An Improved Numerical Approach for American Converts Valuation

[a]Bolujo Joseph Adegboyegun
School of Mathematics and Applied Statistics
University of Wollongong, Australia
Corresponding e-mail: bja998@uowmail.edu.au

[b]Titilayo Omotayo Akinwumi
Department of Mathematics and Computer Science,
Elizade University, Ilara-Mokin, Nigeria
titilayo.akinwumi@elizadeuniversity.edu.ng

Abstract

We solve the free boundary problem of American convert using singularity-separating method(SSM). The discontinuity of the non-smooth payoff function at expiry is removed by introducing a new coordinate which reduce the free boundary problem to a partial differential equation coupled with an ordinary differential equation on a rectangular domain. We observe that the approach is suitable and economical where the price of American converts and its critical asset price are required close to the maturity. Numerical results are presented to show the applicability of this approach.

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1 Introduction

American converts(ACs) or American-style Convertible bonds are innovative financial instruments that usually share some of the risk and return characteristics of ordinary corporate bonds on one hand and equity on the other. They are debt securities that can be converted into the common stock of the issuing firm at any time prior to maturity using a preset conversion ratio with the option to convert solely at the discretion of the bond holder, who will only does so if the market is favorable. From the perspective of the corporate borrower, ACs have the benefit of lower interest rate cost than the straight bond and it offers a relatively cheap way for many companies, particularly fast growing ones, to raise capital when other markets are closed. However, there is a drawback that the borrower faces capital structure uncertainty. In return for a declined yield, an investor will receive a security with considerable upside potential along with downside protection[1].

One of the most attractive features of American converts is that they can be customized to meet the needs of investors and issuers. Such adaptive features include early exercise or conversion, callability by the issuer and putability by the holder.The flexible nature of this derivative makes its valuation very complex in financial market[2]. Strictly speaking, there is no analytical method to price American convert in the literatures, hence numerical methods must be used. Examples of such numerical methods include the finite difference method[9], finite element method[10] and finite volume method[11]. Since the contract can be exercised at any time prior to maturity, there is a free boundary separating the region where it is optimal to hold from that where exercise is optimal and this must be found as part of the solution. To prevent risk-free profit, the value of the ACs must equal to the value of the underlying asset times the conversion ratio. In addition, we have a smoothing condition that the first derivative of the option value with respect to asset price must be continuous across the boundary. However, the second derivative of the solution is not continuous in this region. This make it difficult to accurately track the security value and its optimal exercise price using the prior acts.

Aside the weak singularity on the free boundary, it should be noted that at expiry, the derivative of the final value of the ACs is discontinuous. Hence, there is a faster variation of the price and exercise boundary at $t \approx T$ and the solutions obtained near this domain are prone to relatively large error. Although such problem can be managed by either using the projected method or by increasing the grip points near this region[3]. However, the former cannot give the exact location of the critical asset price while the later requires large computer memory.

In this paper, the singularity-separating approach is applied to single-factor American converts with continuous dividends payment on the underlying stocks. The approach had earlier been applied to barrier option[6] and American exotic option[4]. The idea behind this approach was presented in the celebrated work of You-lan Zhu et al[4] in which our work owe part of its derivation to. New coordinate was introduced, under which the free boundary problem was reduced to solving a partial differential equation on a rectangular domain coupled with an ordinary differential equation problem. Here, we compute the difference between the free boundary problem of American convert and its European counterpart whose solution could be obtained analytically. It should be obvious that both systems satisfy the same final condition, thus, their difference has a smoother solution near expiration. The resulting non-linear system with weak or no singularity is discretized using centered finite difference scheme and Gauss- Seidel iterative scheme is employed to solve the system.

The structure of this paper is as follows. We describe the free-boundary problem of single factor American- converts in section 2. In the subsequent section, we reduce the free-boundary problem into a partial differential equation coupled with an ordinary differential equation on a rectangular domain. Afterwards, we develop a numerical scheme to solve the resulting non-linear system. In the last section, we perform some numerical experiments and present our results graphically.

