The Beta Lindley-Poisson Distribution with Applications

Mavis Pararai[1], Broderick O. Oluyede[2], and Gayan Warahena-Liyanage[3]

Abstract

The beta Lindley-Poisson (BLP) distribution which is an extension of the Lindley-Poisson Distribution is introduced and its properties are explored. This new distribution represents a more flexible model for the lifetime data. Some statistical properties of the proposed distribution including the expansion of the density function, hazard rate function, moments and moment generating function, skewness and kurtosis are explored. Rényi entropy and the distribution of the order statistics are given. The maximum likelihood estimation technique is used to estimate the model parameters and finally applications of the model to real data sets are presented for the illustration of the usefulness of the proposed distribution.

Keywords: Beta Lindley-Poisson distribution, Exponentiated Lindley distribution, Lindley distribution, Maximum likelihood estimation.

1 Introduction

Lindley [15] proposed a distribution in the context of fiducial and Bayesian statistics. Properties and applications of the Lindley distribution have been studied in the context of reliability analysis by Ghitany et al. [9], several other authors including Sankaran [28], Ghitany et al. [8] and Nadarajah et al. [20] proposed and developed the mathematical properties of the generalized Lindley distribution. The probability density function (pdf) and cumulative distribution function (cdf) of the Lindley distribution are respectively, given by

$$f(x;\beta) = \frac{\beta^2}{\beta+1}(1+x)e^{-\beta x}, \tag{1.1}$$

and

$$F(x) = 1 - \left(1 + \frac{\beta x}{\beta+1}\right)e^{-\beta x}, \tag{1.2}$$

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for $x > 0, \alpha, \beta > 0$.

The main objective of this article is to construct, develop and explore the properties of the four-parameter beta Lindley Poisson (BLP) distribution. The beauty of this model is the fact that it not only generalizes the Lindley and Lindley Poisson distribution but also exhibits the desirable properties of increasing, decreasing, and bathtub shaped hazard function.

This paper is organized as follows. In section 2, the model, sub-models and some properties including the quantile function, expansion of density, hazard function and reverse hazard function, moments and conditional moments of the BLP distribution are derived. Section 3 contains the mean deviations, Bonferroni and Lorenz curves. In section 4, the distribution of order statistics and Rényi entropy are presented. Maximum likelihood estimates of the model parameters and asymptotic confidence intervals are given in section 5. A Monte Carlo simulation study is conducted in section 6 to examine the bias and mean square error of the maximum likelihood estimators for each parameter. Applications to illustrate the usefulness of the distribution and its sub-models are given in section 7. Some concluding remarks are given in section 8.

2 The Model, Sub-models and Properties

Suppose that the random variable X has the Lindley distribution where its cdf and pdf are given in (1.2) and (1.1). Given N, let $X_1, ..., X_N$ be independent and identically distributed random variables from Lindley distribution. Let N be distributed according to the zero truncated Poisson distribution with pdf

$$P(N = n) = \frac{\theta^n e^{-\theta}}{n!(1 - e^{-\theta})}, \quad n = 1, 2, ..., \theta > 0. \tag{2.1}$$

Let X=max$(Y_1, ..., Y_N)$, then the cdf of $X|N = n$ is given by

$$G_{X|N=n}(x) \quad = \quad \left[1 - \left(1 + \frac{\beta x}{\beta + 1}\right) e^{-\beta x}\right]^n, \quad x > 0,\ \beta > 0,\ \theta > 0, \tag{2.2}$$

which is the Lindley distribution. The Lindley Poisson (LP) distribution denoted by LP(β, θ) is defined by the marginal cdf of X, that is,

$$G_{LP}(x; \beta, \theta) = \frac{1 - \exp\left\{\theta\left[1 - \left(1 + \frac{\beta x}{\beta+1}\right) e^{-\beta x}\right]\right\}}{1 - e^{\theta}} \tag{2.3}$$

for $x > 0, \beta > 0, \theta > 0$. The Lindley Poisson density function is given by

$$g_{LP}(x; \beta, \theta) = \frac{\theta\beta^2(1 + x)e^{-\beta x} \exp\left\{\theta\left[1 - \left(1 + \frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]\right\}}{(\beta + 1)(e^{\theta} - 1)}. \tag{2.4}$$

In this section we present the beta Lindley Poisson (BLP) distribution and derive some properties of this class of distributions including the cdf, pdf, expansion of the density, hazard and reverse hazard functions, shape and sub-models. Let $G(x)$ denote the cdf of a continuous random variable X and define a general class of distributions by

$$F(x) = \frac{B_{G(x)}(a,b)}{B(a,b)}, \tag{2.5}$$

where $B_{G(x)}(a,b) = \int_0^{G(x)} t^{a-1}(1-t)^{b-1}dt$ and $1/B(a,b) = \Gamma(a+b)/\Gamma(a)\Gamma(b)$. The class of generalized distributions above was motivated by the work of Eugene et al. [7]. They proposed and studied the beta-normal distribution. Some beta-generalized distributions discussed in the literature include the work done by Jones [12], Nadarajah and Kotz [18], Nadarajah and Gupta [17], Nadarajah and Kotz [17], Barreto-Souza et al. [1] proposed the beta-Gumbel, beta-Frechet, beta-exponential (BE), beta-exponentiated exponential distributions, respectively. Gusmao et al. [11] presented results on the generalized inverse Weibull. Pescim et al. [25] proposed and studied the beta-generalized half-normal distribution which contains some important distributions such as the half-normal and generalized half normal (Cooray and Ananda) [5] as special cases. Furthermore, Cordeiro et al. [6] presented the generalized Rayleigh distribution and Carrasco et al. [2] studied the generalized modified Weibull distribution with applications to lifetime data. More recently, Oluyede et al. [24] studied the beta generalized Lindley distribution with applications.

By considering $G(x)$ as the cdf of the Lindley Poisson (LP) distribution we obtain the beta Lindley Poisson (BLP) distribution with a broad class of distributions that may be applicable in a wide range of day to day situations including applications in medicine, reliability and ecology. The cdf and pdf of the four-parameter BLP distribution are respectively given by

$$\begin{aligned} F_{BLP}(x;\beta,\theta,a,b) &= \frac{1}{B(a,b)} \int_0^{G_{LP}(x;\theta,\beta)} t^{a-1}(1-t)^{b-1}dt \\ &= \frac{B_{G_{LP}(x)}(a,b)}{B(a,b)}, \end{aligned} \tag{2.6}$$

and

$$
\begin{aligned}
f_{BLP}(x;\beta,\theta,a,b) &= \frac{1}{B(a,b)}\Big[G_{LP}(x)\Big]^{a-1}\Big[1-G_{LP}(x)\Big]^{b-1} g_{LP}(x), \\
&= \frac{\theta\beta^2(1+x)e^{-\beta x}\exp\left\{\theta\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]\right\}}{B(a,b)(\beta+1)(e^{\theta}-1)} \\
&\times \left(1-\exp\left\{\theta\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]\right\}\right)^{a-1} \\
&\times \left(\exp\left\{\theta\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]\right\}-e^{\theta}\right)^{b-1} \\
&\times (1-e^{\theta})^{2-a-b}, \qquad (2.7)
\end{aligned}
$$

for $x>0, \beta>0, \theta>0, a>0, b>0$. Plots of the pdf of the BLP distribution for several values of β, θ, a and b are given in Figure below. Using the substitu-

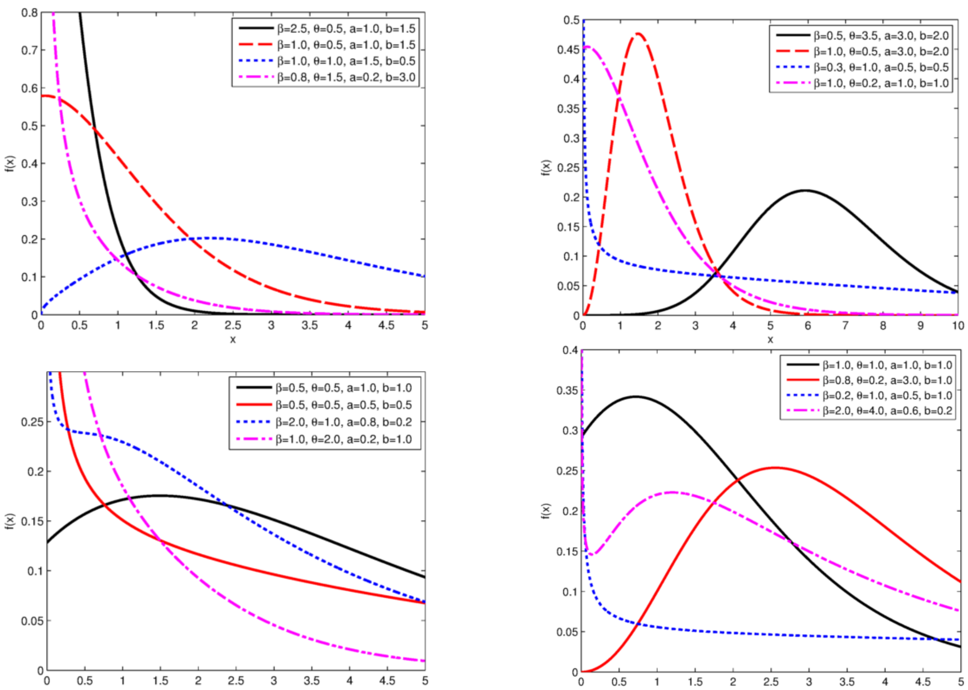

Figure 2.1: Plots of the pdf for different values of β, θ, a and b

tion $\omega = \theta\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]$, we can write the pdf of the BLP distribution as

$$
\begin{aligned}
f(x;\beta,\theta,a,b) &= \frac{\theta\beta^2(1+x)e^{-\beta x}e^{\omega}(1-e^{\omega})^{a-1}(e^{\omega}-e^{\theta})^{b-1}}{B(a,b)(\beta+1)(e^{\theta}-1)} \\
&\times (1-e^{\theta})^{2-a-b}. \qquad (2.8)
\end{aligned}
$$

The cumulative distribution function of of the BLP random variable X is given by

$$F_{BLP}(x) = I_{G_{PL}(x)}(a, b) = I_{\frac{1-e^{\omega}}{1-e^{\theta}}}(a, b). \tag{2.9}$$

2.1 Quantile Function

The quantile function of the BLP distribution, say $x = F^{-1}(y; \beta, \theta, a, b)$, is obtained by solving the non-linear equation

$$\beta x - \ln\left(1 + \frac{\beta x}{\beta + 1}\right) + \ln\left(1 - \ln\left(1 - y\left(1 - e^{\theta}\right)\right)^{\frac{1}{\theta}}\right) = 0, \tag{2.10}$$

where $y = Q_{a,b}(u)$ is the quantile function of the beta distribution with parameters a and b. We can readily simulate from the BLP distribution, since if Y is a random variable having a beta distribution with parameters a and b, then the solution of the equation (2.10) gives the quantiles of the BLP distribution.

