

*Communications in Mathematical Finance* , vol.  $x$ , no.  $x$ , 2015,  $x-x$

ISSN: 2241-1968(print), 2241-195X(online)

©International Scientific Press, 2015

# The Black-Scholes Equation with Variable Volatility Through The Adomian Decomposition Method

O. González-Gaxiola<sup>1</sup>

## Abstract

The Black-Scholes equation is a partial differential equation characterizing the price evolution of a European call option and put option on a stock. In this work, we use the Adomian Decomposition Method (ADM) for the approximation of the solution of the Black-Scholes equation and show how it can be applied to a case in which the volatility is not constant but is dependent on the price of the underlying asset. Finally, we expose a numerical example to validate the developed method.

**Mathematics Subject Classification** : 91G80, 91B25, 91G60

**Keywords:** Adomian decomposition method, Black-Scholes Equation, Volatility, Option pricing.

## 1 Introduction

In finance, a financial derivative (or derivative) is a financial product whose value varies depending on the price of another asset; the asset of which depends takes the name of the underlying asset. In practice, the options are a type of

---

<sup>1</sup>Department of Applied Mathematics and Systems, UAM-Cuajimalpa; Vasco de Quiroga No. 4871, Col. Sta. Fé, Cuajimalpa México, D.F. 05300, MEXICO. e-mail: ogonzalez@correo.cua.uam.mx

derivative, *i.e.*, a contract which we have the option to buy or sell a certain underlying asset at a certain date and at a price established. Therefore, an option can be a purchase option or a selling option, called call option and put option, respectively.

Commonly, an option is valued using the Black-Scholes model, which is a partial differential equation which, when solved, produces a function that allows us to know the price of a derivative, depending on the underlying asset price and time. As the name suggests, this model was proposed by Fischer Black and Myron Scholes [8]. Subsequently, Robert Merton published a paper with a deeper and more consistent development of the mathematical model.

In this paper we will do a study through the Adomian decomposition method of the Black-Scholes equation for a case in which volatility is variable, this case is a more acceptable model to describe the current behavior of finance in the world.

This paper is organised as follows. Section 2 includes a brief description of the Adomian decomposition method (ADM). In the Section 3, the Black-Scholes equation for the standard case of constant volatility is exposed. In Section 4, a recursive solution for the Black-Scholes equation (with variable volatility) is found via the ADM. Finally, in Section 5 we present an example to illustrate the goodness of the Adomian decomposition method applied to mathematical finance and compare the results with solutions previously obtained in the literature for this type of equations.

## 2 Description of the Adomian Decomposition Method (ADM)

The Adomian decomposition method (ADM) will be applied to the following general nonlinear equation

$$L_t u(x, t) + Ru(x, t) + Nu(x, t) = g(x, t) \quad (1)$$

where  $L_t = \frac{\partial}{\partial t}$ ,  $R$  is the linear remainder operator that could include partial derivatives with respect to  $x$ ,  $N$  is a nonlinear operator which is presumed to be analytic and  $g$  is a non-homogeneous term that is independent of the solution  $u$ .

Solving for  $L_t u(x, t)$ , we have

$$L_t u(x, t) = g(x, t) - Ru(x, t) - Nu(x, t). \quad (2)$$

As  $L$  is presumed to be invertible, we can apply  $L_t^{-1}(\cdot) = \int_0^t (\cdot) dr$  to both sides of equation (2), obtaining

$$L_t^{-1} L_t u(x, t) = L_t^{-1} g(x, t) - L_t^{-1} Ru(x, t) - L_t^{-1} Nu(x, t). \quad (3)$$

An equivalent expression to (3) is

$$u(x, t) = f(x) + L_t^{-1} g(x, t) - L_t^{-1} Ru(x, t) - L_t^{-1} Nu(x, t). \quad (4)$$

where  $f(x)$  is the constant of integration with respect to  $t$  that satisfies  $L_t f = 0$ . In equations where the initial value  $t = t_0$ , we can conveniently define  $L^{-1}$ . The ADM proposes a decomposition series solution  $u(x, t)$  given as

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t). \quad (5)$$

The nonlinear term  $Nu(x, t)$  is given as

$$Nu(x, t) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \dots, u_n) \quad (6)$$

where  $\{A_n\}_{n=0}^{\infty}$  is the Adomian polynomials sequence established in [4, 12].

The Adomian polynomials have been studied in a formal manner in [12].

