Value-at-Risk Estimation Using an Interpolated Distribution of Financial Returns Series

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Keywords: Value-at-Risk, Non-parametric estimation, Empirical distribution, Spline Interpolation.

Abstract

This paper develops a model for estimating Value-at-Risk (VaR) from the historical return series. The proposed method uses spline interpolation to represent the empirical probability distribution of the return series. The approach developed in this paper is easy to implement using available programming platforms, and it can be generalized to other applications that involve estimating empirical distribution. In order to check the validity of the model, I use established back-testing methods and show that the model is robust to the changes in sample size and significance levels used to estimate VaR. I test the model against some similar distribution-based models using historical data from S&P500 index. I show that Value-at-Risk estimation based on the proposed method can outperform common historical, parametric, and kernel-based methods. As a result, the method can be useful in the context of validation of market risk models.

Keywords: Value at Risk, cubic spline, probability distribution, Back-testing.

Introduction

Value-at-risk (VaR) is a widely used measure in risk management. It is defined as the highest possible loss over a certain period of time at a given significance level, which is typically chosen to be 1% or 5%. Establishment of VaR as a practical methodology in risk measurement got initiated back in 1994, when J.P. Morgan published RiskMetrics as its first systematically developed risk-measurement procedure. According to the Basle Committee on Banking Supervision, banks are allowed to calculate their risks based on a VaR concept (Basle (1995), Basle (1996)). VaR has been increasingly adopted in various contexts with significant improvements on its calculation techniques; (see Jorion (2007)). However, VaR models suffer from some inefficiencies in risk measurement.

In order to satisfy the Basle requirements, banks and financial institutions were computing their VaR using either historical simulation, Variance-Covariance technique (Parametric) or Monte Carlo simulation (See more detailed VaR methodologies in Dowd (2002) and Alexander (2009)). With the emergence of the 2008 financial crisis, modifying the VaR methodology has become an essential requirement. VaR has already received lots of attention in the literature (see Jorion (2007)) and the literature dealing with different modeling issues is large enough, but few studies

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have been specifically concentrated on quantifying the uncertainty: Jorion (1996), Chritoffersen & Gonclaves (2005) and Chan et al. (2007). Inefficiency of the VaR estimates may have different reasons: The major one is due to data, such as lack of sufficient data; Further, uncertainty due to poorly characterized parameters in a specified mathematical model which are reflected in the VaR calculation (Dowd (1998)). A considerable amount of research has been devoted to exploring the VaR limitations (Krause (2003) and Danielsson (2002)) and producing more accurate VaR estimates, Khindanova et al. (2001), Sun et al. (2009), Lonnbark (2010), Huang (2010), Shaker-Akhtekhane and Mohammadi (2012), and Shaker-Akhtekhane et al. (2018) among others.

In this paper, a non-parametric approach using historical data based on cubic spline smoothing (CSS) is proposed to calculate VaR. I will use three widely used tests to examine the validity of the proposed method; Binomial, unconditional coverage and conditional coverage tests. Additionally, I will compare the performance of the proposed method to that of some popular Value-at-Risk measurement methods. I will use models with characteristics similar to that of the proposed method. The models include parametric (Normal and student's *t*), plain historical simulation and Epanechnikov kernel estimation method. Some other popular kernel estimators are: Epanechnikov, Biweight, Triweight, Triangular, Normal and Uniform kernels. It is noteworthy that unimodal densities have the same performance when used as a kernel. Also, uniform kernels are not very popular in practice since the corresponding density estimation is piecewise constant (Wand & Jones (1995)). Considering all aspects and the similarities between these kernels, the Epanechnikov kernel is chosen to be examined and compared in this paper. It should also be noted that, I have used all the kernels and obtained the results, but I have decided not to report all kernel methods because they produce very similar results.

The rest of the paper is organized as follows. Section 2 provides a brief explanation of cubic splines. In Section 3, cubic smoothing spline VaR estimation method is discussed, and Section 4 explains the statistical testing procedures and examines the reliability of the model. Finally, section 5 provides concluding remarks.