Note also that the quantile function of the BLP distribution can be obtained via the Lambert W function as follows: Let $Z(p) = -1 - \beta - \beta Q(p)$, then from $1 - \left(1 + \frac{\beta x}{\beta+1}\right) \exp\left(-\beta x\right) = \ln\left(1 - y\left(1 - e^{\theta}\right)\right)^{\frac{1}{\theta}}$, we have

$$1 + \frac{Z(p)}{\beta + 1} \exp\left(Z(p) + 1 + \beta\right) = \ln\left(1 - y\left(1 - e^{\theta}\right)\right)^{\frac{1}{\theta}}, \tag{2.11}$$

that is,

$$Z(p) \exp\left(Z(p)\right) = \frac{-(\beta + 1)}{\exp(\beta + 1)} \left(1 - \ln\left(1 - y\left(1 - e^{\theta}\right)\right)^{\frac{1}{\theta}}\right). \tag{2.12}$$

Thus,

$$Z(p) = W\left(\frac{-(\beta + 1)}{\exp(\beta + 1)} \left(1 - \ln\left(1 - y\left(1 - e^{\theta}\right)\right)^{\frac{1}{\theta}}\right)\right), \tag{2.13}$$

$0 < p < 1$, where $W(.)$ is the Lambert function. The quantile function of the BLP distribution is obtained by solving for $Q(p)$ and is given by

$$Q(p) = -1 - \frac{1}{\beta} - \frac{1}{\beta} W\left(\frac{-(\beta + 1)}{\exp(\beta + 1)} \left(1 - \ln\left(1 - y\left(1 - e^{\theta}\right)\right)^{\frac{1}{\theta}}\right)\right). \tag{2.14}$$

Consequently, random number can be generated based on equation (2.14). Table 2.1 lists the quantile for selected values of the parameters of the BLP distribution.

Table 2.1: BLP quantile for selected values

	(β,θ,a,b)				
u	(1.5,1.5,0.5,0.5)	(1.5,0.5,1.5,0.5)	(0.5,0.5,1.5,1.5)	(0.5,1.0,0.5,1.5)	(1.0,1.0,1.0,1.0)
0.1	0.0615	0.5394	1.0102	0.0624	0.3219
0.2	0.2266	0.8739	1.5698	0.2386	0.6202
0.3	0.4610	1.1846	2.0652	0.5072	0.9137
0.4	0.7445	1.5008	2.5501	0.8539	1.2153
0.5	1.0755	1.8427	3.0550	1.2773	1.5384
0.6	1.4665	2.2334	3.6105	1.7902	1.9008
0.7	1.9509	2.7106	4.2621	2.4283	2.3323
0.8	2.6066	3.3541	5.1021	3.2799	2.8967
0.9	3.6832	4.4124	6.4126	4.6304	3.7921

2.2 Expansion of the Density Function

The expansion of the pdf of BLP distribution is presented in this section. For $b > 0$ a real non-integer, we use the series representation

$$(1 - G_{LP}(x))^{b-1} = \sum_{i=0}^{\infty} \binom{b-1}{i} (-1)^i \left[G_{LP}(x)\right]^i, \tag{2.15}$$

where $G_{_{LP}}(x) = G_{LP}(x;\beta,\theta)$ is the Lindley Poisson cdf. If $b > 0$ is a real non-integer and $a > 0$ is also a real non-integer, from Equation (2.15) and the above expansion, we can rewrite the density of the BLP distribution as

$$\begin{aligned} f_{BLP}(x) &= \sum_{j=0}^{\infty} \binom{b-1}{j} (-1)^j \frac{g_{LP}(x)}{B(a,b)} \left[G_{LP}(x)\right]^{a+j-1} \qquad (2.16)\\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{b-1}{j}\binom{a+j-1}{k} \frac{(-1)^{a+2j+k-1}[e^{\theta(1+k)}-1]}{(1+k)B(a,b)(e^{\theta}-1)^{a+j}} \\ &\times \frac{\beta^2\theta(1+k)(1+x)e^{-\beta x}}{(\beta+1)[e^{\theta(1+k)}-1]} \exp\left\{\theta(k+1)\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]\right\} \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \lambda_{\theta,a,b}(j,k) f(x;\theta(1+k),\beta), \qquad (2.17) \end{aligned}$$

where $\lambda_{\theta,a,b}(j,k) = \binom{b-1}{j}\binom{a+j-1}{k}\frac{(-1)^{a+2j+k-1}[e^{\theta(1+k)}-1]}{(1+k)B(a,b)(e^{\theta}-1)^{a+j}}$. This shows that the BLP distribution can be written as a linear combination of Lindley Poisson density functions. Hence mathematical properties of the BLP distribution can be obtained from those of the LP properties.

2.3 Some sub-models of the BLP distribution

In this section we present some sub-models of the BLP distribution for selected values of the parameters β, θ, a and b.

- If $a = b = 1$ and $\theta \to 0^+$, we obtain Lindley (L) distribution.
- When $a = b = 1$, we obtain the Lindley Poisson (LP) distribution whose cdf and pdf are given by equations (2.3) and (2.4), respectively.
- When $b = 1$, we obtain the exponentiated Lindley Poisson (ELP) distribution.
- When $\theta \to 0^+$, we get the beta Lindley (BL) distribution (Oluyede and Yang [24]).
- When $b = 1$ and $\theta \to 0^+$, we get the exponentiated Lindley (EL) distribution.

2.4 Hazard and Reverse Hazard Rate Functions

The hazard and reverse hazard functions for the BLP distribution will be presented in this section. Using some selected values of β, θ, a and b, some plots of the hazard function will be presented. The hazard and reverse hazard functions of the BLP distribution are given by

$$h_{BLP}(x;\beta,\theta,a,b) = \frac{f_{BLP}(x;\beta,\theta,a,b)}{1 - F_{BLP}(x;\beta,\theta,a,b)} = \frac{g_{LP}(x)[G_{LP}(x)]^{a-1}[1 - G_{LP}(x)]^{b-1}}{B(a,b) - B_{G_{LP}(x)}(a,b)}, \tag{2.18}$$

and

$$\tau_{BLP}(x;\beta,\theta,a,b) = \frac{f_{BLP}(x;\beta,\theta,a,b)}{F_{BLP}(x;\beta,\theta,a,b)} = \frac{g_{LP}(x)[G_{PL}(x)]^{a-1}[1 - G_{LP}(x)]^{b-1}}{B_{G_{LP}(x)}(a,b)}, \tag{2.19}$$

respectively for $x > 0, \beta > 0, \theta > 0, a > 0$ and $b > 0$. The graphs of the hazard rate function exhibit increasing, decreasing, bathtub, bathtub followed by upside down bathtub shapes for the selected values of the model parameters.

2.5 Moments

The r^{th} moment of a continuous random variable X, denoted by μ'_r, is ,

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx \quad \text{for } r = 0, 1, 2, \ldots \quad . \tag{2.20}$$

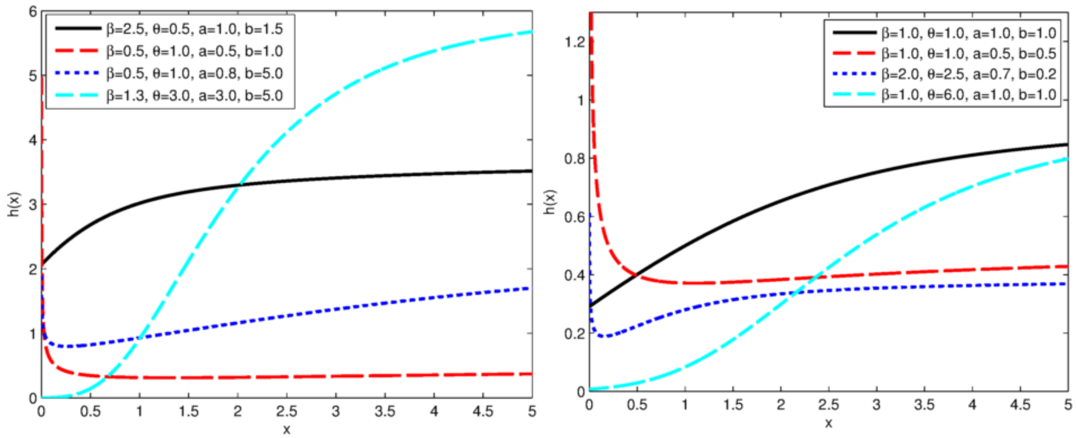

Figure 2.2: Plots of the hazard function for different values of β, θ, a and b

In order to find the moments, consider the following lemma.

