^2\int_0^T\int_\Omega \xi\left|\nabla\widetilde{\psi}\right|^2,$$

$$\left|\int_0^T\int_\Omega \nabla\widetilde{\psi}.\mathcal{A}\nabla\left(\widetilde{\psi}\right)\nabla\left(\alpha\right).\mathcal{A}\nabla\left(\alpha\right)\right| \leq C\lambda^2\int_0^T\int_\Omega \xi^2\left|\nabla\widetilde{\psi}\right|^2.$$

It then follows

$$(a) \geq -Cs\lambda^2\int_0^T\int_\Omega \xi\left|\nabla\widetilde{\psi}\right|^2 - C(s+s^2\lambda^2)\int_0^T\int_\Omega \xi^2|\nabla\widetilde{\psi}|^2 \quad (i)$$

To estimate (b) ones first evaluate $\nabla\left(\nabla\left(\alpha\right).\mathcal{A}\nabla\left(\alpha\right)\right)$. In fact from (3.5), we have

$$\begin{aligned}
\nabla\left(\nabla\left(\alpha\right).\mathcal{A}\nabla\left(\alpha\right)\right) &= \lambda^2\nabla\left(\xi^2\nabla\left(\eta^0\right).\mathcal{A}\nabla\left(\eta^0\right)\right)\\
&= 2\lambda^2\xi\nabla(\xi)\nabla\left(\eta^0\right).\mathcal{A}\nabla\left(\eta^0 + \lambda^2\xi^2\nabla\left(\nabla\left(\eta^0\right).\mathcal{A}\nabla\left(\eta^0\right)\right)\right)\\
&= 2\lambda^3\xi^2\nabla(\eta^0)\nabla\left(\eta^0\right).\mathcal{A}\nabla\left(\eta^0 + \lambda^2\xi^2\nabla\left(\nabla\left(\eta^0\right).\mathcal{A}\nabla\left(\eta^0\right)\right)\right)
\end{aligned}$$

Thanks to Young inequality and for larger values of λ one obtains

$$\left|\int_0^T\int_\Omega \widetilde{\psi}\mathcal{A}\nabla\left(\widetilde{\psi}\right).\nabla\left(\nabla\left(\alpha\right).\mathcal{A}\nabla\left(\alpha\right)\right)\right|\leq C\lambda^3\left(\int_0^T\int_\Omega \xi^2\widetilde{\psi}^2+\int_0^T\int_\Omega \xi^2\left|\nabla\widetilde{\psi}\right|^2\right).$$

As done previously, we also have

$$\left|\int_0^T\int_\Omega \frac{\partial\xi}{\partial t}\widetilde{\psi}.\mathcal{A}\nabla\left(\widetilde{\psi}\right).\nabla(\eta^0)\right|\leq C\left(\int_0^T\int_\Omega \xi^2\widetilde{\psi}^2+\int_0^T\int_\Omega \xi^2\left|\nabla\widetilde{\psi}\right|^2\right)$$

so that we can state

$$(b)\quad\geq\quad -C(s^2\lambda^3+s\lambda)\left(\int_0^T\int_\Omega \xi^2\widetilde{\psi}^2+\int_0^T\int_\Omega \xi^2\left|\nabla\widetilde{\psi}\right|^2\right)\qquad (ii)$$

To estimate (c), we have to evaluate $\nabla\left(\Delta'\left(\alpha\right)\right).$ Since

$$\Delta'\left(\alpha\right)=-\lambda^2\xi.\nabla(\eta^0).\mathcal{A}\nabla(\eta^0)-\lambda\xi\Delta'(\eta^0),$$

we have

$$\nabla\left(\Delta'\left(\alpha\right)\right)=-\lambda^3\xi^2\left(\nabla(\eta^0).\mathcal{A}\nabla(\eta^0)\right)\nabla(\eta^0)-\lambda^2\xi^2\Delta'(\eta^0)\nabla(\eta^0).$$