Cubic Splines

Cubic splines are powerful mathematical tools for interpolating discrete data using a reasonably smooth curve. Given data points across two-dimensional space $\{(x_1, y_1), \dots, (x_n, y_n)\}$ with $x_1 < \dots < x_n$, a cubic spline for these data points is defined as the following piecewise function

$$S(x) = s_i(x) \quad \text{if} \quad x_i \le x < x_{i+1}, \quad i \in \{1, \dots, n-1\} \tag{1}$$

where the s_i's are cubic polynomials, i.e. polynomials of degree three, defined by

$$s_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i, \quad i \in \{1, \dots, n-1\} \tag{2}$$

and where the following conditions hold:

1. $S(x)$ interpolates the data points $s_i(x_i) = y_i$ and $s_i(x_{i+1}) = y_{i+1}$ for $i \in \{1, \dots, n-1\}$.
2. $S(x)$ is twice continuously differentiable in the interval (x_1, x_n)

The choice of degree three offers a compromise between simplicity and flexibility. In order to uniquely determine a cubic spline, one needs to introduce additional conditions. For example, one can demand that the limit of the second derivative is zero in both endpoints x_1 and x_n. Cubic splines for which this condition holds are called *natural*. For detailed information on cubic splines the reader is referred to textbooks on numerical analysis, e.g. (Stoer *et al*. 2010).

There is another type of cubic splines which is called "cubic smoothing spline" (CSS). The smoothing spline f minimizes

$$p \sum_{j=1}^{n} \gamma_j \, |y_j - f(x_j)|^2 + (1-p) \int \lambda(t) |D^2 f(t)|^2 \, dt \tag{3}$$

where, $|z|^2$ stands for the sum of the squares of all the entries of z. n is the number of the points, and the integral is over the smallest interval containing all the entries of x. Also, γ is the weight vector in the error measure, and λ is the piecewise constant weight function in the roughness measure. $D^2 f$ denotes the second derivative of the function f, and p is the smoothing parameter. For $p = 0$, f is the least-squares straight line fit to the data, while, at the other extreme, i.e., for $p = 1$, f is the *natural* cubic spline interpolant. As p moves from 0 to 1, the smoothing spline changes from one extreme to the other. In the following, cubic smoothing spline (CSS) with $p = 0.5$ (to compromise between smoothness and interpolation) and equal weights($\lambda = 1, \gamma_i = 1$, for $i = 1, \dots, n()$) will be employed. I do this to keep things simple as well as to provide proper insights on how this simple, non-optimized version of the model works compared to other popular counterparts. Then one can think of calibrating these parameters according to the case at hand.

Value-at-Risk Model

Let $\{x_t\}$ be a sequence of prices, then, corresponding returns sequence is

$$r_t = 100 \, log \frac{x_t}{x_{t-1}}$$

The Value-at-Risk for d days ahead at α significance level, $VaR_{d,\alpha}$, is defined as

$$F(VaR_d(\alpha)) = P(r_t < VaR(\alpha)) = \alpha \tag{4}$$

or

$$VaR_d(\alpha) = inf\{v | P(r_t < v) = \alpha\} = F^{-1}(\alpha) \tag{5}$$

where $F(r_t)$ is the cumulative distribution function of the returns. This function can be estimated either parametrically and/or non-parametrically. In the next subsection, I describe the cubic smoothing spline (CSS VaR) method, which is of importance here.

Estimating Value-at-Risk using CSS

Suppose that we want to calculate 1-day ahead VaR at α significant level, i.e., $VaR_{1,\alpha}$. Let $\boldsymbol{x} = \{x_1, x_2, \dots, x_N\}$ be the return series, (where N is the sample size). In an algorithmic manner, I perform the following steps to obtain the empirical cumulative distribution of the data:

- Assign the weights of $\{w_0, w_1, w_2, \dots, w_N, w_{N+1}\}$ to each data-point $\{x_0, x_1, x_2, \dots, x_N, x_{N+1}\}$. Where, $x_0 = min(x_i) - \sigma$ and $x_{N+1} = max(x_i) + \sigma$, $i = 1,2, \dots, N$. σ is the standard deviation of $\boldsymbol{x}$.