Lemma 1
Let

$$L_1(\beta,\theta,a,b,r) = L_1 = \int_0^\infty x^r(1+x)\exp\left\{\theta(k+1)\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]\right\}dx, \tag{2.21}$$

then

$$\begin{aligned} L_1(\beta,\theta,a,b,r) &= \sum_{p=0}^{\infty}\sum_{q=0}^{p}\sum_{s=0}^{q}\sum_{m=0}^{s+1}\frac{(-1)^q\theta^p(k+1)^p\beta^s}{p!(\beta+1)^q}\binom{p}{q}\binom{q}{s}\binom{s+1}{m} \\ &\times \frac{\Gamma(r+m+1)}{[\beta q]^{r+m+1}}. \end{aligned} \tag{2.22}$$

Proof. We apply the following series expansions: If b is a positive real non-integer and $|z|<1$, then

$$(1-z)^{b-1} = \sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(b)}{\Gamma(b-j)j!}z^j, \tag{2.23}$$

and $e^z = \sum_{p=0}^{\infty}\frac{z^p}{p!}$, so that by considering the right hand side of equation (2.21), we have

$$
\begin{aligned}
L_1 &= \sum_{p=0}^{\infty} \frac{\theta^p (k+1)^p}{p!} \int_0^{\infty} x^r (1+x) \left[1 - \left(1 + \frac{\beta x}{\beta+1}\right) e^{-\beta x}\right]^p dx \\
&= \sum_{p=0}^{\infty} \sum_{q=0}^{p} \frac{(-1)^q \theta^p (k+1)^p}{p!(\beta+1)^q} \binom{p}{q} \int_0^{\infty} [1+\beta(1+x)]^q x^r (1+x) e^{-\beta x q} dx \\
&= \sum_{p=0}^{\infty} \sum_{q=0}^{p} \sum_{s=0}^{q} \sum_{m=0}^{s+1} \frac{(-1)^q \theta^p (k+1)^p \beta^s}{p!(\beta+1)^q} \binom{p}{q} \binom{q}{s} \binom{s+1}{m} \\
&\times \int_0^{\infty} x^{r+m} e^{-\beta x q} dx. \qquad (2.24)
\end{aligned}
$$

By letting $u = \beta x q$, we have $x = \frac{u}{\beta q}$ and $dx = \frac{du}{\beta q}$. Thus,

$$
\int_0^{\infty} x^{r+m} e^{-\beta x q} dx = \frac{1}{[\beta q]^{r+m+1}} \int_0^{\infty} u^{r+m} e^{-u} du = \frac{\Gamma(r+m+1)}{[\beta q]^{r+m+1}}.
$$

□

By using Lemma 1, the r^{th} moment of the BLP distribution is

$$
\mu'_r = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\beta^2 \theta(k+1)}{(\beta+1)[e^{\theta(k+1)} - 1]} \lambda_{\theta,a,b}(j,k) L_1(\beta, \theta, a, b, r), \qquad (2.25)
$$

where

$$
\lambda_{\theta,a,b}(j,k) = \binom{b-1}{j} \binom{a+j-1}{k} \frac{(-1)^{a+2j+k-1}[e^{\theta(1+k)} - 1]}{(1+k)B(a,b)(e^{\theta} - 1)^{a+j}}. \qquad (2.26)
$$

Plots of the skewness and kurtosis for selected choices of the parameter b as a function of a, as well as for some selected choices of a as a function of b are displayed in Figures 2.3 and 2.4. These plots clearly indicate that the skewness and kurtosis depend on the shape parameters a and b. Table 2.2 lists the first four moments of the BLP distribution for selected values of the parameters, by fixing $\beta = 1.5$ and $\theta = 1.5$, and Table 2.3 lists the first four moments of the BLP distribution for selected values of the parameters, by fixing $a = 1.5$ and $b = 1.5$. These values can be determined numerically using R and MATLAB.

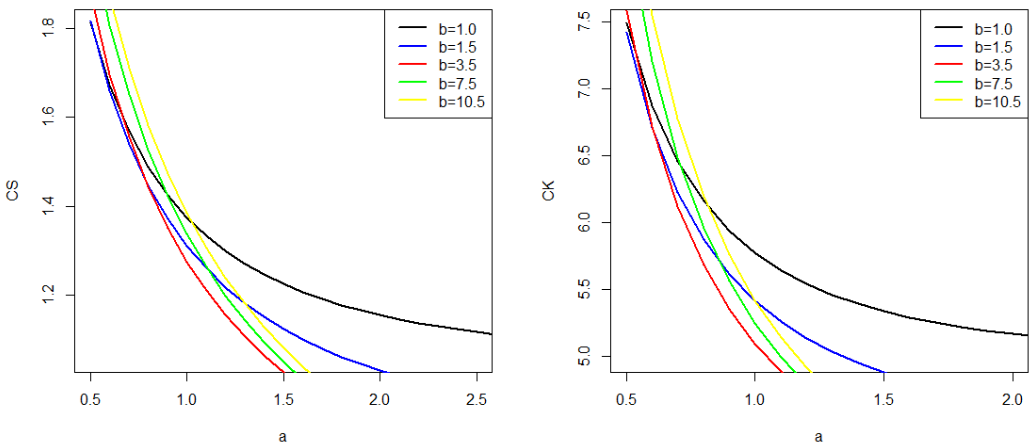

Figure 2.3: Skewness and Kurtosis of BLP distribution as a function of a for some values of b with $\beta = 0.5$ and $\theta = 0.5$.

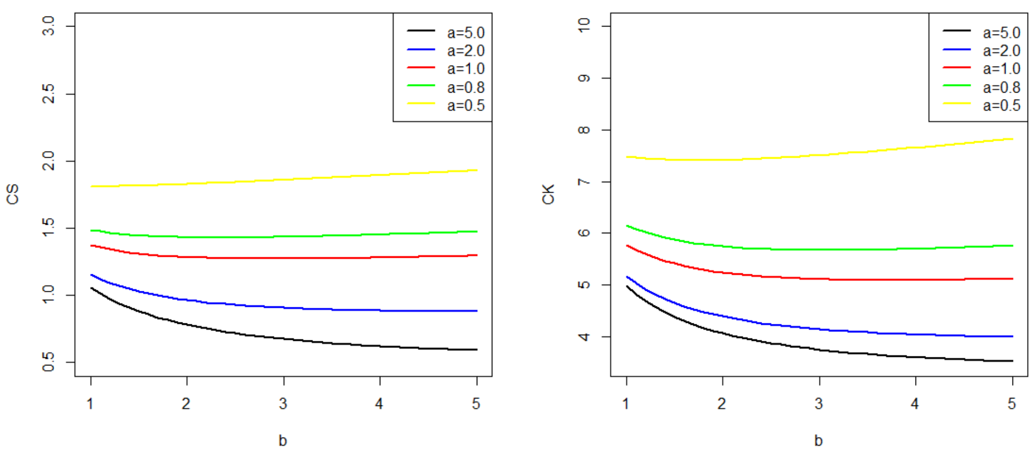

Figure 2.4: Skewness and Kurtosis of BLP distribution as a function of b for some values of a with $\beta = 0.5$ and $\theta = 0.5$.

Table 2.2: Moments of the *BLP* distribution for some parameter values; $\beta = 1.5$ and $\theta = 1.5$.

μ'_s	$a = 0.5, b = 1.0$	$a = 1.0, b = 1.5$	$a = 1.0, b = 1.0$	$a = 1.5, b = 1.0$
μ'_1	0.8272	0.9427	1.2764	1.5739
μ'_2	1.4564	1.3998	2.5600	3.4525
μ'_3	3.6466	2.7287	6.8107	9.6269
μ'_4	11.6674	6.5182	22.4884	32.6293
SD	0.8787	0.7149	0.9648	0.9876
CV	1.0623	0.7583	0.7559	0.6275
CS	1.7160	1.2192	1.2995	1.1657
CK	7.0045	5.0662	5.5140	5.1799

Table 2.3: Moments of the *BLP* distribution for some parameter values; $a = 1.5$ and $b = 1.5$.

μ'_s	$\beta = 1.0, \theta = 1.0$	$\beta = 0.5, \theta = 0.5$	$\beta = 0.5, \theta = 1.5$	$\beta = 1.5, \theta = 0.5$
μ'_1	1.7341	3.4566	4.2328	0.9601
μ'_2	4.2358	16.8027	23.5029	1.3879
μ'_3	13.0924	103.6617	158.8613	2.6361
μ'_4	48.7020	771.9723	1260.5950	6.1728
SD	1.1085	2.2033	2.3635	0.6827
CV	0.6392	0.6374	0.5584	0.7111
CS	1.0909	1.1238	0.9154	1.2836
CK	4.7591	4.8794	4.3074	5.4133

The moment generating function of a random variable X having the BLP distribution is given by

$$
\begin{aligned}
E(e^{tX}) &= \sum_{l=0}^{\infty} \frac{t^l}{l!} \int_0^{\infty} x^l f(x) dx \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{t^l \beta^2 \theta(k+1)}{l!(\beta+1)[e^{\theta(k+1)} - 1]} \lambda_{\theta,a,b}(j,k) L_1(\beta, \theta, a, b, l),
\end{aligned}
$$

by Lemma 1, where $\lambda_{\theta,a,b}(j,k)$ is defined in (2.26).

2.6 Conditional Moments

For lifetime models, it may be useful to know about the conditional moments which can be defined as $E(X^r \mid X > x)$. In order to calculate these, we consider the following lemma whose proof is along the same lines as Lemma 1.

Lemma 2

Let

$$L_2(\beta,\theta,a,b,r,t) = \int_t^{\infty} x^r(1+x)\exp\left\{\theta(k+1)\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]\right\}dx,$$

then

$$L_2(\beta,\theta,a,b,r,t) = \sum_{p=0}^{\infty}\sum_{q=0}^{p}\sum_{s=0}^{q}\sum_{m=0}^{s+1}\frac{(-1)^q\theta^p(k+1)^p\beta^s}{p!(\beta+1)^q}\binom{p}{q}\binom{q}{s}\binom{s+1}{m}\frac{\Gamma\left[(r+m+1,\beta tq)\right]}{[\beta q]^{r+m+1}}.$$

Note that

$$L_2(\beta,\theta,a,b,r,t) = \sum_{p=0}^{\infty}\sum_{q=0}^{p}\sum_{s=0}^{q}\sum_{m=0}^{s+1}\frac{(-1)^q\theta^p(k+1)^p\beta^s}{p!(\beta+1)^q}\binom{p}{q}\binom{q}{s}\binom{s+1}{m}\int_t^{\infty}x^{r+m}e^{-\beta xq}dx.$$

By letting $u=\beta xq$, we have $x=\frac{u}{\beta q}$ and $dx=\frac{du}{\beta q}$. Thus when $x=t, u=\beta tq$.

$$\begin{aligned}\int_{\beta tq}^{\infty}x^{r+m}e^{-\beta x(q)}dx &= \frac{1}{[\beta(q+1)]^{r+m+1}}\int_{\beta tq}^{\infty}u^{r+m}e^{-u}du\\ &= \frac{\Gamma\left[(r+m+1,\beta tq)\right]}{[\beta q]^{r+m+1}}.\end{aligned}$$

We therefore have

$$\begin{aligned}L_2(\beta,\theta,a,b,r,t) &= \sum_{p=0}^{\infty}\sum_{q=0}^{p}\sum_{s=0}^{q}\sum_{m=0}^{s+1}\frac{(-1)^q\theta^p(k+1)^p\beta^s}{p!(\beta+1)^q}\binom{p}{q}\binom{q}{s}\binom{s+1}{m}\\ &\times \frac{\Gamma\left[(r+m+1,\beta tq)\right]}{[\beta q]^{r+m+1}}.\end{aligned}$$

By applying Lemma 2, the rth conditional moment of the BLP distribution is given by

$$\begin{aligned}E(X^r|X>x) &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\lambda_{\beta,\theta,a,b}(j,k)\beta^2\theta(k+1)}{(\beta+1)[e^{\theta(k+1)}-1]}\frac{L_2(\beta,\theta,a,b,r,x)}{1-F_{BPL}(x)}\\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\lambda_{\beta,\theta,a,b}(j,k)\beta^2\theta(k+1)}{(\beta+1)[e^{\theta(k+1)}-1]}\frac{L_2(\beta,\theta,a,b,r,x)}{1-I_{\frac{1-e^{\omega}}{1-e^{\theta}}}(a,b)},\end{aligned}$$

where $\lambda_{\beta,\theta,a,b}(j,k)$ is defined in (2.26).