Applying again Young inequality ones finally obtains

$$(c)\quad\geq\quad -C(s\lambda^3+s\lambda^2)\left(\int_0^T\int_\Omega \xi^2\widetilde{\psi}^2+\int_0^T\int_\Omega \xi^2\left|\nabla\widetilde{\psi}\right|^2\right).\qquad (iii)$$

Also, we know from (3.10) that

$$\mathcal{A}\nabla\left(\widetilde{\psi}\right)\mathbf{n}=e^{-s\alpha}g+s\lambda\widetilde{\psi}\mathcal{A}\nabla\left(\eta^0\right)\mathbf{n}\text{ on }\Sigma_T.$$

Then we have

$$-s\int_{\Sigma_T}\frac{\partial\alpha}{\partial t}\widetilde{\psi}\mathcal{A}\nabla\left(\widetilde{\psi}\right)\mathbf{n}d\sigma dt=-s\int_{\Sigma_T}\frac{\partial\alpha}{\partial t}\widetilde{\psi}e^{-s\alpha}g\mathbf{n}d\sigma dt-s^2\lambda\int_{\Sigma_T}\frac{\partial\alpha}{\partial t}\widetilde{\psi}^2\mathcal{A}\nabla\left(\eta^0\right)\mathbf{n}d\sigma dt$$

$$s\int_{\Sigma_T}\Delta'\left(\alpha\right)\widetilde{\psi}\mathcal{A}\nabla\left(\widetilde{\psi}\right)\mathbf{n}d\sigma dt=s\int_{\Sigma_T}\Delta'\left(\alpha\right)\widetilde{\psi}e^{-s\alpha}g\mathbf{n}d\sigma dt+s^2\lambda\int_{\Sigma_T}\Delta'\left(\alpha\right)\widetilde{\psi}^2\mathcal{A}\nabla\left(\eta^0\right)\mathbf{n}d\sigma dt$$

and

$$\begin{aligned}s^2\int_{\Sigma_T}\nabla\left(\alpha\right).\mathcal{A}\nabla\left(\alpha\right)\widetilde{\psi}\mathcal{A}\nabla\left(\widetilde{\psi}\right)\mathbf{n}d\sigma dt\quad&=\quad s^2\int_{\Sigma_T}\nabla\left(\alpha\right).\mathcal{A}\nabla\left(\alpha\right)\widetilde{\psi}e^{-s\alpha}g\mathbf{n}d\sigma dt\\&\quad+s^3\lambda\int_{\Sigma_T}\nabla\left(\alpha\right).\mathcal{A}\nabla\left(\alpha\right)\widetilde{\psi}^2\mathcal{A}\nabla\left(\eta^0\right)\mathbf{n}d\sigma dt\end{aligned}$$

so that we deduce the following inequality:

$$\begin{aligned}(d)\quad\geq\quad &-Cs\lambda^2\int_{\Sigma_T}\xi\left|\widetilde{\psi}\right|e^{-s\alpha}|g|d\sigma dt-C(s+s^2\lambda^2)\int_{\Sigma_T}\xi^2\left|\widetilde{\psi}\right|e^{-s\alpha}|g|d\sigma dt-Cs^2\lambda^3\int_{\Sigma_T}\xi\widetilde{\psi}^2d\sigma dt\\&-Cs^3\lambda^3\int_{\Sigma_T}\xi^2\widetilde{\psi}^2d\sigma dt\end{aligned}\qquad (iv)$$

Finally, (i), (ii), (iii) and (iv) yealds

$$
\begin{aligned}
(a)+(b)+(c)+(d) \geq\; & -Cs\lambda^2 \int_0^T \int_\Omega \xi \left|\nabla\widetilde{\psi}\right|^2 - C(s+s^2\lambda^2)\int_0^T\int_\Omega \xi^2 |\nabla\widetilde{\psi}|^2 \\
& -C(s\lambda + s\lambda^2 + s\lambda^3 + s^2\lambda^3)\left(\int_0^T\int_\Omega \xi^2\widetilde{\psi}^2 + \int_0^T\int_\Omega \xi^2\left|\nabla\widetilde{\psi}\right|^2\right) \\
& -Cs\lambda^2 \int_{\Sigma_T} \xi \left|\widetilde{\psi}\right| e^{-s\alpha}|g| d\sigma dt - C(s+s^2\lambda^2)\int_{\Sigma_T}\xi^2\left|\widetilde{\psi}\right| e^{-s\alpha}|g|d\sigma dt \\
& -Cs^2\lambda^3 \int_{\Sigma_T}\xi\widetilde{\psi}^2 d\sigma dt - Cs^3\lambda^3\int_{\Sigma_T}\xi^2\widetilde{\psi}^2 d\sigma dt.
\end{aligned}
$$