Substituting (5) and (6) into equation (4), we obtain

$$\sum_{n=0}^{\infty} u_n(x, t) = f(x) + L_t^{-1} g(x, t) - L_t^{-1} R \sum_{n=0}^{\infty} u_n(x, t) - L_t^{-1} \sum_{n=0}^{\infty} A_n(u_0, u_1, \dots, u_n), \quad (7)$$

with  $u_0$  identified as  $f(x) + L_t^{-1} g(x, t)$ , and therefore, we can write

$$\begin{aligned} u_0(x, t) &= f(x) + L_t^{-1} g(x, t), \\ u_1(x, t) &= -L_t^{-1} R u_0(x, t) - L_t^{-1} A_0(u_0), \\ &\vdots \\ u_{n+1}(x, t) &= -L_t^{-1} R u_n(x, t) - L_t^{-1} A_n(u_0, \dots, u_n). \end{aligned}$$

From which we can establish the following recurrence relation, that is obtained in a explicit way for instance in reference [13],

$$\begin{cases} u_0(x, t) = f(x) + L_t^{-1}g(x, t), \\ u_{n+1}(x, t) = -L_t^{-1}Ru_n(x, t) - L_t^{-1}A_n(u_0, u_1, \dots, u_n), \quad n = 0, 1, 2, \dots \end{cases} \quad (8)$$

Using (8), we can obtain an approximate solution of (1), subject to the initial condition  $u(x, 0) = f(x)$  as

$$u(x, t) \approx \sum_{n=0}^k u_n(x, t), \quad \text{where} \quad \lim_{k \rightarrow \infty} \sum_{n=0}^k u_n(x, t) = u(x, t). \quad (9)$$

ADM requires far less work in comparison with traditional methods [1]. This method considerably decreases the volume of calculations. The decomposition procedure of Adomian [7] easily obtains the solution without linearizing the problem by implementing the decomposition method rather than the standard methods. In this approach, the solution is found in the form of a convergent series with easily computed components; in many cases, the convergence of this series is extremely fast and consequently only a few terms are needed in order to have an idea of how the solutions behave. Convergence conditions of this series have been investigated by several authors, e.g., [5, 6, 2, 3].

### 3 The Black-Scholes Equation (with constant volatility)

This model is a partial differential equation whose solution describes the value of an European Option, see [8, 11]. Nowadays, it is widely used to estimate the pricing of options other than the European ones. For an European call or put on an underlying stock paying no dividends, the equation is:

$$V_\tau(S, \tau) + \frac{1}{2}\sigma^2 S^2 V_{SS}(S, \tau) + rSV_S(S, \tau) - rV(S, \tau) = 0, \quad (10)$$

where  $V$  is the price of the option as a function of underlying price  $S$  and time  $\tau$ ; with  $0 \leq \tau < T$ , and  $S \geq 0$ ; here the risk-free interest rate  $r$  and the volatility  $\sigma$  are assumed to be constant.

There are many varieties of options. European options may only be exercised on the maturity date. American options may be exercised any time up to

and including the maturity date. The Asian option is an option whose payoff depends on the average price of the underlying asset during the period since the issue of the option until its expiration date, e.g., see [10] for details.

In the case of a European option, the value of the like-call option can be obtained from (10) with the boundary conditions:

$$\begin{cases} V_c(S, T) = \max(S - K, 0) & : \tau = T \\ V_c(0, \tau) = 0 & : S = 0 \\ V_c(S, \tau) \rightarrow S - Ke^{-r(T-\tau)} & : S \rightarrow \infty. \end{cases}$$

We have that

$$\begin{aligned} V_c(S, \tau) = & SN \left( \frac{1}{\sigma\sqrt{T-\tau}} \left[ \ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-\tau) \right] \right) \\ & - e^{-r(T-\tau)} KN \left( \frac{1}{\sigma\sqrt{T-\tau}} \left[ \ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-\tau) \right] \right), \end{aligned}$$

where  $N(\eta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\eta} e^{-\frac{x^2}{2}} dx$  and  $K > 0$  is the strike price.

Here  $S = S(\tau)$  for  $\tau \in [0, T]$ ; the solution of (10) provides both an option pricing formula for a European option and a hedging portfolio that replicates the contingent claim assuming that: The asset price  $S$  or the value of the underlying asset follows a geometric Brownian motion and the volatility  $\sigma$  which measures the standard deviation of the returns and the riskless interest rate  $r$  are all constant for  $0 \leq \tau \leq T$ , and no dividends are paid in that time period. In the next section we will do a study through ADM of the Black-Scholes equation for a case in which volatility is not constant.