2.7 Reliability

We derive the reliability R when X and Y have independent BLP$(\theta_1, \beta_1, a_1, b_1)$ and BLP$(\theta_2, \beta_2, a_2, b_2)$ distributions, respectively. Note that from (2.6), the BLP cdf can be written as

$$\begin{aligned} F_{BLP}(x;\beta,\theta,a,b) &= \frac{1}{B(a,b)}\int_0^{G_{LP}(x;\beta,\theta)} t^{a-1}(1-t)^{b-1}dt \\ &= \frac{1}{B(a,b)}\sum_{m=0}^{\infty}\binom{b-1}{m}\frac{(-1)^m}{a+m} \\ &\times \left[\frac{1-\exp\left\{\theta\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]\right\}}{1-e^{\theta}}\right]^{a+m}. \end{aligned} \tag{2.27}$$

From equation (2.27) of the pdf of the BLP and the above equation, the reliability to be given by

$$R = P(X > Y) = \int_0^{\infty} f_X(x;\beta_1,\theta_1,a_1,b_1)F_Y(x;\beta_2,\theta_2,a_2,b_2)dx.$$

Thus after applying some series expansions we have

$$\begin{aligned} F_Y(x;\beta_2,\theta_2,a_2,b_2) &= \frac{1}{B(a_2,b_2)}\sum_{m=0}^{\infty}\binom{b_2-1}{m}\frac{(-1)^m}{(a_2+m)(1-e^{\theta_2})^{a_2+m}} \\ &\times \left[1-\exp\left\{\theta_2\left[1-\left(1+\frac{\beta_2 x}{\beta_2+1}\right)e^{-\beta_2 x}\right]\right\}\right]^{a_2+m}. \\ &= \frac{1}{B(a_2,b_2)}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\binom{b_2-1}{m}\binom{a_2+m}{p} \\ &\times \frac{(-1)^{m+p}}{(a_2+m)(1-e^{\theta_2})^{a_2+m}} \\ &\times \left\{\exp\left(\theta_2 p\left[1-\left(\frac{1+\beta_2[1+x]}{1+\beta_2}\right)e^{-\beta_2 x}\right]\right)\right\} \\ &= \frac{1}{B(a_2,b_2)}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\binom{b_2-1}{m}\binom{a_2+m}{p}\frac{(-1)^{m+p}(\theta_2 p)^q}{(a_2+m)(1-e^{\theta_2})^{a_2+m}q!} \\ &\times \left[1-\left(\frac{1+\beta_2[1+x]}{1+\beta_2}\right)e^{-\beta_2 x}\right]^q \\ &= \frac{1}{B(a_2,b_2)}\sum_{m,p,q=0}^{\infty}\sum_{r=0}^{q}\sum_{s=0}^{r}\sum_{t=0}^{s}\binom{b_2-1}{m}\binom{a_2+m}{p}\binom{q}{r} \\ &\times \binom{r}{s}\binom{s}{t}\frac{(-1)^{m+p+r}(\theta_2 p)^q\beta_2^s x^t e^{-\beta_2 r x}}{(a_2+m)(1-e^{\theta_2})^{a_2+m}q!(1+\beta_2)^r}. \end{aligned} \tag{2.28}$$

From the expansion of the pdf in (2.2), we have

$$\begin{aligned}
f_X(x;\beta_1,\theta_1,a_1,b_1) &= \frac{1}{B(a_1,b_1)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{b_1-1}{i}\binom{a_1+i-1}{j} \\
&\times \frac{(-1)^{a_1+2i+j-1}\theta_1\beta_1^2(1+x)e^{-\beta_1 x}}{(\beta_1+1)(e^{\theta_1}-1)^{a_1+i}} \\
&\times \exp\left\{\theta_1(j+1)\left[1-\left(\frac{1+\beta_1[1+x]}{\beta_1+1}\right)e^{-\beta_1 x}\right]\right\} \\
&= \frac{1}{B(a_1,b_1)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{u=0}^{l}\sum_{v=0}^{u+1}\binom{b_1-1}{i}\binom{a_1+i-1}{j} \\
&\times \binom{k}{l}\binom{l}{u}\binom{u+1}{v}\frac{(-1)^{a_1+j+l-1}\theta_1^{k+1}(j+1)^k\beta_1^{u+2}}{(\beta_1+1)^{l+1}(e^{\theta_1}-1)^{a_1+j}k!} \\
&\times x^v e^{-\beta_1(l+1)x} \\
&= \frac{1}{B(a_1,b_1)}\sum_{i,j,k=0}^{\infty}\sum_{w=0}^{\infty}\sum_{l=0}^{k}\sum_{u=0}^{l}\sum_{v=0}^{u+1}\binom{b_1-1}{i}\binom{a_1+i-1}{j} \\
&\times \binom{k}{l}\binom{l}{u}\binom{u+1}{v}\frac{(-1)^{a_1+j+l+w-1}\theta_1^{k+1}(j+1)^k\beta_1^{u+w+2}}{(\beta_1+1)^{l+1}(e^{\theta_1}-1)^{a_1+j}k!w!} \\
&\times (l+1)^w x^{v+w}. \qquad (2.29)
\end{aligned}$$

By using parts of the last two equations and the substitution $u=\beta_2 rx$, we thus have

$$\begin{aligned}
\int_0^{\infty} x^{t+v+w}e^{-\beta_2 rx}dx &= \frac{1}{(\beta_2 r)^{t+v+w+1}}\int_0^{\infty} u^{t+v+w}e^{-u}du \\
&= \frac{\Gamma(t+v+w+1)}{(\beta_2 r)^{t+v+w+1}}. \qquad (2.30)
\end{aligned}$$

Using equation (2.28), the reliability of the BLP distribution is given by

$$\begin{aligned}
R &= \frac{1}{B(a_1,b_1)B(a_2,b_2}\sum_{i,j,k=0}^{\infty}\sum_{w=0}^{\infty}\sum_{l=0}^{k}\sum_{u=0}^{l}\sum_{v=0}^{u+1}\binom{b_1-1}{i}\binom{a_1+i-1}{j} \\
&\times \binom{k}{l}\binom{l}{u}\binom{u+1}{v}\frac{\theta_1^{k+1}(j+1)^k\beta_1^{u+w+2}(l+1)^w}{(\beta_1+1)^{l+1}(e^{\theta_1}-1)^{a_1+j}k!w!} \\
&\times \sum_{m,p,q=0}^{\infty}\sum_{r=0}^{q}\sum_{s=0}^{r}\sum_{t=0}^{s}\binom{b_2-1}{m}\binom{a_2+m}{p}\binom{q}{r} \\
&\times \binom{r}{s}\binom{s}{t}\frac{(-1)^{a_1+j+l+w+m+p+r-1}(\theta_2 p)^q\beta_2^s}{(a_2+m)(1-e^{\theta_2})^{a_2+m}q!(1+\beta_2)^r} \\
&\times \frac{\Gamma(t+v+w+1)}{(\beta_2 r)^{t+v+w+1}}. \qquad (2.31)
\end{aligned}$$

3 Mean Deviations, Bonferroni and Lorenz Curves

In this section, we present the mean deviation about the mean and mean deviation about the median of the BLP distribution. Also presented are Bonferroni and Lorenz curves. Bonferroni and Lorenz curves are widely used tools for analyzing and visualizing income inequality. Lorenz curve, $L(p)$ can be regarded as the proportion of total income volume accumulated by those units with income lower than or equal to the volume q, and Bonferroni curve, $B(p)$ is the scaled conditional mean curve, that is, ratio of group mean income of the population.

3.1 Mean Deviations

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. These are known as the mean deviation about the mean and the mean deviation about the median and are defined by

$$\delta_1(x) = \int_0^\infty |x-\mu| f_{_{BLP}}(x)dx \quad \text{and} \quad \delta_2(x) = \int_0^\infty |x-M| f_{_{BLP}}(x)dx, \quad (3.1)$$

respectively, where $\mu = E(X)$ and $M =$Median (X) denotes the median. The measures $\delta_1(x)$ and $\delta_2(x)$ can be calculated using the relationships

$$\delta_1(x) = 2\mu F_{_{BLP}}(\mu) - 2\mu + 2\int_\mu^\infty x f_{_{BLP}}(x)dx, \quad (3.2)$$

and

$$\delta_2(x) = -\mu + 2\int_M^\infty x f_{_{BLP}}(x)dx, \quad (3.3)$$

respectively. When $r = 1$, we get the mean $\mu = E(X)$ from equation (2.20). Note that $T(\mu) = \int_\mu^\infty x f_{_{BLP}}(x)dx$ and $T(M) = \int_M^\infty x f_{_{BLP}}(x)dx$ are given by $T(\mu) = L_2(\beta, \theta, a, b, 1, \mu)$ and $T(M) = L_2(\beta, \theta, a, b, 1, M)$, respectively, so that the mean deviation about the mean and the mean deviation about the median are given by

$$\delta_1(x) = 2\mu F_{_{BLP}}(\mu) - 2\mu + 2T(\mu) \quad \text{and} \quad \delta_2(x) = -\mu + 2T(M),$$

respectively.