- The assigned weights can be either equal, ($\frac{1}{N+1}$) or exponentially declining or any other desired weights which can vary depending on the data and our estimation goals.

  Here, I use equal weights, and I set $w_0 = 0$, $w_j = \frac{1}{N+1}$, $j = 1,2,\ldots,N+1$.
- Next, sort the data from the lowest to the highest to obtain ordered data, $\{x'_0, x'_1, x'_2, \ldots, x'_N, x'_{N+1}\}$.
- Assign the cumulative weights to the ordered data such that the weights $w'_0, w'_1, w'_2, \ldots, w'_N, w'_{N+1}$ are assigned to $x'_0, x'_1, x'_2, \ldots, x'_N, x'_{N+1}$, respectively, where

$$w'_i = \sum_{k=0}^{i} w_k \quad i = 0,1,\ldots,N+1$$

  For example, in our case (equal weights), the cumulative weight corresponding to i^{th} ordered return is ($w_i = \frac{i}{N+1}$).

By implementing these steps and making use of the assigned weights and data series at hand, one can obtain the empirical cumulative distribution shown in Figure1-(a). Now, I smoothly interpolate the cumulative distribution function F using CSS (See Figure1-(b)). Then, the empirical density function can be derived as the first derivative of F, i.e., $f = F'$. Finally, VaR_α is calculated using f, or it can be calculated directly using F, see Figure1-(c) and Figure1-(d). The results are not affected by two extra points, x_0 and x_{N+1}. These points are added to make the end slopes of the smoothed curve zero. This is clear in Figure 1.

The main strength of the proposed method is that it provides the empirical cumulative distribution function and density function without imposing distributional assumptions on the data. As a result, it can be considered as a non-parametric attempt to estimate the distributional shape of the data. In order to check the validity of the proposed approach in the context of VaR measurement, I have selected some benchmark VaR estimation models to compare against my model. As the proposed spline smoothing method estimates the empirical distribution of the data, it would be appropriate to select other distribution-fitting methods (parametric normal and t, and kernel fitting). Also, since the proposed method just uses historical data to estimate the empirical distribution, I have included the historical simulation method which basically uses quantiles of the data in the left tail.

As described above, I estimate the cumulative distribution instead of density function, and as a result, I don't impose restrictive distributional assumptions on the data, e.g., histograms and bins which are needed if we try to estimate the density function directly using splines. Our technique just uses the data at hand and all the available data are involved in creating the cumulative function and the density function. Next section is devoted to back-testing and examining the model's validity and comparing it with four other simple but popular methods.