This achieves the proof. □

Proof of the proposition 3.1. Firstly, replacing $\widetilde{\psi}$ by its expression in (3.7) yealds

$$
\begin{aligned}
\left|\nabla\widetilde{\psi}\right|^2 &= |-se^{-s\alpha}\psi\nabla\alpha + e^{-s\alpha}\nabla\psi|^2 \\
&\leq s^2 e^{-2s\alpha}|\psi|^2|\nabla\alpha|^2 + 2se^{-2s\alpha}|\psi||\nabla\alpha||\nabla\psi| + e^{-2s\alpha}|\nabla\psi|^2 \\
&\leq Ce^{-2s\alpha}\left(s^2|\psi|^2|\nabla\alpha|^2 + 2s|\psi||\nabla\alpha||\nabla\psi| + |\nabla\psi|^2\right)
\end{aligned}
$$

and thanks to Young inequality ones obtains

$$
\begin{aligned}
\left|\nabla\widetilde{\psi}\right|^2 &\leq Ce^{-2s\alpha}\left(s^2|\psi|^2|\nabla\alpha|^2 + |\nabla\psi|^2\right) \\
&\leq Ce^{-2s\alpha}\left(s^2\lambda^2\xi^2|\psi|^2|\nabla\eta^0|^2 + |\nabla\psi|^2\right).
\end{aligned}
$$

Secondly, from relations (3.13) and (3.14), we can write

$$
\langle L(\widetilde{\psi}), G(\widetilde{\psi})\rangle = \langle L(\widetilde{\psi}), \frac{\partial\widetilde{\psi}}{\partial t}\rangle + \langle L(\widetilde{\psi}), e^{-s\alpha}F\left(e^{s\alpha}\widetilde{\psi}\right)\rangle - 2s\langle L(\widetilde{\psi}), \nabla\left(\widetilde{\psi}\right).\mathcal{A}\nabla(\alpha)\rangle - \langle L(\widetilde{\psi}), \Delta'\left(\widetilde{\psi}\right)\rangle
$$

Thus, using inequalities (3.16) to (3.19) ones obtains

$$
\begin{aligned}
\langle L(\widetilde{\psi}), G(\widetilde{\psi})\rangle \geq\; & -C\left(s\lambda + s\lambda^2 + s^2\lambda^2 + s^2\lambda^3 + s\lambda + s^2\lambda^2 + s\lambda^3\right)\int_0^T\int_\Omega \xi^2 e^{-2s\alpha}\psi^2 \\
& -Cs\lambda^2\int_0^T\int_\Omega \xi e^{-2s\alpha}|\nabla\psi|^2 \\
& -C\left(s + s^2\lambda^2 + s^2\lambda^3 + s\lambda + s^2\lambda^2\right)\int_0^T\int_\Omega \xi^2 e^{-2s\alpha}|\nabla\psi|^2 \\
& -C\left(s^3\lambda^2 + s^4\lambda^4 + s^4\lambda^5 + s^3\lambda^3 + s^4\lambda^4\right)\int_0^T\int_\Omega \xi^4 e^{-2s\alpha}\psi^2 \\
& -C(s + s^2\lambda^2 + s^2\lambda + s^3\lambda^3)\int_0^T\int_\Omega \xi^3 e^{-2s\alpha}\psi^2 \\
& -C\left(s^2\lambda + s^3\lambda^3\right)\int_0^T\int_\Omega \xi^3 e^{-2s\alpha}|\nabla\psi|^2 \\
& -C\left(s^4\lambda^3 + s^5\lambda^3\right)\int_0^T\int_\Omega \xi^5 e^{-2s\alpha}\psi^2 \\
& -C(s+s^2\lambda^2)\int_0^T\int_\Omega \xi^2 e^{-2s\alpha}\psi F(\psi) - Cs\lambda^2\int_0^T\int_\Omega \xi e^{-2s\alpha}\psi F(\psi) \\
& -C\left(s + s\lambda^2 + s\lambda^2 + s^2\lambda^2\right)\int_{\Sigma_T}\xi^2 e^{-2s\alpha}|\psi||g|d\sigma dt \\
& -C\left(s^2\lambda + s^2\lambda^2 + s^2\lambda^3 + s^3\lambda^3\right)\int_{\Sigma_T}\xi^3 e^{-2s\alpha}\psi^2 d\sigma dt.
\end{aligned}
$$

