GEOMETRIC SERIES ON FOURIER COSINE-SINE TRANSFORM

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Abstract

The aim of this study is to provide new properties of geometric series on Fourier cosine and sine transform. The authors also presented very short form of general properties of Fourier cosine and sine transform with a product of a power series at a non-negative real number b in a very elementary ways.

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1 Introduction

Fourier startled the mathematicians in France by suggesting that any function could be expressed as an infinite series of sine. This idea started an enormous development of Fourier series. Fourier series and the Fourier transform [6, 7, 8] are concerned with dividing a function into a superposition of sine and cosine, its components of various frequencies. It is a crucial tool for understanding waves, including water waves, sound waves and light waves. Fourier analysis is a mathematical technique which enables us to decompose an arbitrary function into a superposition of oscillations which can be resolved into a sum of sine and cosine. The theory of Fourier series can be used to analyze the flow of heat in a bar and the motion of a vibrating string. Joseph Fourier a 21 years old mathematician and engineer announced a thesis which began a new chapter in the history of mathematics. Fourier's original investigations led to the theory of Fourier series were motivated by an attempt to understand heat flow [1, 2]. Nowadays, the motion of dividing a function into its components with respect to an appropriate orthonormal basis of functions is one of the key ideas of applied mathematics, useful not only as a tool for solving partial differential equations but also for many other purposes as well. In this paper, we provided presumably new general properties regarding to the Fourier cosine-sine transform of a function, $f(x-b)$, inducing the product of a power series in $(x-b)$. Similarly, we provided general properties regarding the Fourier cosine-sine transform inducing the product of a geometric series using known results.

2 Materials and Methods

The basic materials for this study are Fourier transforms. We will consider only the concepts of Fourier cosine-sine transform with some mathematical methods and also power series for this paper. We use also methods such as techniques of integration, higher order derivative of function.

Definition 2.1. Since $e^{-i\omega x} = \cos\omega x - i\sin\omega x$, we have the Fourier transform of the function $f(x)$ is given by

$$\mathcal{F}(f(x)) = \frac{1}{2}[F_c(f(x)) - iF_s(f(x))] = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(x)e^{-i\omega x}dx,$$

which says that the cosine Fourier transform of $f(x)$ is twice the real part of its Fourier transform. That is, the Fourier cosine-sine transform of the function $f(x)$ is given as:

$$\mathcal{F}_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos(\omega x)dx \quad and \quad \mathcal{F}_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin(\omega x)dx,$$

where $0 \leq \omega < \infty$, c and s represent cosine and sine respectively[2]. And for

$$i\partial_\omega \mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}} \int_0^\infty x e^{-i\omega x} f(x)dx = \mathcal{F}(xf(x))$$

we have

$$\mathcal{F}(x^m f(x)) = (i\partial_\omega)^m \mathcal{F}(f(x)).$$

Definition 2.2. A series of the form

$$G(x) = \sum_{k=0}^{\infty} ax^k = \sum_{k\ even} ax^k + \sum_{l\ odd} ax^l = a + ax + ax^2 + ...,$$

is called a geometric series. Where $a \neq 0$ is the coefficient of the series[9].

Definition 2.3. A series of the form

$$P(x) = \sum_{k=0}^{\infty} a_k (x-b)^k = \sum_{r\ even} a_r (x-b)^r + \sum_{l\ odd} a_l (x-b)^l,$$

is called a power series in $(x - b)$ or a power series at b. Where a_k are often called the coefficient of the series and it may depend on k but not on x [3]. From this if a_k is constant for all k and $b = 0$, then we get a geometric series.

It is assumed that both functions $f(x)$ and $f(x - b)$ have Fourier cosine and sine transform and using definitions, then we consider the following theorem.

3 The Main Results

Theorem. Each of the following relationships holds true.

$$\mathcal{F}_c(G(x)f(x)) = \sum_{k\ even} a(-1)^{\frac{k}{2}} \partial_\omega^k \mathcal{F}_c(f(x)) + \sum_{l\ odd} a(-1)^{\frac{l+3}{2}} \partial_\omega^l \mathcal{F}_s(f(x)); \quad (3.1)$$

$$\mathcal{F}_s(G(x)f(x)) = \sum_{k\ even} a(-1)^{\frac{k}{2}} \partial_\omega^k \mathcal{F}_s(f(x)) + \sum_{l\ odd} a(-1)^{\frac{l+1}{2}} \partial_\omega^l \mathcal{F}_c(f(x)). \quad (3.2)$$

Proof. To prove the identities (3.1) and (3.2), we use the above definitions (2.1), (2.2) and the works done[2, 4]. Suppose the function $G(x)f(x)$ has Fourier transform, then we have

$$(\mathcal{F}_c - i\mathcal{F}_s)(G(x)f(x)) = \sum_{k\ even} a(-1)^{\frac{k}{2}}\partial_\omega^k(\mathcal{F}_c - i\mathcal{F}_s)(f(x))$$

$$+\sum_{l\ odd} ia(-1)^{\frac{l-1}{2}}\partial_\omega^l(\mathcal{F}_c - i\mathcal{F}_s)(f(x)).$$

Now, identifications of real and then imaginary terms in both members give the required results (3.1) and (3.2) respectively. Because $(-1)^{\frac{l+3}{2}} = (-1)^{\frac{l-1}{2}}$. Hence we complete the proofs of (3.1) and (3.2).

Corollary. Using the above theorem, we have the following identities.

$$\mathcal{F}_c(P(x)f(x-b)) = \sum_{k\ even} a_k(-1)^{\frac{k}{2}}\partial_\omega^k\mathcal{F}_c(f(x-b))$$

$$+\sum_{l\ odd} a_l(-1)^{(\frac{l+3}{2})}\partial_\omega^l\mathcal{F}_s(f(x-b)) \tag{3.3}$$

and

$$\mathcal{F}_s(P(x)f(x-b)) = \sum_{k\ even} a_k(-1)^{\frac{k}{2}}\partial_\omega^k\mathcal{F}_s(f(x-b))$$

$$+\sum_{l\ odd} a_l(-1)^{(\frac{l+1}{2})}\partial_\omega^l\mathcal{F}_c(f(x-b)). \tag{3.4}$$

Proof. Similarly, the proof of identities (3.3) and (3.4) run parallel to that of (3.1), (3.2) and the recent work done [5]. We omit the details.

4 Concluding and Remarks

We presented the Fourier cosine and Fourier sine transform of a function, $f(x-b)$, after multiplying the given function by a power series $(x-b)$ as well as by multiplying the given function of a geometric series. This provided the relationship between Fourier cosine and Fourier sine transform. Significance of this study will help to represent the solutions of ODEs, PDEs, and integral equations that involves power series terms in the integral form of functions of cosine and sine. Moreover, the approach adopted in this paper was meant to reach not only researchers but also undergraduate students.

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