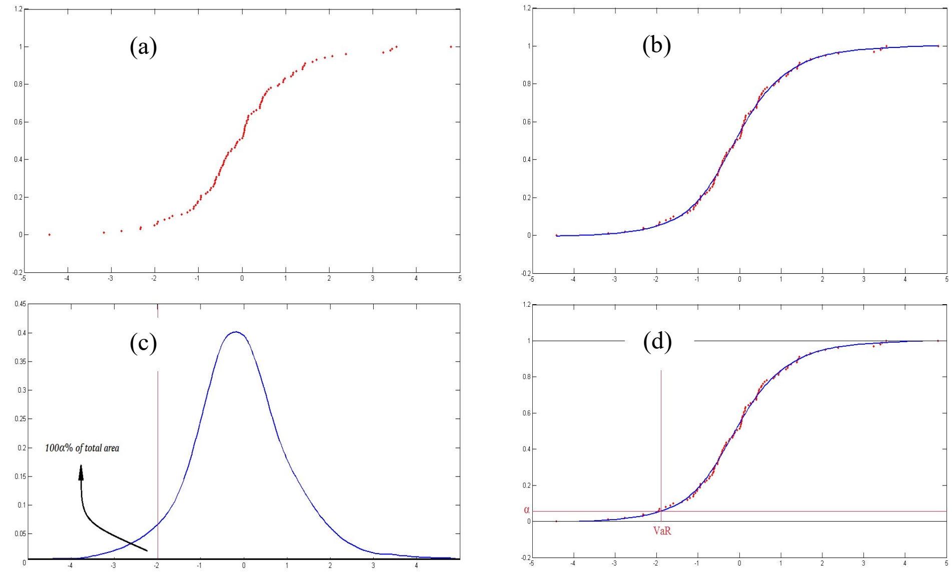

Fig. 1 VaR measurement Procedure using CSS. Panel (a) illustrates the cumulative distribution of the data; panel (b) shows the interpolated and smoothed cumulative distribution function; panels (c) and (d) display the VaR calculation using density function and cumulative distribution function, respectively.

Validity of the CSS VaR model

This section discusses reliability of the proposed model using three back-testing approaches. I also apply the same tests to other competing methods in order to get an understanding of how well the CSS VaR model works compared to the other similar, widely used models. Here, I introduce the back-testing methods briefly followed by implementation of the back-testings on the VaR models in the next subsections.

Back-testing is based on a rolling window for sub-samples data selection. The estimation sample is held constant, and it is rolled over the entire sample starting at the first data point. The length of the risk horizon is kept constant (here I take it equal to 1 day), and the test sample starts at the end of the estimation sample. We roll the estimation and test periods forward 1 day and keep rolling the estimation and test samples over the entire sample until we reach the last observation. Then we record the calculated VaR and the realized value for the entire period of window movement. The result of this procedure will provide two time series covering all the consecutive rolling test periods. One series contains the 1-day VaR estimates and the other contains the 1-day realized returns. The Back-test is performed based on these two series. Note that this rolling window approach is standard in risk literature which can easily be found in any VaR related textbook, e.g. see Alexander (2009).

Binomial back-testing

Most Back-tests on daily VaR are based on the assumption that the daily returns are generated by an *i.i.d.* Bernoulli process. A Bernoulli variable may take only two values, i.e., 1 and 0, or ’success’ and ’failure’. Here, we will call ’success’ an exceedance of the VaR on the return. That is, the calculated VaR exceeds the corresponding realized return value. The exceedances are assigned value of 1. We can define an indicator function $I_{\alpha,t}$ over the time series of daily returns relative to the $100\alpha\%$ daily VaR by:

$$I_{\alpha,t+1} = \begin{cases} 1, & \text{if } \; Y_{t+1} < -VaR_{1,\alpha,t}, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Where Y_{t+1} is the 'realized' daily return on the portfolio from the time *t*, when the VaR is estimated, to the time (*t*+1). If the VaR model is accurate and $I_{\alpha,t}$ follows an *i.i.d.* Bernoulli process, the probability of 'success' at any time *t* is α. Thus, the expected number of successes in a test sample with *n* observations is $n\alpha$. Let's denote the number of successes by the random variable $X_{n,\alpha}$. These assumptions imply that $X_{n,\alpha}$ follows a binomial distribution with parameters *n* and α. Therefore we have the following:

$$E(X_{n,\alpha}) = n\alpha \tag{7}$$

$$V(X_{n,\alpha}) = n\alpha(1-\alpha) \tag{8}$$

It is obvious that if the number of exceedances is closer to the mean, $n\alpha$, then the model can be regarded as more accurate. We can also use confidence intervals instead of exact values to assess the validity of the model. When *n* is very large the distribution of $X_{n,\alpha}$ is approximately normal, so a two-sided $1-\rho$ confidence interval for $X_{n,\alpha}$ under the null hypothesis that the VaR model is accurate is given by the following:

$$\left(n\alpha - z_{1-\frac{\rho}{2}}\sqrt{n\alpha(1-\alpha)}, n\alpha + z_{1-\frac{\rho}{2}}\sqrt{n\alpha(1-\alpha)}\right). \tag{9}$$