Since we have taken s and λ large enough this inequality becomes

$$\begin{aligned}
\langle L(\widetilde{\psi}), G(\widetilde{\psi})\rangle \geq & -Cs\lambda^2 \int_0^T \int_\Omega \xi e^{-2s\alpha} |\nabla\psi|^2 - Cs^2\lambda^3 \left(\int_0^T \int_\Omega \xi^2 e^{-2s\alpha} \psi^2 + \int_0^T \int_\Omega \xi^2 e^{-2s\alpha} |\nabla\psi|^2 \right) \\
& -Cs^3\lambda^3 \left(\int_0^T \int_\Omega \xi^3 e^{-2s\alpha} \psi^2 + \int_0^T \int_\Omega \xi^3 e^{-2s\alpha} |\nabla\psi|^2 \right) \\
& -Cs^4\lambda^5 \int_0^T \int_\Omega \xi^4 e^{-2s\alpha} \psi^2 - Cs^5\lambda^3 \int_0^T \int_\Omega \xi^5 e^{-2s\alpha} \psi^2 \\
& -Cs^2\lambda^2 \int_0^T \int_\Omega \xi^2 e^{-2s\alpha} \psi F(\psi) - Cs\lambda^2 \int_0^T \int_\Omega \xi e^{-2s\alpha} \psi F(\psi) \\
& -Cs^2\lambda^2 \int_{\Sigma_T} \xi^2 e^{-2s\alpha} |\psi| \, |g| d\sigma dt \\
& -Cs^3\lambda^3 \int_{\Sigma_T} \xi^3 e^{-2s\alpha} \psi^2 d\sigma dt.
\end{aligned}$$

This achieves the proof. □

Proposition 3.2. *Assume that $0 < \lambda < \lambda_0$ and $0 < s < s_0$ for given λ_0 and s_0. Then there exists a positive constant C such that for every $\psi \in \mathcal{C}([0,\ T]; L^2(\Omega)) \bigcap L^2(0, T; H^1(\Omega))$ we have*

$$\begin{aligned}
\langle L(\widetilde{\psi}),\ L(\widetilde{\psi})\rangle \geq & \ Cs^2\lambda^4 \int_0^T \int_\Omega \xi^2 \widetilde{\psi}^2 + Cs^4\lambda^4 \int_0^T \int_\Omega \xi^4 e^{-s\alpha} \psi^2 \\
& -Cs^3\lambda^2 \int_0^T \int_\Omega \xi^4 e^{-s\alpha} \psi^2 - Cs^3\lambda^4 \int_0^T \int_\Omega \xi^3 e^{-s\alpha} \psi^2.
\end{aligned} \tag{3.20}$$