2 Free-boundary problem for single-factor American converts

In this section, we present mathematical formula for pricing single-factor American converts. For simplicity, we shall consider an American convert without any embedded feature, zero coupon payment on bond and that the bond can be converted to one unit of the stock(Conversion ratio=1). Let $V(S,t)$ denote the value of an AC, S be the price of the underlying asset and t be the current time. Then, under the Black-Scholes framework[7], the value of a convertible bond $V(S,t)$ should satisfy the partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV = 0 \tag{2.1}$$

where r is the risk-free interest rate, σ is the volatility of the underlying asset price and D_0 is the rate of dividend paid to the underlying asset. Here, we shall further assume r and σ are constants.

Equation(2.1) needs to be solved together with a set of appropriate boundary and terminal conditions. As presented in the previous work of Adegboyegun[8], the valuation of an American convert is considered the solution to a free boundary problem with a parabolic partial differential equation. We suppose that the optimal exercise boundary $S_f(t)$ is monotonically decreasing with $S_f(T) = Z$. The region where it is optimal to hold, generally called the continuation region, is defined as $[0, S_f] \times [0, T]$, and the region where it is optimal to exercise, generally called the exercise region, is defined as $(S_f, \infty) \times [0, T]$. The payoff function of an American convert at expiry, $t = T$ is defined as

$$V(S,T) = \max(S, Z) \tag{2.2}$$

For arbitrage considerations, the value of the bond must equal the stock value on the free boundary and we have a high contact condition tangential to the bond value. Thus, the boundary conditions as $S \to \infty$ should be replaced by two conditions:

$$\begin{aligned} &V(S_f(t), t) = S_f(t) \\ &\frac{\partial V}{\partial S}(S_f, t) = 1 \end{aligned} \tag{2.3}$$

In the absence of default issues, the boundary condition as $S = 0$ for American convert without put option and zero coupon payment is

$$V(S,t) = Ze^{-r(T-t)} \tag{2.4}$$

The convertible valuation problem is now completely defined by a differential system composed of equations(2.1)-(2.4). To solve this problem using SSM approach, we first introduce its European counterpart as follows

$$\begin{aligned}
&\frac{\partial V_e}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_e}{\partial S^2} + (r - D_0)S\frac{\partial V_e}{\partial S} - rV_e = 0 \\
&V_e(S,T) = \max(S,Z) \\
&\lim_{S\to\infty} V_e(S,t) = S \\
&V_e(S,t) = Ze^{-r(T-t)}
\end{aligned} \tag{2.5}$$

where $0 \leq x \leq \infty$ and $0 \leq t \leq T$, its solution $V_e(S,t)$ has the following explicit expression:

$$V_e(S,t) = C(S,t,Z) + Ze^{-r(T-t)} \tag{2.6}$$

Here, $C(S,t,Z/n)$ is the price of a European call option and the exercise price $E = Z$
The solution $C(S,t,Z)$ is given by[6,7]:
$C(S,t) = Se^{-D_0(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2)$
where

$$N(z) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{z} e^{-(1/2)\xi^2 \, \mathrm{d}\xi} \tag{2.7}$$

and

$$\begin{aligned}
d_1 &= \left[\ln\frac{Se^{-D_0(T-t)}}{E} + \frac{1}{2}\sigma^2(T-t)\right]/(\sigma\sqrt{T-t}) \\
d_2 &= \left[\ln\frac{Se^{-D_0(T-t)}}{E} - \frac{1}{2}\sigma^2(T-t)\right]/(\sigma\sqrt{T-t})
\end{aligned} \tag{2.8}$$