3.2 Bonferroni and Lorenz Curves

In this subsection, we present Bonferroni and Lorenz curves. Bonferroni and Lorenz curves have applications not only in economics for the study of income and poverty, but also in other fields such as reliability, demography, insurance and medicine. Bonferroni and Lorenz curves for the BLP distribution are given by

$$B(p) = \frac{1}{p\mu}\int_0^q x f_{BLP}(x)dx = \frac{1}{p\mu}[\mu - T(q)],$$

and

$$L(p) = \frac{1}{\mu}\int_0^q x f_{BLP}(x)dx = \frac{1}{\mu}[\mu - T(q)],$$

respectively, where $T(q) = \int_q^\infty x f_{BLP}(x)dx = L_2(\beta, \theta, a, b, 1, q)$, $q = F_{BLP}^{-1}(p)$, $0 \leq p \leq 1$.

4 Order Statistics and Rényi Entropy

In this section, the distribution of order statistics and Rényi entropy for the BLP distribution are presented. The concept of entropy plays a vital role in information theory. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty.

4.1 Distribution of Order Statistics

Suppose that $X_1, \cdots, X_n$ is a random sample of size n from a continuous pdf, $f(x)$. Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the corresponding order statistics. If $X_1, \cdots, X_n$ is a random sample from BLP distribution, it follows from the equations (2.6) and (2.7) that the pdf of the k^{th} order statistics, say $Y_k = X_{k:n}$ is given by

$$\begin{aligned} f_k(y_k) &= \frac{n!}{(k-1)!(n-k)!}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\left(\frac{B_{G_{LP}(y_k;\beta,\theta)}(a,b)}{B(a,b)}\right)^{k-1+l} \\ &\times \frac{\theta\beta^2(1+y_k)e^{-\beta y_k}e^{\omega}(1-e^{\omega})^{a-1}(e^{\omega}-e^{\theta})^{b-1}}{B(a,b)(\beta+1)(e^{\theta}-1)}(1-e^{\theta})^{2-a-b}. \end{aligned} \tag{4.1}$$

The corresponding cdf of Y_k is

$$\begin{aligned} F_k(y_k) &= \sum_{j=k}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l[F(y_k)]^{j+l} \\ &= \sum_{j=k}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l\left[\frac{B_{G_{LP}(y_k;\beta,\theta)}(a,b)}{B(a,b)}\right]^{j+l}. \end{aligned}$$

4.2 Rényi Entropy

Rényi entropy [26] is an extension of Shannon entropy. Rényi entropy is defined to be $H_v(f_{BLP}(x;\beta,\theta,a,b)) = \frac{\log\left(\int_0^\infty f_{BLP}^v(x;\beta,\theta,a,b)dx\right)}{1-v}$, where $v > 0$, and $v \neq 1$. Rényi entropy tends to Shannon entropy as $v \to 1$. Note that

$$\begin{aligned}
\int_0^\infty f_{BLP}^v(x)dx &= \left(\frac{1}{B(a,b)}\right)^v \sum_{j=0}^{\infty} (-1)^j \binom{bv-v}{j} \int_0^\infty g_{PL}^v(x)[G_{PL}(x)]^{j+av-v}dx \\
&= \left[\frac{\theta\beta^2}{B(a,b)(\beta+1)(e^\theta-1)}\right]^v \sum_{j=0}^{\infty} (-1)^j \binom{bv-v}{j} \\
&\times \int_0^\infty \exp\left\{\theta v\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]\right\} \\
&\times \frac{(-1)^{j+av-v}(1+x)^v e^{-\beta v x}}{(e^\theta-1)^{j+av-v}} \\
&\times \left(1-\exp\left\{\theta\left[1-\left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right]\right\}\right)^{j+av-v} dx \\
&= C \sum_{j,k,m=0}^{\infty} \sum_{p=0}^{m} \sum_{q=0}^{p} \binom{bv-v}{j}\binom{j+av-v}{k}\binom{m}{p}\binom{p}{q} \\
&\times \frac{(-1)^{av-v+k+p}[\theta(v+k]^m\beta^q}{(e^\theta-1)^{j+av-v}(\beta+1)^p m!} \int_0^\infty (1+x)^{q+v} e^{-\beta(p+v)x}dx \\
&= C \sum_{j,k,m=0}^{\infty} \sum_{p=0}^{m} \sum_{q=0}^{p} \sum_{r=0}^{q+v} \binom{bv-v}{j}\binom{j+av-v}{k}\binom{m}{p}\binom{p}{q}\binom{q+v}{r} \\
&\times \frac{(-1)^{av-v+k+p}[\theta(v+k]^m\beta^q}{(e^\theta-1)^{j+av-v}(\beta+1)^p m!} \int_0^\infty x^r e^{-\beta(p+v)x}dx, \qquad (4.2)
\end{aligned}$$

where $C = \left[\frac{\theta\beta^2}{B(a,b)(\beta+1)(e^\theta-1)}\right]^v$. Letting $u = \beta x(p+v) \Rightarrow x = \frac{u}{\beta(p+v)}$, so that $dx = \frac{du}{\beta(p+v)}$. Hence

$$\int_0^\infty x^r e^{-\beta(p+v)x}dx = \frac{1}{[\beta(p+v)]^{r+1}} \int_0^\infty x^r e^{-u}du = \frac{\Gamma(r+1)}{[\beta(p+v)]^{r+1}}. \qquad (4.3)$$

Consequently, Rényi entropy for the BLP distribution is given by

$$\begin{aligned} H_v(f_{BLP}) &= \frac{1}{1-v} \log \Bigg[\sum_{j,k,m=0}^{\infty} \sum_{p=0}^{m} \sum_{q=0}^{p} \sum_{r=0}^{q+v} \binom{bv-v}{j} \binom{j+av-v}{k} \\ &\times \binom{m}{p} \binom{p}{q} \binom{q+v}{r} \frac{(-1)^{av-v+k+p} [\theta(v+k]^m \beta^q}{(e^\theta - 1)^{j+av-v} (\beta+1)^p m!} \\ &\times \left(\frac{\theta \beta^2}{B(a,b)(\beta+1)(e^\theta - 1)} \right)^v \frac{\Gamma(r+1)}{[\beta(p+v)]^{r+1}} \Bigg], \end{aligned} \tag{4.4}$$

for $v > 0, v \neq 1$.

5 Maximum Likelihood Estimation

Let $x_1, \cdots, x_n$ be a random sample from the BLP distribution.The log-likelihood function is given by

$$\begin{aligned} L &= n \log(\theta) + 2n \log(\beta) + \sum_{i=0}^{n} \log(1+x_i) - \beta \sum_{i=0}^{n} x_i + \sum_{i=0}^{n} \omega_i \\ &+ (a-1) \sum_{i=0}^{n} \log(1 - e^{\omega_i}) + (b-1) \sum_{i=0}^{n} \log(e^{\omega_i} - e^\theta) \\ &+ n(2-a-b) \log(1-e^\theta) + n \log(\Gamma(a+b)) - n \log(\beta+1) \\ &- n \log(e^\theta - 1) - n \log \Gamma(a) - n \log \Gamma(b). \end{aligned} \tag{5.1}$$

The elements of the score vector are given by

$$\frac{\partial L}{\partial a} = n[\psi(a+b) - \psi(a)] + \sum_{i=0}^{n} \log(1 - e^{\omega_i}) - n \log(1 - e^\theta),$$

$$\frac{\partial L}{\partial b} = n[\psi(a+b) - \psi(b)] + \sum_{i=0}^{n} \log(e^{\omega_i} - e^\theta) - n \log(1 - e^\theta),$$

$$\begin{aligned} \frac{\partial L}{\partial \beta} &= \frac{2n}{\beta} - \frac{n}{\beta+1} - \sum_{i=0}^{n} x_i + \sum_{i=0}^{n} \frac{\partial \omega_i}{\partial \beta} - (a-1) \sum_{i=0}^{n} \frac{e^{\omega_i} \frac{\partial \omega_i}{\partial \beta}}{1 - e^{\omega_i}} \\ &+ (b-1) \sum_{i=0}^{n} \frac{e^{\omega_i} \frac{\partial \omega_i}{\partial \beta}}{e^{\omega_i} - e^\theta}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= \frac{n}{\theta} + \sum_{i=0}^{n} \frac{\partial \omega_i}{\partial \theta} - (a-1) \sum_{i=0}^{n} \frac{e^{\omega_i} \frac{\partial \omega_i}{\partial \theta}}{1 - e^{\omega_i}} + (b-1) \sum_{i=0}^{n} \frac{e^{\omega_i} \frac{\partial \omega_i}{\partial \theta} - e^\theta}{e^{\omega_i} - e^\theta} \\ &+ \frac{n(1-a-b)e^\theta}{e^\theta - 1} - \frac{ne^\theta}{e^\theta - 1}, \end{aligned}$$

respectively. Note that since $\omega = \theta\left[1 - \left(1 + \frac{\beta x}{\beta+1}\right) e^{-\beta x}\right]$, we have

$$\frac{\partial \omega}{\partial \beta} = \theta e^{-\beta x}\left[\left(1 + \frac{\beta x}{\beta + 1}\right) - \frac{1}{(\beta+1)^2}\right]$$

and

$$\frac{\partial \omega}{\partial \theta} = \left[1 - \left(1 + \frac{\beta x}{\beta + 1}\right) e^{-\beta x}\right].$$

The maximum likelihood estimates, $\hat{\Theta}$ of $\Theta = (a, b, \beta, \theta)$ are obtained by solving the nonlinear equations $\frac{\partial l}{\partial a} = 0$, $\frac{\partial l}{\partial b} = 0$, $\frac{\partial l}{\partial \beta} = 0$, and $\frac{\partial l}{\partial \theta} = 0$. These equations are not in closed form and the values of the parameters a, b, β and θ must be found by using iterative methods.

6 Simulation

In this section, we study the performance and accuracy of maximum likelihood estimates of the BLP model parameters by conducting various simulations for different sample sizes and different parameter values. Equation (2.14) is used to generate random data from the BLP distribution. The simulation study is repeated for $N = 5,000$ times each with sample size $n = 25, 100, 200, 400, 800, 1000$ and parameter values $I : \beta = 2.0, \theta = 1.0, a = 0.2, b = 0.5$ and $II : \beta = 2.0, \theta = 5.0, a = 0.5, b = 0.1$. Four quantities are computed in this simulation study.

(a) Average bias of the MLE $\hat{\vartheta}$ of the parameter $\vartheta = \beta, \theta, a, b$:

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{\vartheta} - \vartheta).$$

(b) Root mean squared error (RMSE) of the MLE $\hat{\vartheta}$ of the parameter $\vartheta = \beta, \theta, a, b$:

$$\sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\vartheta} - \vartheta)^2}.$$

Table 6.1 presents the Average Bias and RMSE values of the parameters β, θ, a and b for different sample sizes. From the results, we can verify that as the sample size n increases, the RMSEs decay toward zero. We also observe that for all the parametric values, the biases decrease as the sample size n increases.