Unconditional and conditional coverage Back-testing

In addition to the basic binomial test, I use unconditional coverage (UCC) and conditional coverage (CC) tests introduced by Kupiec (1995) and Christoffersen (1998) to further examine the validity of the VaR models. The unconditional coverage test is a likelihood ratio statistic which is given by

$$LR_{uc} = \frac{\pi}{\frac{\pi_{\text{obs}}^{n_1}(1-\pi_{\text{obs}})^{n_0}}{exp_{n_1 n_{0exp}}}}, \tag{10}$$

where π_{exp} is the expected proportion of exceedances, π_{obs} is the observed proportion of exceedances from VaR, n_1 is the observed number of exceedances from VaR and $n_0 = n - n_1$ is total cases which the indicator function of returns is zero, where n stands for total sample size. The asymptotic distribution of $-2\,ln\,L\,R_{uc}$ is chi-square with one degree of freedom, and the null hypothesis is that the VaR estimation method is accurate in the sense that the total number of exceedances is close to the expected number. The hypothesis is rejected if computed $-2\,ln\,L\,R_{uc}$ is greater than the corresponding critical value.

In a similar way the conditional coverage test statistic is given by

$$LR_{cc} = \frac{\pi}{\frac{\pi_{01}^{n_{01}}(1-\pi_{01})^{n_{00}}\pi_{11}^{n_{11}}(1-\pi_{11})^{n_{10}}}{exp_{n_1 n_{0exp}}}}, \tag{11}$$

where π_{exp} is the expected proportion of exceedances, n_1 is the observed number of exceedances from VaR and $n_0 = n - n_1$ in which n stands for total sample size, therefore n_0 is total cases which the indicator function of returns equals zero and n_{ij} is the number of returns with indicator value i followed by indicator value j. Also

$$\pi_{01} = \frac{n_{01}}{n_{00} + n_{10}} \quad \text{and} \quad \pi_{11} = \frac{n_{11}}{n_{10} + n_{11}}.$$

The asymptotic distribution of $-2\, ln\, L\, R_{cc}$ is chi-square with two degrees of freedom.

Back-testing Results

For back-testing, I use S&P500 index's daily observations from Jun 1990 to Dec 2006. I have not included the period leading to 2008's financial crisis as that period represents a different market regime which would not be appropriate for testing simple parametric and non-parametric approaches. It can be noted that, the proposed method can be coupled with different weighting schemes or with some modern volatility models in order to capture varying market regimes, which would be beyond the scope of this paper. The whole sample size is 4287 which is enough to have several sizes for the estimation sample as well as enough estimated VaR's and realized return values to perform reliable tests on the models. I run the back-testing at two significance levels (0.05 and 0.01) and four Rolling Window Sizes, denoted by RWSs, (1, 2, 3 and 4 years of length, respectively 250, 500, 750 and 1000 daily observations). Table 1 contains the binomial back-testing results of the methods. According to Table 1, assuming a 5% rejection rate for the accuracy of the methods, (i.e., reject the null hypothesis of the model being accurate if the p-value is less than 0.05), the proposed model passes the binomial test under all significance levels and sample sizes (all 8 cases) while all other models fail at least once.

Table 2 reports the test results for the unconditional and conditional coverage back-testing methods. According to Table 2, again assuming 5% rejection rate for our hypothesis testing, the proposed model passes the unconditional coverage test for all eight cases. For this test historical simulation and epanechnikov kernel methods also pass the test for all cases while parametric t fails two and parametric normal fails four out of eight cases. For conditional coverage test also our model outperforms the others. Although this time our model fails the test in two out of eight cases, the second best model is the epanechnikov kernel which fails in four out of eight cases. Table 3 summarizes the failures of the models under 1% and 5% rejection rates for the null hypothesis that the model is accurate. If we take a look at the overall performance of the models in all three back-tests, i.e. add the failures under the 5% rejection rate, we can rank the models as[1]: 1. Cubic smoothing spline, 2. Epanechnikov kernel, 3. Plain historical simulation, 4. Parametric t, and 5. Parametric normal.