3 Singularity-separating approach

We have earlier remarked that there is no acceptable analytical solution to free boundary problem of American converts. In addition, the existing numerical methods are prone to large errors since the derivative of ACs values are discontinuous at some points. To address these problem, we follow the techniques of You-lan Zhu et al[4], by introducing a European convert in section two. It should be obvious that both converts have the same payoff at maturity, $V(S,T) = V_e(S,T) = \max(S,Z)$. Thus, their difference should be a smooth function or at worst a function with weaker singularity which can be easily

tracked numerically. Since $V_e(S,t)$ has an explicit expression, as soon has we obtain their difference numerically, we can obtain $V(S,t)$
Let us define their difference by

$$\overline{V}(S,t) = V(S,t) - V_e(S,t) \tag{3.1}$$

On the domain $[0, S_f(t)] \times [0,T]$, both $V(S,t)$ and $V_e(S,t)$ have the same payoff function at $t = T$, then $\overline{V}(S,T) = 0$. Moreover, inspired by the property of linear homogeneity of partial differential equation, since both functions satisfy this property so it is expected that their difference does. On the free boundary $S = S_f(t)$, we have $\overline{V}(S_f(t),t) = S_f(t) - V_e(S_f(t),t)$ and $\frac{\partial \overline{V}}{\partial S}(S_f(t),t) = 1 - \frac{\partial V_e}{\partial S}(S_f(t),t)$. Therefore, the solution, $\overline{V}(S,t)$ can be obtained by equivalently solve the free boundary problem

$$\begin{aligned}
&\frac{\partial \overline{V}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \overline{V}}{\partial S^2} + (r - D_0) S \frac{\partial \overline{V}}{\partial S} - r\overline{V} = 0 \\
&\overline{V}(S,T) = 0 \\
&\overline{V}(S_f(t),t) = S_f(t) - V_e(S_f(t),t) \\
&\frac{\partial \overline{V}}{\partial S}(S_f(t),t) = 1 - \frac{\partial V_e}{\partial S}(S_f(t),t) \\
&S_f(T) = Z
\end{aligned} \tag{3.2}$$

Two remarks should be made here. Firstly, equation(3.2) is a simplified version of the free boundary problem of an American convert and the boundary conditions appear to be smoother with the presence of unknown region $S = S_f(t)$, an independent variable of European convert. We must also point out that the domain where equation(3.2) is defined is not a rectangular one. Therefore, an attempt shall be made to convert the final boundary-value problem to initial boundary value problem. To do this, we introduce a new coordinate system (ξ, τ) through the transformation defined by
$\xi = \frac{S}{S_f(t)}, \quad \tau = T - t, \quad S = \xi S_f(t)$
The above transformation maps the domain $[0, S_f(t)] \times [0,T]$ in the (S,t)-space onto the domain $[0,1] \times [0,T]$ in the space (ξ, τ). It should be obvious that the problem is now an initial boundary-value problem on a rectangular domain. Having transformed from a free boundary to a fixed boundary coordinates, we shall proceed to transform the convertible bond $\overline{V}(S,t)$, the analytic solution of European convert $V_e(S,t)$, and the unknown region $S_f(t)$ into the new coordinates using the following substitution:

$$\overline{V}(S,t) = Zu(\xi,\tau), \quad S_f(t) = Z\bar{S}_f(\tau), \quad V_e(S,t) = Zv(\xi,\tau)$$

Using $v(\xi,\tau) = V_e(S,t)/Z$, we have from equation(6)

$$\begin{aligned} v(\xi,\tau) &= \frac{C(S,t;Z)}{Z} + e^{-r(T-t)} \\ &= \frac{S}{Z}e^{-D\tau}N(d_1) - e^{-r\tau}N(d_2) + e^{-r\tau} \\ &= \frac{\xi\bar{S}_f(t)}{Z}e^{-D\tau}N(d_1) + e^{-r\tau}N(-d_2) \end{aligned} \tag{3.3}$$

where d_1 and d_2 are given in equation(2.8)
It should be noted that under the new coordinates, the value of the American convert with weaker singularity is $u(\xi,\tau) = \frac{\overline{V}(S,t)}{Z}$. If we use the variable $\xi = \frac{S}{S_f(t)}$ and the fact that $Z\bar{S}_f(\tau) = S_f(t)$ we obtain the followings;