## 4 The Black-Scholes equation through ADM

There are several transaction cost models from the most relevant class of Black-Scholes equations for European and American options with a constant interest rate  $r$  and a nonconstant modified volatility function

$$\hat{\sigma}^2 = \hat{\sigma}^2(\tau, S, V_S, V_{SS}). \quad (11)$$

There have been many approaches to improve the aforementioned model by treating the volatility in different ways, e.g., using a modified volatility function  $\hat{\sigma}$  to model the effects of transaction costs, illiquid markets and large traders,

which is the reason for the nonlinearity of (10). In the present paper, we will consider the variable volatility  $\sigma(S) = \sqrt{2}\sigma S$  with  $\sigma$  a constant, replacing this volatility in the equation (10) becomes

$$\begin{cases} V_\tau(S, \tau) + \sigma^2 S^4 V_{SS}(S, \tau) + rSV_S(S, \tau) - rV(S, \tau) = 0, \\ V(S, T) = f(S), \quad S \in [0, \infty). \end{cases} \quad (12)$$

Note by considering the translation  $t = T - \tau$ , and denoting  $V(S, \tau) = u(S, t)$ , problem (12) assumes the form

$$\begin{cases} u_t(S, t) + \sigma^2 S^4 u_{SS}(S, t) + rSu_S(S, t) - ru(S, t) = 0, \\ u(S, 0) = f(S), \quad S \in [0, \infty). \end{cases} \quad (13)$$

#### 4.1 ADM for Black-Scholes

Comparing (13) with equation (1), we have that  $g(S, t) = 0$ , and

$$L_t = \frac{\partial(\cdot)}{\partial t}, \quad R = \sigma^2 S^4 \frac{\partial^2(\cdot)}{\partial S^2} + rS \frac{\partial(\cdot)}{\partial S} - r, \quad N = 0. \quad (14)$$

Applying  $L^{-1} = \int_t^T (\cdot) dz$  in both sides of the equation (13), we obtain,

$$L^{-1}u_t(S, t) = -\sigma^2 L^{-1}S^4 u_{SS}(S, t) - rL^{-1}Su_S(S, t) + rL^{-1}u(S, t), \quad (15)$$

equivalently

$$u(S, T) - u(S, t) = -\sigma^2 \int_t^T S^4 u_{SS}(S, z) dz - r \int_t^T Su_S(S, z) dz + r \int_t^T u(S, z) dz,$$

from where

$$u(S, t) = f(S) + \sigma^2 \int_t^T S^4 u_{SS}(S, z) dz + r \int_t^T Su_S(S, z) dz - r \int_t^T u(S, z) dz.$$

By ADM assuming that the solution could be expressed in terms of a series

$$u(S, t) = \sum_{i=0}^{\infty} u_i(S, t),$$

we obtain

$$\begin{aligned} \sum_{i=0}^{\infty} u_i(S, t) &= f(S) + \sigma^2 \sum_{i=0}^{\infty} \int_t^T S^4 u_{i,SS}(S, z) dz \\ &\quad + r \int_t^T \sum_{i=0}^{\infty} Su_{i,S}(S, z) dz - r \sum_{i=0}^{\infty} \int_t^T u_i(S, z) dz, \end{aligned}$$

from where, we establish the following recursion relation

$$\begin{cases} u_0(S, t) = f(S), \\ u_{n+1}(S, t) = \int_T^t Ru_n(S, t), \quad n = 0, 1, 2, \dots, \end{cases} \quad (16)$$

then, an approximation is given by the partial sum

$$u(S, t) \approx \sum_{i=0}^k u_i(S, t). \quad (17)$$

## 5 An Illustrative Example

In this example, we consider the Black-Scholes with variable volatility (13) and with the market parameters given in the Table 1

Parameter	Value
Maturity time	$T = 1 \text{ year}$
Strike price	$K = \$15$
Interest rate	$r = 0.01$
Volatility	$\sigma = 0.5$
Payoff function	$f(S) = S^2 - 18S + 45$

Table 1: Data of Black-Scholes model for the case:  $\sigma(S) = (0.5)^2 S^4$ .

In the ADM framework, we choose  $u_0(S, t) = S^2 - 18S + 45$  and therefore, the Adomian approaches are given in the Figure 1 using the formula (17) and calculating with *Mathematica* software package for  $k = 1$ ,  $k = 2$  and  $k = 3$ , *i.e.* the approximate solution is given by

$$\text{for } k = 1; \quad u_{ADM}(S, t) = u_0(S, t) + u_1(S, t)$$

$$\text{for } k = 2; \quad u_{ADM}(S, t) = u_0(S, t) + u_1(S, t) + u_2(S, t)$$

$$\text{for } k = 3; \quad u_{ADM}(S, t) = u_0(S, t) + u_1(S, t) + u_2(S, t) + u_3(S, t)$$

In the Figure 1, we compare the solution of the Black-Scholes equation obtained from (17), for  $t = 0.5$  years, with the exact solution obtained in [9] through ADM for diffusion-convection-reaction type equations. All the numerical work was accomplished with the *Mathematica* software package.