Table 6.1: Monte Carlo Simulation Results: Average Bias, RMSE, CP, and AW

		I		II	
Parameter	n	Average Bias	RMSE	Average Bias	RMSE
β	25	0.5593	1.6583	-0.0050	1.0538
	100	0.1369	0.6980	-0.0866	0.7302
	200	0.0747	0.4119	-0.0753	0.5520
	400	0.0342	0.1853	-0.0530	0.3744
	800	0.0167	0.0900	-0.0221	0.257
	1000	0.0147	0.0688	-0.0095	0.2244
θ	25	2.2498	3.3033	2.6000	10.4471
	100	0.5314	1.0674	0.6384	3.1532
	200	0.2213	0.6090	0.2227	1.9917
	400	0.0679	0.2962	0.0768	1.3389
	800	0.0300	0.1619	0.0184	0.9354
	1000	0.0244	0.1213	0.0384	0.8346
a	25	-0.0046	0.0523	0.0947	0.4295
	100	0.0005	0.0248	0.0427	0.2169
	200	0.0004	0.0172	0.0302	0.1508
	400	0.0010	0.0118	0.0179	0.1039
	800	0.0010	0.0083	0.0096	0.0717
	1000	0.0010	0.0073	0.0055	0.0625
b	25	24.0923	581.3696	27.2032	322.4362
	100	0.5152	7.7664	2.6522	31.0830
	200	0.0717	1.1833	0.8088	6.1785
	400	0.0062	0.0977	0.2167	1.1007
	800	0.0023	0.0335	0.0746	0.3091
	1000	0.0019	0.0264	0.0465	0.2357

7 Applications

In this section, the BLP distribution is applied to two real data sets in order to illustrate the usefulness and applicability of the model. We fit the density functions of the beta Lindley Poisson and its sub-models, namely, the exponentiated Lindley Poisson (ELP), the Lindley-Poisson (LP) (2.4), beta Lindley (BL), exponentiated Lindley (EL) and Lindley (L) (1.1) (Lindley [15]) distributions. Estimates of the parameters of BLP distribution (standard error in parentheses), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), sum of squares (SS) from the probability plots, Cramer Von Mises W^* and Anderson-Darling A^*, statistics are presented for each data set. We also compare the BLP distribution with the beta inverse Weibull distribution (BIW) (Hanook et al. [10]), inverse Weibull Poisson (IWP) distribution (Mahmoudi and Torki [16]) and the Burr XII Poisson (BXIIP) (Silva et al. [33]) distribution. The pdfs of the BIW, IWP and BXIIP distributions are respectively given by

$$g_{BIW}(x) = \frac{\alpha\beta x^{-\beta-1}}{B(a,b)} e^{-a\alpha x^{-\beta}} \left(1 - e^{-a\alpha x^{-\beta}}\right)^{b-1},$$

for $x > 0, \alpha, \beta, a, b > 0$,

$$g_{IWP}(x) = \frac{\theta\alpha\beta x^{-\beta-1} e^{-\alpha x^{-\beta}} \exp(\theta e^{-\alpha x^{-\beta}})}{e^{\theta} - 1}$$

for $x > 0, \alpha, \beta, \theta > 0$, and

$$g_{BXIIP}(x) = \frac{cks^{-c}\lambda}{1 - e^{-\lambda}} x^{c-1} \left[1 + \left(\frac{x}{s}\right)^{c}\right]^{-k-1} \exp\left\{ - \lambda \left[1 - \left(1 + \left(\frac{x}{s}\right)^{c}\right)^{-k}\right]\right\}$$

for $x > 0, c, k, s, \lambda > 0$.

The maximum likelihood estimates (MLEs) of the BLP parameters $\Theta = (\beta, \theta, a, b)$ are computed by maximizing the objective function via the subroutine mle2 in R. The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic, Akaike Information Criterion, $AIC = 2p - 2\ln(L)$, Bayesian Information Criterion, $BIC = p\ln(n) - 2\ln(L)$, and Consistent Akaike Information Criterion, $AICC = AIC + 2\frac{p(p+1)}{n-p-1}$, where $L = L(\hat{\Theta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are presented in Table 5. The goodness-of-fit statistics W^* and A^*, described by Chen and Balakrishnan [4] are also presented in the table. These statistics can be used to verify which distribution fits better to the data. In general, the smaller the values of W^* and A^*, the better the fit. The BLP distribution is fitted to the data sets and these fits are compared to the fits using the Lindley Poisson (LP), Lindley (L), BIW, IWP and BXIIP distributions.

We maximize the likelihood function using NLmixed in SAS as well as the function nlm in R ([27]). These functions were applied and executed for wide range of initial values. This process often results or lead to more than one maximum, however, in these cases, we take the MLEs corresponding to the largest value of the maxima. In a few cases, no maximum was identified for the selected initial values. In these cases, a new initial value was tried in order to obtain a maximum.

The issues of existence and uniqueness of the MLEs are theoretical interest and has been studied by several authors for different distributions including [30], [29], [36], and [34].

We can use the likelihood ratio (LR) test to compare the fit of the BLP distribution with its sub-models for a given data set. For example, to test $a = 1, b = 1$ the LR statistic is $\omega = 2[\ln(L(\hat{\beta}, \hat{\theta}, \hat{a}, \hat{b})) - \ln(L(\tilde{\beta}, \tilde{\theta}, 1, 1))]$, where $\hat{\beta}$, $\hat{\theta}$, $\hat{a}$, and $\hat{b}$ are the unrestricted estimates, and $\tilde{\beta}$, and $\tilde{\theta}$, are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi^2_\epsilon$, where χ^2_ϵ denote the upper $100\epsilon\%$ point of the χ^2 distribution with 2 degrees of freedom.

Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [3]) are given in Figures. For the probability plot, we plotted $F_{BLP}(x_{(j)}; \hat{\beta}, \hat{\theta}, \hat{a}, \hat{b})$ against $\dfrac{j - 0.375}{n + 0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares

$$SS = \sum_{j=1}^{n} \left[F_{BLP}(x_{(j)}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2 .$$

Estimates of the parameters of BLP distribution (standard error in parentheses), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Kolmogorov-Smirnov (KS) and p-values are given in Table 7.4 for the data set.

7.1 Glass Fibers Data

The first data set set consists of 63 observations of the strengths of 1.5 cm glass fibers, originally obtained by workers at the UK National Physical Laboratory. The data was also studied by Smith and Naylor [32]. The data observations are given below:

0.55	0.93	1.25	1.36	1.49	1.52	1.58	1.61	1.64	1.68	1.73	1.81	2.00
0.74	1.04	1.27	1.39	1.49	1.53	1.59	1.61	1.66	1.68	1.76	1.82	2.01
0.77	1.11	1.28	1.42	1.50	1.54	1.60	1.62	1.66	1.69	1.76	1.84	2.24
0.81	1.13	1.29	1.48	1.50	1.55	1.61	1.62	1.66	1.70	1.77	1.84	0.84
1.24	1.30	1.48	1.51	1.55	1.61	1.63	1.67	1.70	1.78	1.89	-	-

Table 7.1: Glass Fiber data set

The results of the fitted estimates from the glass fibers data are given in the table below.

Table 7.2: Estimates of Models for Glass Fibers Data

	Estimates				Statistics								
Model	$\hat{\beta}$	$\hat{\theta}$	$\hat{a}$	$\hat{b}$	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	SS	KS	P-Value
BLP	0.7795	15.0649	0.9653	1299.04	28.2	36.2	36.9	44.8	0.1610	0.9050	0.1536	0.1336	0.2108
	(0.1691)	(4.9931)	(0.6088)	(26.5179)									
ELP	3.0456	14.8150	1.9806	1.0000	59.3	65.3	65.7	71.7	0.7237	3.964912	0.7157	0.2189	0.0048
	(0.2338)	(22.1224)	(2.9033)	-									
LP	3.0456	29.3432	1.0000	1.0000	59.3	63.3	63.5	67.6	0.7237	3.9649	0.7157	0.2189	0.0047
	(0.2338)	(8.3871)	-	-									
BL	0.0445	-	9.9976	2028.85	42.4	48.4	48.9	54.9	0.4753	2.6087	0.4708	0.2038	0.0107
	(0.0039)	-	(1.5952)	(12.4978)									
EL	2.9901	-	26.1716	1.0000	61.2	65.2	65.4	69.5	0.7616	4.1585	0.7609	0.2264	0.0031
	(0.1382)	-	(3.3519)										
L	0.9961	-	-	-	162.6	164.6	164.6	166.7	0.5424	2.9764	3.3017	0.3869	1.35×10^{-8}
	(0.09484)												
	α	β	a	b									
BIW	17.4507	0.7946	0.4075	44019	32.4	40.4	41.1	49.0	0.2825	1.5470	0.3189	0.1792	0.0349
	(2.8918)	(0.2082)	(0.1971)	(621.97)									
	α	β	θ										
IWP	0.6365	3.8769	5.2630		75.8	81.8	82.2	88.2	0.9833	5.3106	0.8549	0.2163	0.0055
	(0.1824)	(0.3096)	(1.3853)										
	c	k	s	λ									
BXIIP	5.8213	136.32	8.4467	107.26	30.4	38.4	39.1	47.0	0.2383	1.3091	0.2108	0.1522	0.1082
	(0.6384)	(5223.30)	(57.9923)	(671.70)									

The estimated variance-covariance matrix for the BLP distribution is given by:

$$\begin{pmatrix} 0.02860 & 0.8389 & -0.09761 & 0.7895 \\ 0.8389 & 24.9308 & -2.9592 & 25.4505 \\ -0.09761 & -2.9592 & 0.3706 & -3.1535 \\ 0.7895 & 25.4505 & -3.1535 & 703.20 \end{pmatrix}$$

and the 95% two-sided asymptotic confidence intervals for β, θ, a and b are given by $0.7795 \pm 0.3315, 15.0649 \pm 9.7862, 0.9653 \pm 1.1932$, and 1299.04 ± 51.974, respectively.