$$\begin{aligned} \frac{\partial \overline{V}}{\partial t} &= Z\Big(-\frac{\partial u}{\partial \tau} + \frac{\xi}{\bar{S}_f(\tau)}\frac{\mathrm{d}\bar{S}_f}{\mathrm{d}\tau}\frac{\partial u}{\partial \xi}\Big) \\ \frac{\partial \overline{V}}{\partial S} &= \frac{1}{\bar{S}_f(\tau)}\frac{\partial u}{\partial \xi} \\ \frac{\partial^2 \overline{V}}{\partial S^2} &= \left(\frac{1}{\bar{S}_f(\tau)}\right)^2 \frac{1}{Z}\frac{\partial^2 u}{\partial \xi^2} \end{aligned} \tag{3.4}$$

Next, we revert to the new coordinates by substituting equation(3.4) into (3.2), hence we have

$$\begin{aligned} &\frac{\partial u}{\partial \tau} - \frac{\sigma^2\xi^2}{2}\frac{\partial^2 u}{\partial \xi^2} + \xi\big[(D_0 - r) - \frac{1}{\bar{S}_f(\tau)}\frac{\mathrm{d}\bar{S}_f}{\mathrm{d}\tau}\big]\frac{\partial u}{\partial \xi} + ru = 0 \\ &u(\xi,0) = 0 \\ &u(1,\tau) = \bar{S}_f(\tau) - v(1,\tau) \\ &\frac{\partial u}{\partial \xi}(1,\tau) = \bar{S}_f(\tau) - \frac{\partial v}{\partial \xi}(1,\tau) \\ &\bar{S}_f(0) = 1 \end{aligned} \tag{3.5}$$

where d_1 and d_2 using the new coordinates become:

$$\begin{aligned} d_1 &= \big[\ln(\xi\bar{S}e^{(r-D_0)\tau}) + \frac{1}{2}\sigma^2\tau\big]/(\sigma\sqrt{\tau}) \\ d_2 &= \big[\ln(\xi\bar{S}e^{(r-D_0)\tau}) - \frac{1}{2}\sigma^2\tau\big]/(\sigma\sqrt{\tau}) \end{aligned} \tag{3.6}$$

The non-linearity of pricing problem of an American convert is well pronounced in equation(3.5). We now have a combination of a parabolic partial differential equation problem for $u(\xi,\tau)$ and an ordinary differential equation for $S_f(\tau)$. The resulting system appears to be complex in nature, however, the non-linearity can easily be resolved with the rectangular domain.

4 Numerical methods

In this section, we discretize equation(3.5) using the centered difference to construct a numerical scheme. Suppose that M and N are two positive integers and we define:
$\Delta\xi = \frac{1}{M}, \quad \Delta\tau = \frac{T}{M}, \quad \xi_m = m\Delta\xi, \quad \tau^n = n\Delta\tau$
where $m = 0, 1, 2, ..., M$, $n = 0, 1, 2, ..., N$. Let u_m^n denote the approximate solution of u at ξ_m and τ^n. We shall further assume that $\bar{S}_f^n$ is the numerical solution of $\bar{S}_f$ at τ^n. At $n = 0$, u_m^n and $\bar{S}_f^n$ are given by the initial condition $u(\xi, 0) = 0$ and $\bar{S}_f(0) = 1$. To find each interior points, u_m^n, $m = 0, 1, 2, ..., M$ and $\bar{S}_f^n$, $n = 1, 2, ..., N$, we discretize the PDE system in equation(3.5) by

$$\begin{aligned}\frac{u_m^{n+1} - u_m^n}{\Delta\tau} &= \frac{\sigma^2 m^2}{4}\left(u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1} + u_{m+1}^n - 2u_m^n + u_{m-1}^n\right) - \frac{r}{2}\left(u_m^{n+1} + u_m^n\right)\\ &+ \frac{m}{4}\left[(r - D) + 2\left(\frac{\bar{S}_f^{n+1} - \bar{S}_f^n}{\Delta\tau\left(\bar{S}_f^{n+1} + \bar{S}_f^n\right)}\right)\right]\left(u_{m+1}^{n+1} - u_m^{n+1} + u_{m+1}^n - u_{m-1}^n\right)\end{aligned} \tag{4.1}$$