Proof. We have

$$\begin{aligned}
\langle L(\widetilde{\psi}),\ L(\widetilde{\psi})\rangle = & \ \|L(\widetilde{\psi})\|^2 = \|s \frac{\partial\alpha}{\partial t} \widetilde{\psi} - s\widetilde{\psi}\Delta'(\alpha) - s^2 \widetilde{\psi} \nabla(\alpha) . \mathcal{A}\nabla(\alpha) \|^2 \\
= & \ s^2 \int_0^T \int_\Omega \left(\frac{\partial\alpha}{\partial t}\right)^2 \widetilde{\psi}^2 + s^2 \int_0^T \int_\Omega \widetilde{\psi}^2 (\Delta'(\alpha))^2 + s^4 \int_0^T \int_\Omega \widetilde{\psi}^2 (\nabla(\alpha) . \mathcal{A}\nabla(\alpha))^2 \\
& -2s^2 \int_0^T \int_\Omega \frac{\partial\alpha}{\partial t} \widetilde{\psi}^2 \Delta'(\alpha) - 2s^3 \int_0^T \int_\Omega \frac{\partial\alpha}{\partial t} \widetilde{\psi}^2 \nabla(\alpha) . \mathcal{A}\nabla(\alpha) \\
& +2s^3 \int_0^T \int_\Omega \Delta'(\alpha) \widetilde{\psi}^2 \nabla(\alpha) . \mathcal{A}\nabla(\alpha).
\end{aligned}$$

We know that

$$s^2 \int_0^T \int_\Omega \left(\frac{\partial\alpha}{\partial t}\right)^2 \widetilde{\psi}^2 \geq 0$$

Also, for larger value of λ, from the last inequality in (3.6), we can write

$$(\Delta'\alpha)^2 \sim C\lambda^4\xi^2,$$

thus

$$s^2 \int_0^T \int_\Omega \widetilde{\psi}^2 (\Delta'(\alpha))^2 \geq Cs^2\lambda^4 \int_0^T \int_\Omega \xi^2 \widetilde{\psi}^2.$$

As $\mathcal{A}$ is positive defined and satisfies the ellipticity property, we then obtain the following inequality:

$$s^4 \int_0^T \int_\Omega \widetilde{\psi}^2 (\nabla(\alpha) . \mathcal{A}\nabla(\alpha))^2 \geq Cs^4\lambda^4 \int_0^T \int_\Omega \xi^4 \widetilde{\psi}^2.$$

From (3.1), we have

$$\left| \int_0^T \int_\Omega \frac{\partial\alpha}{\partial t} \widetilde{\psi}^2 \Delta'(\alpha) \right| \leq C\lambda^2 \int_0^T \int_\Omega \xi^3 \widetilde{\psi}^2,$$

thus

$$-2s^2\int_0^T\int_\Omega \frac{\partial\alpha}{\partial t}\widetilde{\psi}^2\Delta'(\alpha)\geq -Cs^2\lambda^2\int_0^T\int_\Omega \xi^3\widetilde{\psi}^2.$$

In the same way, ones successively obtains

$$-2s^3\int_0^T\int_\Omega \frac{\partial\alpha}{\partial t}\widetilde{\psi}^2\nabla(\alpha).\mathcal{A}\nabla(\alpha)\geq -Cs^3\lambda^2\int_0^T\int_\Omega \xi^4\widetilde{\psi}^2$$

and

$$2s^3\int_0^T\int_\Omega \Delta'(\alpha)\widetilde{\psi}^2\nabla(\alpha).\mathcal{A}\nabla(\alpha)\geq -Cs^3\lambda^4\int_0^T\int_\Omega \xi^3\widetilde{\psi}^2.$$

We finally deduce

$$\begin{array}{rcl}\langle L(\widetilde{\psi}),\ L(\widetilde{\psi})\rangle & \geq & Cs^2\lambda^4\int_0^T\int_\Omega \xi^2\widetilde{\psi}^2+Cs^4\lambda^4\int_0^T\int_\Omega \xi^4\widetilde{\psi}^2\\ && -Cs^2\lambda^2\int_0^T\int_\Omega \xi^3\widetilde{\psi}^2-Cs^3\lambda^2\int_0^T\int_\Omega \xi^4\widetilde{\psi}^2\\ && -Cs^3\lambda^4\int_0^T\int_\Omega \xi^3\widetilde{\psi}^2.\end{array}$$

For larger values of λ and s, we have

$$-Cs^2\lambda^2\int_0^T\int_\Omega \xi^3\widetilde{\psi}^2-Cs^3\lambda^4\int_0^T\int_\Omega \xi^3\widetilde{\psi}^2\geq -Cs^3\lambda^4\int_0^T\int_\Omega \xi^3\widetilde{\psi}^2.$$

The proof is then achieved by replacing $\widetilde{\psi}$ by its expression. □

4 The main result

The main result is given by the theorem below.