[1] Note that this method is not a conventional ranking approach and is merely used to provide a better understanding of which models fails in fewer/more tests compared to others.

TABLE 1 Binomial back-testing results of the VaR models, (ρ=0.05).

VaR model	α	RWS	Expected value	Number of exceedances	Difference	p-value
Cubic smoothing spline	0.05	250	202	190	12	0.3921
		500	189	182	7	0.5837
		750	177	181	4	0.7488
		1000	164	188	24	0.0584
	0.01	250	40	33	7	0.2437
		500	38	36	2	0.7601
		750	35	39	4	0.5396
		1000	33	38	5	0.3685
Epanechnikov kernel	0.05	250	202	179	23	0.0989
		500	189	179	10	0.4403
		750	177	182	5	0.6911
		1000	164	189	25	0.0485
	0.01	250	40	38	2	0.7077
		500	38	40	2	0.7279
		750	35	42	7	0.2625
		1000	33	41	8	0.1541
Parametric normal	0.05	250	202	183	19	0.1734
		500	189	169	20	0.1292
		750	177	173	4	0.7664
		1000	164	172	8	0.5404
	0.01	250	40	62	22	0.0006
		500	38	60	22	0.0003
		750	35	68	33	0.0000
		1000	33	66	33	0.0000
Parametric student's *t*	0.05	250	202	170	32	0.0214
		500	189	162	27	0.0414
		750	177	168	9	0.4947
		1000	164	170	6	0.6511
	0.01	250	40	41	1	0.9206
		500	38	41	3	0.6092
		750	35	34	1	0.8169
		1000	33	36	3	0.5832
Plain HS	0.05	250	202	196	6	0.6727
		500	189	183	6	0.6359
		750	177	183	6	0.6352
		1000	164	182	18	0.1578
	0.01	250	40	50	10	0.1277
		500	38	48	10	0.0980
		750	35	46	12	0.0724
		1000	33	45	12	0.0335

Note: RWS: rolling window size. α is the significance level of VaR, (VaR_α).

TABLE 2 Coverage back-testing results of the VaR models.

VaR model	α	RWS	-2lnLR_{uc}	p-value(UCC)	-2lnLR_{cc}	p-value(CC)
Cubic smoothing spline	0.05	250	0.7463	0.3877	5.7366	0.0568
		500	0.3041	0.5813	7.1087	0.0286
		750	0.1018	0.7497	5.9034	0.0523
		1000	3.7201	0.0538	12.2097	0.0023
	0.01	250	1.4493	0.2286	2.5977	0.2728
		500	0.0948	0.7581	0.9116	0.6339
		750	0.3642	0.5462	3.4201	0.1809
		1000	0.7701	0.3802	3.7594	0.1526
Epanechnikov kernel	0.05	250	2.8261	0.0927	8.3132	0.0157
		500	0.6061	0.4363	11.4991	0.0032
		750	0.1564	0.6925	5.7675	0.0559
		1000	3.7201	0.0538	12.2097	0.0022
	0.01	250	0.1433	0.7050	0.9030	0.6367
		500	0.1188	0.7303	6.9359	0.0312
		750	1.1842	0.2765	3.7743	0.1515
		1000	1.8833	0.1700	4.3979	0.1109
Parametric normal	0.05	250	1.9103	0.1669	8.1333	0.0171
		500	2.3848	0.1225	7.2399	0.0268
		750	0.0888	0.7657	4.6426	0.0981
		1000	0.3694	0.5433	9.9169	0.0070
	0.01	250	10.0592	0.0015	15.6548	0.0004
		500	11.0931	0.0009	20.0254	0.0000
		750	23.9406	0.0000	30.2755	0.0000
		1000	26.0949	0.0000	29.7245	0.0000
Parametric student's *t*	0.05	250	5.5768	0.0182	16.4111	0.0003
		500	4.3631	0.0367	12.1515	0.0023
		750	0.4737	0.4913	5.9148	0.0520
		1000	0.2023	0.6529	6.9682	0.0307
	0.01	250	0.0099	0.9208	11.4134	0.0033
		500	0.2545	0.6140	6.8083	0.0332
		750	0.0534	0.8157	4.0226	0.1338
		1000	0.2920	0.5889	3.6321	0.1627
Plain HS	0.05	250	0.1801	0.6713	5.4790	0.0646
		500	0.2266	0.6341	5.4109	0.0668
		750	0.2227	0.6370	7.0570	0.0293
		1000	1.9310	0.1647	10.5802	0.0050
	0.01	250	2.1568	0.1419	6.9983	0.0302
		500	2.5234	0.1122	11.1581	0.0038
		750	2.9478	0.0860	7.9703	0.0186
		1000	3.9415	0.0636	8.5478	0.0139