The LR test statistics of the hypotheses $H_0 : LP$ vs $H_a : BLP$ and $H_0 : L$ vs $H_a : BLP$ are 31.1 (p-value=$1.8 \times 10^{-7} < 0.0001$) and 134.4 (p-value=$6.1 \times 10^{-29} < 0.0001$). Also tested were $H_0 : BL$ vs $H_a : BLP$ and $H_0 : EL$ vs $H_a : BLP$ which resulted in p-values of 1.6×10^{-4} and 6.8×10^{-8} respectively. The BLP distribution is significantly better than its sub-models namely LP,L,BL and EL distributions based on the likelihood ratio tests. The BLP distribution was compared to non-nested BIW, IWP and BXIIP distributions using the AIC, AICC, BIC, W^*, A^* and SS statistics. The model with the smallest value for each of the statistics will be the best one to be used in fitting the data. Comparing the BLP distribution with the non-nested BIW, IWP and BXIIP distributions, we note that the BLP distribution is better based on the AIC, AICC and BIC values. The BLP distribution has the smallest goodness of fit statistic W^* and A^* values as well as the smallest SS value among all the models that were fitted. Moreover, the BLP model has points closer to the diagonal line corresponding to the smallest SS value for the probability plots when compared to the non-nested distributions. The values of KS statistic as well as the corresponding p-value clearly indicate that the BLP is better that the sub-models and the non-nested distributions for the glass fibers data set. Hence, the BLP distribution is the “best” fit for the glass fibers data when compared to all the other models that were considered.

Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 7.1. The plots also show that the BLP distribution is the “best” fit for the glass fibers data.

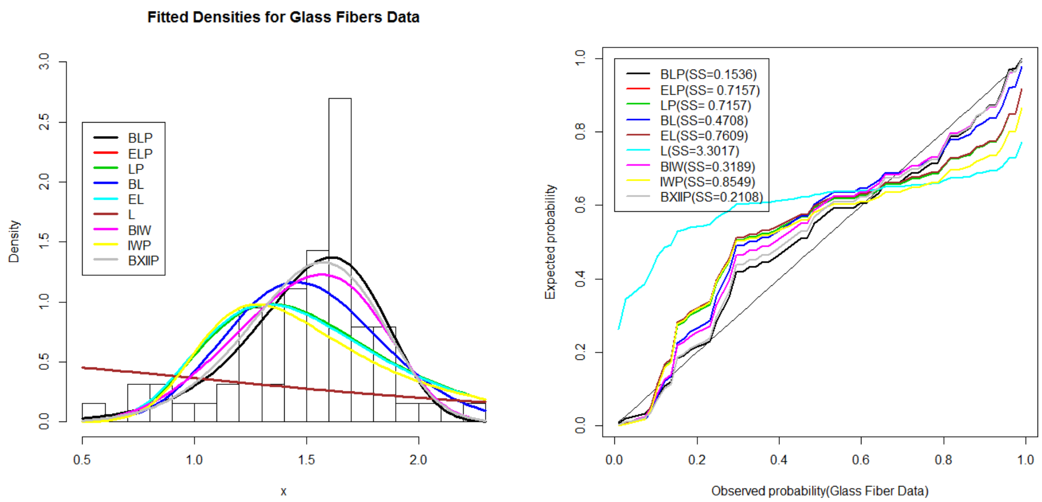

Figure 7.1: Histogram, Fitted Density and Probability Plots for Glass Fiber Data

7.2 Taxes Data

The second real data set represents the taxes data set. This data was previously analyzed by Nassar and Nada [21]. The data consists of the monthly actual taxes revenue in Egypt from January 2006 to November 2010. The data is highly skewed to the right. The actual taxes revenue data (in million Egyptian pounds) are:

5.90	20.4	14.9	16.2	17.2	7.80	6.10	9.20	10.2	9.60	13.3	8.50	21.6
18.5	5.10	6.70	17.0	8.60	9.70	39.2	35.7	15.7	9.70	10.0	4.10	36.0
8.50	8.00	9.20	26.2	21.9	16.7	21.3	35.4	14.3	8.50	10.6	19.1	20.5
7.10	7.70	18.1	16.5	11.9	7.0	8.60	12.5	10.3	11.2	6.10	8.40	11.0
11.6	11.9	5.20	6.80	8.90	7.10	10.8	-	-	-	-	-	-

Table 7.3: Taxes data set

The results of the fitted estimates from the taxes data are given in the table below.

Table 7.4: Estimates of Models for Taxes Data

	Estimates				Statistics								
Model	$\hat{\beta}$	$\hat{\theta}$	$\hat{a}$	$\hat{b}$	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	SS	KS	P-Value
BLP	0.5152	120.19	0.1359	0.2407	374.8	382.8	383.5	391.1	0.0234	0.1812	0.0221	0.0537	0.9957
	(0.1011)	(86.1168)	(0.08455)	(0.06973)									
ELP	0.2606	33.8110	0.1410	1.0000	392.2	398.2	398.6	404.4	0.2446	1.4957	0.2538	0.1341	0.2390
	(0.02358)	(849.45)	(3.5411)	-									
LP	0.2449	3.9650	1.0000	1.0000	390.8	394.8	395.0	398.9	0.2442	1.4954	0.2198	0.1326	0.2506
	(0.02883)	(1.1896)	-	-									
BL	0.9436	-	40.5032	0.1479	376.0	382.0	382.5	388.3	0.0470	0.3042	0.0506	0.0761	0.8842
	(0.1234)	-	(14.1156)	(0.03542)									
EL	0.2210	-	2.8125	1.0000	384.6	388.6	388.8	392.7	0.1718	1.0499	0.1884	0.1261	0.3057
	(0.02610)	-	(0.6984)										
L	0.1392	-	-	-	401.3	403.3	403.3	405.3	0.2091	1.3027	0.4463	0.1922	0.0256
	(0.01286)												
	α	β	a	b									
BIW	22.9693	1.4162	2.0904	2.3427	376.7	384.7	385.5	393.0	0.0426	0.2588	0.0429	0.0648	0.9654
	(13.3203)	(0.8463)	(2.4243)	(2.8068)									
	α	β	θ										
IWP	16.3803	2.3788	12.6655		377.4	383.4	383.8	389.6	0.0354	0.2427	0.0326	0.0680	0.9481
	(70.1303)	(0.4900)	(39.1263)										
	c	k	s	λ									
BXIIP	5.2146	0.01674	8.2517	26.1080	378.8	386.8	387.5	395.1	0.0490	0.3182	0.0456	0.0850	0.7869
	(1.5604)	(0.02677)	(1.2993)	(36.2426)									

The estimated variance-covariance matrix for the BLP distribution is given by:

$$\begin{pmatrix} 0.01021 & 3.0640 & 0.0031181 & -0.00447 \\ 3.0640 & 7416.11 & -5.1954 & -3.9193 \\ 0.003118 & -5.1954 & 0.007149 & 0.001393 \\ -0.00447 & -3.9193 & 0.001393 & 0.004863 \end{pmatrix}$$

and the 95% two-sided asymptotic confidence intervals for β, θ, a and b are given by 0.5152 ± 0.1982, 120.19 ± 168.7889, 0.1359 ± 0.1657, and 0.2407 ± 0.1367, respectively.

The LR test statistics of the hypotheses $H_0 : LP$ vs $H_a : BLP$ and $H_0 : L$ vs $H_a : BLP$ are 16.0 (p-value$=3.4\times10^{-4} < 0.0001$) and 26.5 (p-value$=7.5\times10^{-6} < 0.0001$). Also tested were $H_0 : ELP$ vs $H_a : BLP$ and $H_0 : EL$ vs $H_a : BLP$ which resulted in p-values of 3.02×10^{-5} and 7.4×10^{-3} respectively. The BLP distribution is significantly better than its sub-models namely LP, L, ELP and EL distributions based on the likelihood ratio tests. The BLP distribution was compared to non-nested BIW, IWP and BXIIP distributions using the AIC, AICC, BIC, W^*, A^* and SS statistics. The model with the smallest value for each of the statistics will be the best one to be used in fitting the data. Comparing the BLP distribution with the non-nested BIW, IWP and BXIIP distributions, we note that the BLP distribution is better based on the AIC, AICC and BIC values. The BLP distribution has the smallest goodness of fit statistic W^* and A^* values as well as the smallest SS value among all the models that were fitted. Moreover, the BLP model has points closer to the diagonal line corresponding to the smallest SS value for the probability plots when compared to the non-nested distributions. The values of KS statistic as well as the corresponding p-value clearly indicate that the BLP is better that the sub-models and the non-nested distributions for the taxes data set. Hence, the BLP distribution is the "best" fit for the taxes data when compared to all the other models that were considered.

Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 7.2. The plots also show that the BLP distribution is the "best" fit for the taxes data.

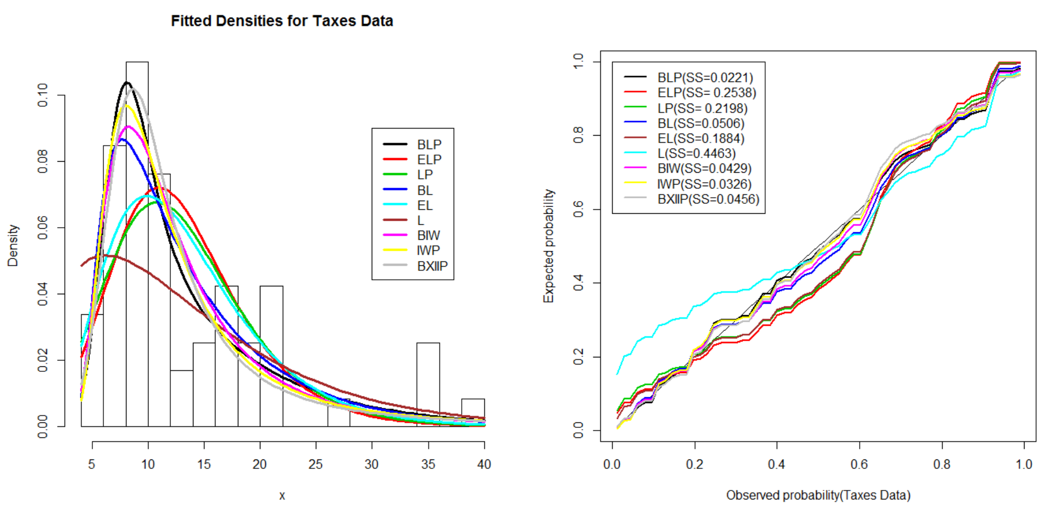

Figure 7.2: Histogram, Fitted Density and Probability Plots for Taxes Data

8 Concluding Remarks

A new and generalized Lindley Poisson distribution called the beta Lindley Poisson (BLP) distribution is proposed and studied. The BLP distribution has several well known distributions including the Lindley, exponentiated Lindley-Poisson and Lindley-Poisson distributions as special cases. The density of this new class of distributions can be expressed as a linear combination of LP density functions. The BLP distribution possesses a hazard function with flexible behavior. We also obtain closed form expressions for the moments, mean and median deviations, distribution of order statistics and Rényi entropy. Maximum likelihood estimation technique is used to estimate the model parameters. Finally, the BLP distribution is fitted to real data sets to illustrate its applicability and usefulness.