Theorem 4.1. *Assume that hypotheses* (2.4), (2.5), (2.6) *and* (2.7) *hold. Then there exists a non negative constant* C *depending of* Ω, ω *and* T *such that for every*

$$\psi\in\mathcal{C}\left([0,\ T];L^2(\Omega)\right)\bigcap L^2\left(0,T;H^1(\Omega)\right)$$

solution of (2.1), (2.2) *and* (2.3) *with* $g\in L^2(\Sigma_T)$, *we have*

$$\begin{array}{rcl}s^2\lambda^4\int_0^T\int_\Omega \xi^2e^{-s\alpha}\psi^2+s^4\lambda^4\int_0^T\int_\Omega \xi^4e^{-s\alpha}\psi^2 & \leq & Cs^2\lambda^3\left(\int_0^T\int_\Omega \xi^2e^{-2s\alpha}\psi^2+\int_0^T\int_\Omega \xi^2e^{-2s\alpha}\left|\nabla\psi\right|^2\right)\\ && +Cs\lambda^2\int_0^T\int_\Omega \xi\left|\nabla\psi\right|^2+Cs^3\lambda^4\int_0^T\int_\Omega \xi^3e^{-2s\alpha}\psi^2\\ && +Cs^3\lambda^3\int_0^T\int_\Omega \xi^3e^{-2s\alpha}\left|\nabla\psi\right|^2\\ && +Cs^4\lambda^5\int_0^T\int_\Omega \xi^4e^{-2s\alpha}\psi^2\\ && +Cs^5\lambda^3\int_0^T\int_\Omega \xi^5e^{-2s\alpha}\psi^2\\ && +Cs^3\lambda^2\int_0^T\int_\Omega \xi^4e^{-s\alpha}\psi^2+Cs^3\lambda^3\int_0^T\int_\Omega \xi^3e^{-s\alpha}\psi^2\\ && +Cs^2\lambda^2\int_0^T\int_\Omega \xi^2e^{-2s\alpha}\psi F(\psi)\\ && +Cs\lambda^2\int_0^T\int_\Omega \xi e^{-2s\alpha}\psi F(\psi)\\ && +Cs^2\lambda^2\int_{\Sigma_T}\xi^2e^{-2s\alpha}\left|\psi\right|\left|g\right|d\sigma dt\\ && +Cs^3\lambda^3\int_{\Sigma_T}\xi^3e^{-2s\alpha}\psi^2d\sigma dt\end{array}\tag{4.1}$$

where C may denote different constant.

Proof. Knowing that

$$\|L(\widetilde{\psi})\|^2 + 2\langle L(\widetilde{\psi}), G\left(\widetilde{\psi}\right)\rangle + \|G(\widetilde{\psi})\|^2 = 0$$

its follows

$$\|L(\widetilde{\psi})\|^2 + 2\langle L(\widetilde{\psi}), G\left(\widetilde{\psi}\right)\rangle \leq 0$$

and combining inequalities (3.15) and (3.20) of propositions (3.1) and (3.2), we deduce