Note: RWS: rolling window size, α is the significance level of VaR, ($VaR_α$).

UCC: unconditional coverage, CC: conditional coverage.

TABLE 3 Summary of accuracy hypothesis rejection

VaR model	BB		UCC		CC	
	0.01	0.05	0.01	0.05	0.01	0.05
Cubic smoothing spline	0	0	0	0	1	2
Parametric normal	4	4	4	4	5	7
Parametric student's *t*	0	2	0	2	3	5
Plain historical	0	1	0	0	2	6
Epanechnikov kernel	0	1	0	0	2	4

***Note*:** BB: binomial back-testing. UCC: unconditional coverage. CC: conditional coverage.
0.01 and 0.05 stand for rejection rate of hypothesis testing, ρ.

Conclusion

In this paper, I have proposed a cubic smoothing spline procedure to approximate the empirical distribution of a given series and have applied this approach to estimate Value-at-Risk. The approach presented in this paper provides an innovative way of estimating PDF and CDF functions without relying on kernels or histogram-based methods. The proposed method uses historical data to approximate the distribution of the data and calculates the VaR using the density function or cumulative function estimation using cubic splines. Conventional approaches of estimating Value-at-Risk rely heavily on accurate estimation of the probability distribution of returns and that's why I have used the cubic smoothing splice approach to estimate Value-at-Risk. I compared the accuracy of several VaR approaches with similar characteristics to that of the proposed method. I have used the following models as benchmarks for comparison against the Cubic smoothing spline: (1) Parametric normal with unconditional mean and variance, (2) Parametric student's *t* with unconditional mean and variance, (3) Plain historical simulation, and (4) Epanechnikov kernel method.

I have used three different back-testing methods to test the validity of the proposed model. The back-testing methods used in this paper are: binomial, unconditional coverage and conditional coverage tests. Also, in order to perform a better evaluation of the models, I have used back-tests for different sample sizes and VaR significance levels, making a total of eight different cases for each model. The proposed model passes binomial test and unconditional coverage test for all eight cases under a 5% rejection rate. It fails the conditional coverage test only twice for the same rejection rate. Despite failing the conditional coverage test twice, our proposed method still outperforms the other competing models in all three tests. If we rank the models used in this paper in terms of their performance in back-tests across all eight cases, we find that the proposed method performs the best whereas the parametric normal model performs the worst amongst the models used in this paper. Epanechnikov kernel method as well as other kernel methods in general (because all kernel methods produce very similar results) show a satisfactory degree of accuracy compared to other models and ranks second.

As discussed earlier in the paper, I have used a very simple version of the cubic smoothing splines in order to evaluate the model in its simplest form. One can obtain optimal parameter values for the model such as weights and the smoothing parameter depending on the data, to improve on our results.

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