For the boundary condition at $\xi = 0$, i.e $m = 0$, we evaluate equation(4.1) at $m = 0$ and this gives

$$\frac{u_0^{n+1} - u_0^n}{\Delta\tau} = -\frac{r}{2}\left(u_0^{n+1} + u_0^n\right) \tag{4.2}$$

Here, no special boundary condition is required since the coefficient of ξ derivatives vanish and the equation itself should be used at this boundary.
At $\xi = 1$ which corresponds to $m = M$, the boundary condition is approximated as

$$u_M^{n+1} = \bar{S}_f^{n+1}(\tau^{n+1})\left(1 - e^{(r-D)\tau^{n+1}}N(d_1) + e^{-r\tau^{n+1}}N(-d_2)\right) \tag{4.3}$$

and the smoothing-condition is discretized as

$$\frac{3u_M^{n+1} - 4u_{M-1}^{n+1} + u_{M-2}^{n+1}}{2\Delta\xi} = \bar{S}_f^{n+1}(\tau^{n+1})\left(1 - e^{(r-D)\tau^{n+1}}N(d_1)\right) \tag{4.4}$$

The resulting equations(4.1)-(4.4) is a non-linear system with unknowns u_m^{n+1}, $m = 1, 2, ..., M$, and $\bar{S}_f^{n+1}$ if u_m^n and $\bar{S}_f^n$ are known. We shall therefore attempt to solve the system for $n = 0, 1, 2, ..., N - 1$ in order to obtain the unknown results. To achieve this task, iteration methods need to be used. One of the earlier approach in the literatures is the combination of secant method with the LU decomposition[3]. This approach seems promising but requires additional extrapolation algorithm to get an improved results.

Here, we adopt the approach of Quang Shi[5], the Gauss-Siedel-type iteration method.

Let $u_m^{(i)}$ and $\bar{S}_f^{(i)}$ be the ith iteration value of u_m^{n+1} and $\bar{S}_f^{n+1}$ respectively. With these new notations, the nonlinear system of equations(4.1)-(4.4) can be written in the following iteration form:

$$\begin{aligned}\frac{u_m^{(i)}-u_m^n}{\Delta\tau} &= \frac{\sigma^2 m^2}{4}\Big(u_{m+1}^{(i)}-2u_m^{(i)}+u_{m-1}^{(i)}+u_{m+1}^n-2u_m^n+u_{m-1}^n\Big)-\frac{r}{2}\Big(u_m^{(i)}+u_m^n\Big)\\ &+\frac{m}{4}\Big[(r-D)+2\Big(\frac{\bar{S}_f^{(i)}-\bar{S}_f^n}{\Delta\tau\big(\bar{S}_f^{(i)}+\bar{S}_f^n\big)}\Big)\Big]\Big(u_{m+1}^{(i)}-u_m^{(i)}+u_{m+1}^n-u_{m-1}^n\Big)\end{aligned} \tag{4.5}$$

for $m = 1, 2, ..., M$

$$\frac{u_0^{(i)}-u_0^n}{\Delta\tau} = -\frac{r}{2}\Big(u_0^{(i)}+u_0^n\Big) \quad \text{for} \quad m=0 \tag{4.6}$$

$$u_M^{(i)} = \bar{S}_f^{(i)}\Big(1-e^{-D\tau}N(d_1)+e^{-r\tau}N(-d_2)\Big) \quad \text{for} \quad m=M \tag{4.7}$$

and

$$\frac{3u_M^{(i)}-4u_{M-1}^{(i)}+u_{M-2}^{(i)}}{2\Delta\xi} = \bar{S}_f^{(i)}\Big(1-e^{-D\tau}N(d_1)\Big) \tag{4.8}$$

where

$$\begin{aligned} d_1 &= \big[\ln(\bar{S}_f^{(i)}e^{(r-D_0)\tau^{n+1}})+\frac{1}{2}\sigma^2\tau^{n+1}\big]/(\sigma\sqrt{\tau^{n+1}})\\ d_2 &= d_1-\sigma\sqrt{\tau^{n+1}}\end{aligned} \tag{4.9}$$