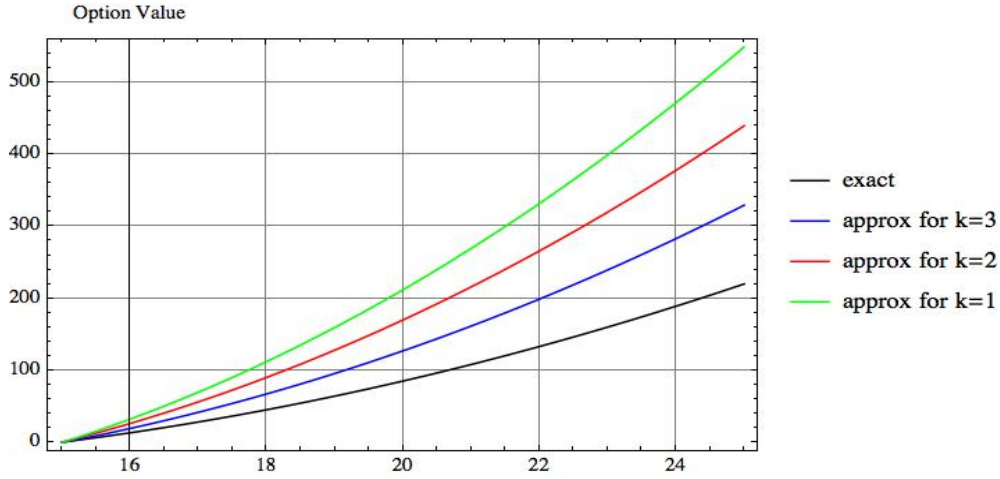


Figure 1: Graph of the values of  $u_{ADM}$  for  $k = 1, 2, 3$  and  $u_{ex}$  for  $t = 0.5$  years

## 6 Summary

In this paper, we used the Adomian decomposition method for approximating the solution of the Black-Scholes equation, for a situation characterized by a non-constant volatility, but that depends on asset price  $S$  (*i.e.*, a modified case). Such nonlinear models frequently occur in financial markets characterized by a lack of liquidity or with high volatility. The approach was made to  $t = 0.5$ , that is half of the expiration date  $T$  and taking into account the volatility  $\sigma(S) = (0.5)^2 S^4$ . In addition, we compare the approximate solutions with the solution for this type of equations obtained in the literature by other methods.

**ACKNOWLEDGEMENTS.** The authors are grateful to the referees for their suggestions which helped to improve the paper.

## References

- [1] Abdelrazec, A., Pelinovsky, D.: Convergence of the Adomian decomposition method for initial-value problems. Numer. Methods Partial Differential Eq. **27**, 749-766 (2011)



- [2] Cherruault, Y.: Convergence of Adomian's method. *Kybernetes* **18**(2), 31-38 (1989)
- [3] Cherruault, Y., Adomian, G.: Decomposition methods: a new proof of convergence. *Math. Comput. Modelling* **18**(12), 103-106 (1993)
- [4] Babolian, E., Javadi, Sh.: New method for calculating Adomian polynomials. *Applied Math. and Computation* **153**, 253-259 (2004)
- [5] Abbaoui, K., Cherruault, Y.: Convergence of Adomian's method applied to differential equations. *Comput. Math. Appl.* **28**(5), 103-109 (1994)
- [6] Abbaoui, K., Cherruault, Y.: New ideas for proving convergence of decomposition methods. *Comput. Math. Appl.* **29**(7), 103-108 (1995)
- [7] Adomian, G.: *Solving Frontier Problems of Physics: The Decomposition Method*, Boston, MA: Kluwer Academic Publishers (1994)
- [8] Black, F., Scholes, M.: The Pricing Options and Corporate Liabilities. *Journal of Political Economy* **81**(3), 637-654 (1973)
- [9] El-Wakil, S. A., Abdou, M. A., Elhanbaly, A.: Adomian decomposition method for solving the diffusion-convection-reaction equations. *Applied Math. and Comp.* **177**, 729-736 (2006)
- [10] Hull, J.C.: *Options, Futures and Other Derivatives*, 5th ed., Prentice Hall, New Jersey (2002)
- [11] Merton, R.C.: Theory of Rational Options Pricing. *Bell Journal of Economic and Management Science* **4**(1), 141-183 (1973)
- [12] Wazwaz, A. M.: A new algorithm for calculating Adomian polynomials for nonlinear operators. *Appl. Math. and Computation* **111**(1), 53-69 (2000)
- [13] Wenhai, C., Zhengyi L.: An algorithm for Adomian decomposition method. *Applied Mathematics and Computation* **159**, 221-235 (2004)