$$\begin{aligned}
0 \geq \|L(\widetilde{\psi})\|^2 + 2\langle L(\widetilde{\psi}), G\left(\widetilde{\psi}\right)\rangle \geq & -Cs^2\lambda^3\left(\int_0^T\int_\Omega \xi^2 e^{-2s\alpha}\psi^2 + \int_0^T\int_\Omega \xi^2 e^{-2s\alpha}\left|\nabla\psi\right|^2\right) \\
& -Cs\lambda^2\int_0^T\int_\Omega \xi\left|\nabla\psi\right|^2 - Cs^3\lambda^4\int_0^T\int_\Omega \xi^3 e^{-2s\alpha}\psi^2 \\
& -Cs^3\lambda^3\int_0^T\int_\Omega \xi^3 e^{-2s\alpha}\left|\nabla\psi\right|^2 \\
& -Cs^4\lambda^5\int_0^T\int_\Omega \xi^4 e^{-2s\alpha}\psi^2 - Cs^5\lambda^3\int_0^T\int_\Omega \xi^5 e^{-2s\alpha}\psi^2 \\
& -Cs^2\lambda^2\int_0^T\int_\Omega \xi^2 e^{-2s\alpha}\psi F\left(\psi\right) - Cs\lambda^2\int_0^T\int_\Omega \xi e^{-2s\alpha}\psi F\left(\psi\right) \\
& -Cs^2\lambda^2\int_{\Sigma_T} \xi^2 e^{-2s\alpha}\left|\psi\right|\left|g\right| d\sigma dt \\
& -Cs^3\lambda^3\int_{\Sigma_T} \xi^3 e^{-2s\alpha}\psi^2 d\sigma dt \\
& +Cs^2\lambda^4\int_0^T\int_\Omega \xi^2 e^{-s\alpha}\psi^2 + Cs^4\lambda^4\int_0^T\int_\Omega \xi^4 e^{-s\alpha}\psi^2 \\
& -Cs^3\lambda^2\int_0^T\int_\Omega \xi^4 e^{-s\alpha}\psi^2 - Cs^3\lambda^3\int_0^T\int_\Omega \xi^3 e^{-s\alpha}\psi^2.
\end{aligned}$$

This finally implies the inequality (4.1) and the proof is achieved. □

From the above result one can deduce the following estimate.

Corollaire 4.2. *Under hypotheses of the theorem* (4.1)*, we have:*

$$\begin{aligned}
s^2\lambda^4\int_0^T\int_\Omega \xi^2 e^{-s\alpha}\psi^2 \leq & \; Cs^5\lambda^5\left(\int_0^T\int_\Omega \xi^5 e^{-s\alpha}\psi^2 + \int_0^T\int_\Omega \xi^5 e^{-s\alpha}\left|\nabla\psi\right|^2\right) \\
& +Cs^2\lambda^2\int_0^T\int_\Omega \xi^2 e^{-s\alpha}\psi F\left(\psi\right) \\
& +Cs^2\lambda^2\int_{\Sigma_T} \xi^2 e^{-s\alpha}\left|\psi\right|\left|g\right| d\sigma dt \\
& +Cs^3\lambda^3\int_{\Sigma_T} \xi^3 e^{-s\alpha}\psi^2 d\sigma dt.
\end{aligned} \quad (4.2)$$

Proof. Knowing that $\xi^4 e^{-s\alpha}\psi^2 \geq 0$, we have:

$$s^2\lambda^4\int_0^T\int_\Omega \xi^2 e^{-s\alpha}\psi^2 \leq s^2\lambda^4\int_0^T\int_\Omega \xi^2 e^{-s\alpha}\psi^2 + s^4\lambda^4\int_0^T\int_\Omega \xi^4 e^{-s\alpha}\psi^2$$

and the inequality (4.1) is written

$$
\begin{aligned}
s^2\lambda^4\int_0^T\int_\Omega \xi^2 e^{-s\alpha}\psi^2 \;\leq\; & Cs^2\lambda^3\left(\int_0^T\int_\Omega \xi^2 e^{-2s\alpha}\psi^2+\int_0^T\int_\Omega \xi^2 e^{-2s\alpha}\left|\nabla\psi\right|^2\right)\\
&+Cs\lambda^2\int_0^T\int_\Omega \xi\left|\nabla\widetilde{\psi}\right|^2+Cs^3\lambda^4\int_0^T\int_\Omega \xi^3 e^{-2s\alpha}\psi^2\\
&+Cs^3\lambda^3\int_0^T\int_\Omega \xi^3 e^{-2s\alpha}\left|\nabla\psi\right|^2\\
&+Cs^4\lambda^5\int_0^T\int_\Omega \xi^4 e^{-2s\alpha}\psi^2+Cs^5\lambda^3\int_0^T\int_\Omega \xi^5 e^{-2s\alpha}\psi^2\\
&+Cs^3\lambda^2\int_0^T\int_\Omega \xi^4 e^{-s\alpha}\psi^2+Cs^3\lambda^3\int_0^T\int_\Omega \xi^3 e^{-s\alpha}\psi^2\\
&+Cs^2\lambda^2\int_0^T\int_\Omega \xi^2 e^{-2s\alpha}\psi F\left(\psi\right)\\
&+Cs\lambda^2\int_0^T\int_\Omega \xi e^{-2s\alpha}\psi F\left(\psi\right)\\
&+Cs^2\lambda^2\int_{\Sigma_T} \xi^2 e^{-2s\alpha}\left|\psi\right|\left|g\right| d\sigma dt\\
&+Cs^3\lambda^3\int_{\Sigma_T} \xi^3 e^{-2s\alpha}\psi^2 d\sigma dt
\end{aligned}
$$