Suppose $u_m^{(i-1)}$ and $\bar{S}_f^{(i-1)}$ are given, this is a linear system for $u_m^{(i)}$ and $\bar{S}_f^{(i)}$, $m = 1, 2, ..., M-1$. Using equation(4.6), we can obtain $u_0^{(i)}$. Since $u_{m-1}^{(i)}$ are known, we can employ (4.5) to obtain $u_m^{(i)}$. The same procedure can be used to obtain $\bar{S}_f^{(i)}$. Finally, using equations(4.7-4.8) we find $u_M^{(i)}$ and $\bar{S}_f^{(i)}$. This system of linear equation can be solved iteratively by using Gauss-Seidel method, which is essentially a kind of iterative method. With u_m^0 and $\bar{S}_f^0$ already given as 0 and 1 respectively from equation(3.5), u_m^{n+1} and $\bar{S}_f^{n+1}$ can be obtained iteratively from u_m^n and $\bar{S}_f^n$ for $n = 0, 1, 2, ..., N-1$.

It is important to point out that at time not close to expiry, there is no fast variation of the price and optimal boundary of an American convert in this domain, hence $V(S,t)$ is quite smooth. Implicit finite difference method is preferred instead of SSM which may require large computer memory.

5 Numerical experiments and results

In this section, we perform some numerical experiments and we present our results graphically. To help readers who may not be used to discussing financial problem with dimensionless quantities, all our results are converted to original financial variables. For the numerical computation, we choose the volatility $\sigma = 40\%$, we suppose that the bond has a face value $Z = \$100$, risk free interest rate $r = 30\%$ and dividend yields of underlying asset $D_0 = 10\%$.

Figure 1 depicts a 3D plot of the today's value of the American convert with $T = 1$. The solution is computed for $0 \leq S \leq S_f(t)$ and the result is shown as a net, the nodes of which are the values of the numerical results on the computational grid.

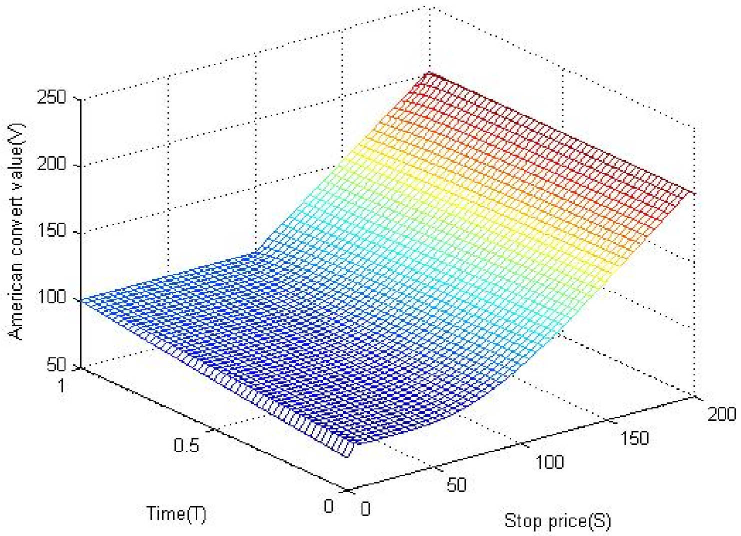

Figure 1: American convert value

Figure 2 gives another view of the American convert values $V(S, t)$ as a function of S for $t = 0, 0.25, 0.5$. The results are obtained using a grid with $M = N = 50$. We have compared different grids and we observed that variant in our result is very small. This further attest to the suitability of SSM approach for American converts near maturity. As expected, the value of an American convert is an increasing function of S and t. This trend can be seen in Figure 2