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Appendix A Observed Information

Elements of the observed information matrix can be readily obtained from the mixed partial derivatives of the log-likelihood function. The mixed second partial derivatives of the log-likelihood function are given by

$$\frac{\partial^2 L}{\partial a^2} = n[\psi'(a+b) - \psi'(a)], \quad \frac{\partial^2 L}{\partial a \partial b} = n[\psi'(a+b)],$$

$$\frac{\partial^2 L}{\partial a \partial \beta} = -\sum_{i=0}^{n} \frac{e^{\omega_i} \frac{\partial \omega_i}{\partial \theta}}{1 - e^{\omega_i}}, \quad \frac{\partial^2 L}{\partial a \partial \theta} = -\sum_{i=0}^{n} \frac{e^{\omega_i} \frac{\partial \omega_i}{\partial \theta}}{1 - e^{\omega_i}} + \frac{ne^{\theta}}{1 - e^{\theta}},$$

$$\frac{\partial^2 L}{\partial b^2} = n[\psi'(a+b) - \psi'(b)], \quad \frac{\partial^2 L}{\partial b \partial \beta} = \sum_{i=0}^{n} \frac{e^{\omega_i} e^{\omega_i}}{e^{\omega_i} - e^{\theta}},$$

$$\frac{\partial^2 L}{\partial b \partial \theta} = \sum_{i=0}^{n} \frac{e^{\omega_i} \frac{\partial \omega_i}{\partial \theta} - e^{\theta}}{e^{\omega_i} - e^{\theta}} + \frac{ne^{\theta}}{1 - e^{\theta}},$$

$$\begin{aligned}
\frac{\partial^2 L}{\partial \beta^2} &= \frac{-2n}{\beta^2} + \frac{n}{(\beta+1)^2} - \sum_{i=0}^{n} \frac{\partial^2 \omega_i}{\partial \beta^2} \\
&- (a-1) \sum_{i=0}^{n} \frac{(1 - e^{\omega_i}) e^{\omega_i} \left[\left(\frac{\partial \omega_i}{\partial \beta} \right)^2 + \frac{\partial^2 \omega_i}{\partial \beta^2} \right] + e^{2\omega_i} \left(\frac{\partial \omega_i}{\partial \theta} \right)^2}{(1 - e^{\omega_i})^2} \\
&+ (b-1) \sum_{i=0}^{n} \frac{(e^{\omega_i} - e^{\theta}) e^{\omega_i} \left[\left(\frac{\partial \omega_i}{\partial \beta} \right)^2 + \frac{\partial^2 \omega_i}{\partial \beta^2} \right] - 2e^{2\omega_i} \left(\frac{\partial \omega_i}{\partial \theta} \right)^2}{(e^{\omega_i} - e^{\theta})^2},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 L}{\partial \beta \partial \theta} &= -\sum_{i=0}^{n} \frac{\partial^2 \omega_i}{\partial \beta \partial \theta} \\
&- (a-1) \sum_{i=0}^{n} \frac{e^{\omega_i}(1 - e^{\omega_i}) \left[\frac{\partial \omega_i}{\partial \theta} \frac{\partial \omega_i}{\partial \beta} + \frac{\partial^2 \omega_i}{\partial \beta \partial \theta} \right] + e^{2\omega_i} \left(\frac{\partial \omega_i}{\partial \theta} \frac{\partial \omega_i}{\partial \beta} \right)}{(1 - \omega_i)^2} \\
&+ (b-1) \sum_{i=0}^{n} \frac{e^{\omega_i}(e^{\omega_i} - e^{\theta}) \left[\frac{\partial \omega_i}{\partial \theta} \frac{\partial \omega_i}{\partial \beta} + \frac{\partial^2 \omega_i}{\partial \beta \partial \theta} \right] - e^{\omega_i} \frac{\partial \omega_i}{\partial \beta} \left[e^{\omega_i} \frac{\partial e^{\omega_i}}{\partial \theta} - e^{\theta} \right]}{(\omega_i - e^{\theta})^2},
\end{aligned}$$

and

$$\frac{\partial^2 L}{\partial \theta^2} = \frac{-n}{\theta^2} + \sum_{i=0}^{n} \frac{\partial^2 \omega_i}{\partial \theta^2} + \frac{(n-a-b)e^{\theta}}{(e^{\theta}-1)^2}$$
$$- (a-1)\sum_{i=0}^{n} \frac{e^{\omega_i}(1-e^{\omega_i})\left[\left(\frac{\partial \omega_i}{\partial \theta}\right)^2 + \frac{\partial^2 \omega_i}{\partial \theta^2}\right] + e^{2\omega_i}\left(\frac{\partial \omega_i}{\partial \theta}\right)^2}{(1-\omega_i)^2}$$
$$+ (b-1)\sum_{i=0}^{n} \frac{(e^{\omega_i}-e^{\theta})\left[e^{\omega_i}\left(\frac{\partial \omega_i}{\partial \theta}\right)^2 + e^{\omega_i(\frac{\partial^2 \omega_i}{\partial \theta^2})-e^{\theta}}\right] - \left[e^{\omega_i}(\frac{\partial \omega_i}{\partial \theta}) - e^{\theta}\right]^2}{(e^{\omega_i}-e^{\theta})^2}.$$

Notice that

$$\frac{\partial^2 \omega_i}{\partial \beta^2} = \theta\left\{e^{-\beta x}\left[\frac{x}{(\beta+1)^2} + \frac{2}{(\beta+1)^3}\right] - xe^{-\beta x}\left[\left(1+\frac{\beta x}{\beta+1}\right) - \frac{1}{(\beta+1)^2}\right]\right\},$$

$$\frac{\partial^2 \omega_i}{\partial \beta \partial \theta} = \left[1 - \left(1+\frac{\beta x}{\beta+1}\right)e^{-\beta x}\right] = \frac{1}{\theta}\frac{\partial \omega_i}{\partial \beta} \quad \text{and} \quad \frac{\partial^2 \omega_i}{\partial \theta^2} = 0.$$

R Algorithms

In this appendix, we present R codes to compute cdf, pdf, moments, reliability, Rényi entropy, mean deviations, maximum likelihood estimates and variance-covariance matrix for the BLP distribution.

```
#Define the pdf of BLP distribution

f1=function(x,beta,theta,a,b){
GLP=(1-exp(theta*(1-(1+beta*x/(beta+1))*exp(-beta*x))))/(1-exp(theta))
glp=theta*beta^2*(1+x)*exp(-beta*x)*(exp(theta*
(1-(1+beta*x/(beta+1))*exp(-beta*x))))/((beta+1)*(exp(theta)-1))
y=GLP^(a-1)*(1-GLP)^(b-1)*glp/beta(a,b)
return(y)
}

#Define the cdf of BLP distribution
F1=function(x,beta,theta,a,b){
y=pbeta((1-exp(theta*(1-(1+beta*x/(beta+1))*exp(-beta*x))))
/(1-exp(theta)),a,b)
return(y)
}

#Define the quantile of BLP distribution
```

```
library(LambertW)
quantile=function(beta,theta,a,b,u){
y=qbeta(u,a,b)
result=-1-1/beta-W_1(-(beta+1)*(theta-log(1-y*
(1-exp(theta))))/(exp(beta+1)*theta))/beta
#check
error=F1(result,beta,theta,a,b)-u
return(list("result"=result,"error"=error))
}

#Define the moments of BLP distribution
moment=function(beta,theta,a,b,r){
f=function(x,beta,theta,a,b,r)
{(x^r)*(f1(x,beta,theta,a,b))}
y=integrate(f,lower=0,upper=Inf,subdivisions=1000,
beta=beta,theta=theta,a=a,b=b,r=r)
return(y)
}

#Define Mean Deviation about the mean of BLP distribution
DU=function(beta,theta,a,b){
mu=moment(beta,theta,a,b,1)$ value
f=function(x,beta,theta,a,b)
{(abs(x-mu)*f1(x,beta,theta,a,b)}
y=integrate(f,lower=0,upper=Inf,subdivisions=100
,beta=beta,theta=theta,a=a,b=b)
return(y)
}

#Define Mean Deviation about the median of BLP distribution
DM=function(beta,theta,a,b){
M=median(c(X)) #X is the data set
f=function(x,beta,theta,a,b)
{(abs(x-M)*f1(x,beta,theta,a,b)}
y=integrate(f,lower=0,upper=Inf,subdivisions=100
,beta=beta,theta=theta,a=a,b=b)
return(y)
}

Define the Renyi entropy of BLP distribution
t=function(alpha,beta,theta,a,b,v){
f=function(x,beta,theta,a,b,v)
{(f1(x,beta,theta,a,b))^(v)}
y=integrate(f,lower=0,upper=Inf,subdivisions=100
,beta=beta,theta=theta,a=a,b=b)$ value
```

```
return(y)
}
Renyi=function(beta,theta,a,b,v){
y=log(t(beta,theta, lambd,v))/(1-v)
return(y)
}

#Calculate the maximum likelihood estimators
#of BLP distribution
library('bbmle')
xvec<-c(X) #X is the data set
ln<-function(beta,theta,a,b){
GLP=(1-exp(theta*(1-(1+beta*xvec/(beta+1))
*exp(-beta*xvec))))/(1-exp(theta))
glp=theta*beta^2*(1+xvec)*exp(-beta*xvec)
*(exp(theta*(1-(1+beta*xvec/(beta+1))
*exp(-beta*xvec))))/((beta+1)*(exp(theta)-1))
mle=-sum(log(GLP^(a-1)*(1-GLP)^(b-1)*glp/beta(a,b)))
return(mle)
}
mle.results1<-mle2(ln,start=list
(beta=beta,theta=theta,a=a,b=b),hessian.opt=TRUE)
summary(mle.results1)

# Variance-covariance matrx of BLP distribution
vcov(mle.results1)
```