Note that the first five terms of the second member of the inequality below may be absorbed by the term $Cs^5\lambda^5\left(\int_0^T\int_\Omega \xi^5 e^{-s\alpha}\psi^2+\int_0^T\int_\Omega \xi^5 e^{-s\alpha}\left|\nabla\psi\right|^2\right)$ so that one gets

$$
\begin{aligned}
s^2\lambda^4\int_0^T\int_\Omega \xi^2 e^{-s\alpha}\psi^2 \;\leq\; & Cs^5\lambda^5\left(\int_0^T\int_\Omega \xi^5 e^{-s\alpha}\psi^2+\int_0^T\int_\Omega \xi^5 e^{-s\alpha}\left|\nabla\psi\right|^2\right)\\
&+Cs^2\lambda^2\int_0^T\int_\Omega \xi^2 e^{-2s\alpha}\psi F\left(\psi\right)\\
&+Cs\lambda^2\int_0^T\int_\Omega \xi e^{-2s\alpha}\psi F\left(\psi\right)\\
&+Cs^2\lambda^2\int_{\Sigma_T} \xi^2 e^{-2s\alpha}\left|\psi\right|\left|g\right| d\sigma dt\\
&+Cs^3\lambda^3\int_{\Sigma_T} \xi^3 e^{-2s\alpha}\psi^2 d\sigma dt.
\end{aligned}
$$

The corollary is established by noting that the term $s\lambda^2\int_0^T\int_\Omega \xi e^{-2s\alpha}\psi F\left(\psi\right)$ can be absorbed by the term $s^2\lambda^2\int_0^T\int_\Omega \xi^2 e^{-2s\alpha}\psi F\left(\psi\right)$ and that $e^{-2s\alpha}\leq e^{-s\alpha}$. □

Next, to obtain the observability inequality type, we will refer to [4] to the state that for every $\psi\in\mathcal{C}\left([0,\ T];L^2(\Omega)\right)\bigcap L^2\left(0,T;H^1(\Omega)\right)$ and thanks to the property (3.1), the term

$$
s^5\lambda^5\left(\int_0^T\int_\Omega \xi^5 e^{-s\alpha}\psi^2+\int_0^T\int_\Omega \xi^5 e^{-s\alpha}\left|\nabla\psi\right|^2\right)
$$

can be absorbed by $Cs^5\lambda^5\int_0^T\int_\omega \xi^5 e^{-s\alpha}\psi^2$. Consequently, we have the following result.

Theorem 4.3. *There is a positive constant C depending on T, Ω et ω such that*

$$
\begin{aligned}
s^2\lambda^4\int_0^T\int_\Omega \xi^2 e^{-s\alpha}\psi^2 \;\leq\; & C\left(s^2\lambda^2\int_0^T\int_\Omega \xi^2 e^{-s\alpha}\psi F\left(\psi\right)+s^5\lambda^5\int_0^T\int_\omega \xi^5 e^{-s\alpha}\psi^2\right.\\
&\left.+s^2\lambda^2\int_{\Sigma_T} \xi^2 e^{-s\alpha}\left|\psi\right|\left|g\right| d\sigma dt+s^3\lambda^3\int_{\Sigma_T} \xi^3 e^{-s\alpha}\psi^2 d\sigma dt\right).
\end{aligned}
\tag{4.3}
$$

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