Figure 3 depicts the graph of optimal exercise boundary against the time to maturity. As expected, the critical asset price increases monotonically with time to expiry τ. In order word, the critical asset price of American convert decreases with time (t)

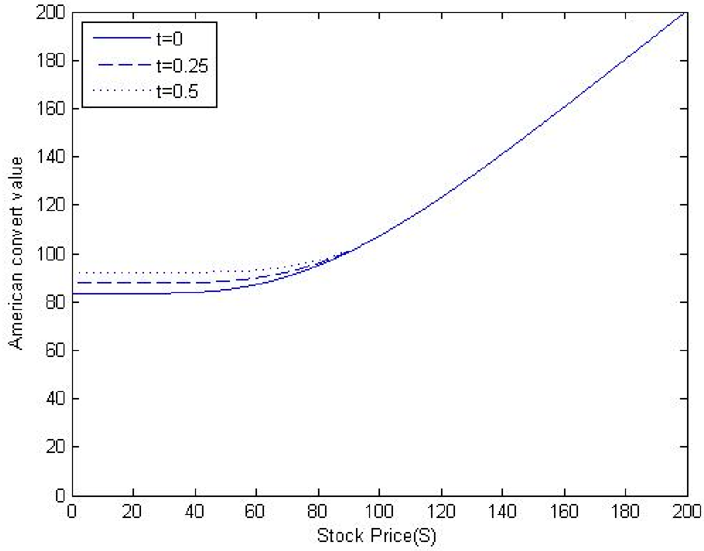

Figure 2: American convert value at t=0, 0.25, 0.5

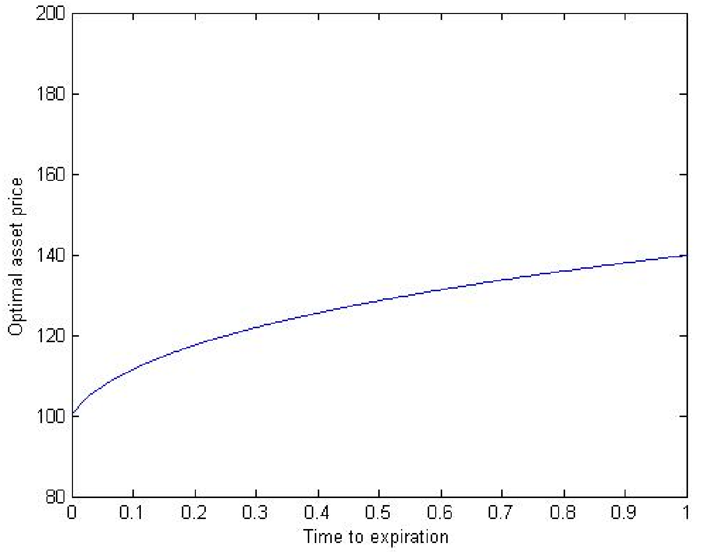

Figure 3: Optimal exercise boundary versus time to expiry

6 Conclusion

We have analyzed the pricing problem of American converts using SSM. Since a faster variation of the price and exercise boundary occur near the expiry of the contract, it is very difficult to track the solutions using the conventional numerical methods from the free boundary problem formulation. Using the SSM approach, we have separated the singularity at the the maturity from our computation. In our analysis, we have considered a fairly simple model without any optionality so that there is just only one exercise boundary on which the bond is exchanged for equity. The case of embedded features lead to additional free boundaries which are currently being worked out and the result is to be presented in a forthcoming paper. Since European converts have the same pay off function at expiry, their difference is expected to have a weak singularity at time $t \approx T$. We track the location of the free boundary from the new rectangular domain and our solutions computed numerically are quite smooth near maturity. The results obtained complement numerical approaches used to find the American convert values and the optimal exercise prices at times that are not close to expiry.

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