**On Quadratic Abstract**

**Measure integro different equations**

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**Abstract**

In this paper a nonlinear abstract measure quadratic integro – differential equation is studied in Banach Algebra. The existence of solution abstract measure integro – differential is proved for extremal solutions for Caratheodory as well as discontinuous cases of the non-linearities involved in the equations.

**(2000) mathematics subject Classifications:**

**Key Works & Phrases :-**

Measures integro – differential equations existence theorem, Extermal solutions.

**5.1 Introduction :-**

For a given closed and bounded interval  in R the set of real numbers, consider the integro differential equation in short (IGDE)



Where  is continuous,



The existence of solutions of IGDE (5.1.1) is proved in Dhage  by using a new non-linear alternative of Leray – Schauder type developed in same paper. In this chapter we apply a nonlinear alternative of Leray – Schauder type involving the product of two operators in a Banach algebra under some weaker conditions than that given in Dhage and Regan  to a quadratic measure differential equation related to IGDE (5.1.1) for proving the existence results. The existence of exterimal solutions is also proved using a fixed point theorem of in ordered Banach algebras.

In the first section introduction is given in section II we state the abstract measure integro differentiate equation to be discussed in this paper. The section III the auxiliary results are given and the existence result is discussed in section IV. Finally the existence results for extermal solutions for the integro differential equations is discussed in section V.

**5.2 Quadratic Integro differential equations :-**

Let X be a real Banach algebra with a convenient norm .

Let  Then the line segment  in X is defined by



Let  be a fixed point and .

Then for any  we define the sets  and  in X by



Let be arbitrary. We say  if , or equivalently . In this case we also write  let M denote the -algebra of all subsets of X such that  is a measureable space. Let  be the space of all vector measures (real signed measures) and define a norm  on by



Where  is a total variation measure of p and is given by



Where supremum is taken over all possible partition  of X. It is known that  is a Banach space with respect to the norm , given by (5.2.3). For any nonempty subset S of X, let  denote the space of -integrable real valued functions on S which is equipped with the norm  given by



For . Let  and define a multiplication composition in  by



Lemma 5.2.1  is a Banach algebra.

**Proof :-** Let  be two elements. Let  be a disjoint partition of X. Then by (5.2.3) – (5.2.4),



Hence  is a Banach algebra.

Let  be a  finite measure on X, and let . We say p is a absolutely continuous with respect to the measures  if  implies  for some . In this case we also write 

Let  be fixed and let  denote the -algebra on  let  be such that  and let  denote the -algebra of all sets containing M. and the sets of the form  Given a  with , consider the abstract measure integro - differential equation (AMIGDE) of the form



Where q is a given known vector measure,  is a singed measure such that ,  is a Radon Nikodym derivative of  with respect to ,



 and the map



is -integrable for each 

**Definition 5.2.1 :-** Given an initial real number  on , a vector  is said to be a solution of AMIGDE (5.2.5) – (5.2.6), if

i) 

ii)  on , &

iii) satisfies  on 

**Remark 5.2.1 :-** The AMIGDE (5.2.5) (5.2.6) is equivalent to the abstract measure integral equation (in short AMIE).



A solution p of abstract measure AMIGDE (5.2.5) – (5.2.6) in  will be denoted by .

Note that the above equations includes the abstract measure differential equation considered in Dhage and Bellale  as a special case. The see this, define  for all  &  then AMIGD (5.2.5) – (5.2.6) reduces to



Thus our AMIGDE (5.2.5) – (5.2.6) is more general & we claim that it is a new to the literature on measure differential equations.

Now we shall prove the existence theorem.

**5.3 Auxiliary Results**

Let X be a Banach space & let . T is called compact if  is a compact subset of X. T is called totally bounded if for any bounded subset S of X, T (S) is a totally bounded subset of X. T is called completely continuous if T is continuous & totally bounded on X. Every compact operator is totally bounded, but the converse may not be true, however two notions are equivalent on a bounded subset of X.

An operator  is called D – Lipschitz if these exists a continuous & non-decreasing function  such that  for all  where . The function  is called a D – function of T on X. In particular if  T is called a Lipschitz with the Lipschitz constants  further if , then T is called a contraction with contraction constant . Again if  for , then T is called a non-linear contraction on X with D – function .

**Theorem 5.3.1 :-** Let U &  denote respectively the open and closed bounded subset of a Banach algebra X such that  let  be two operators such that

i) A is D – Lipschitz

ii) B is completely continuous and

iii)  where .

Then either

a) The equation  has a solution in , as

b) These is a point  such that  for some ,

where  is a boundary of U in X.

**Corollary 5.3.1 :-** Let  denote respectively the open and closed balls in a Banach algebra centred at origin 0 of radius r for some real number . Let  be two operators such that

i) A is Lipschitz with Lipschitz content .

ii) B is compact & continuous, &

iii) , where  then either

a) The operator equation  has a solution x in X with  or

b) These is an  with  such that  for some .

We define an order relation  in  with the help of the cone K in  given by



Thus for any  are have  if and only if  or equivalently

 for all .

Obviously the cone K is positive in . To see this, let .

Then  &  for all . By multiplication composition  for all  & so K is a positive come in .

The following lemma follow immediately from the definition of the positive cone K in 

Lemma (5.3.1). Dhage . If  are such that  & , then .

Lemma 5.3.2 The cone K is normal in .

**Proof :-** To prove it is enough to show that norm  is semi monotone on K. Let  be such that  on . Then we have



For all .

Now for a countable partition



As a result  is semi monotone on k & consequently the cone K is normal in . The proof of the lemma is complete.

An operator  is called positive if the range  of T is contained in the cone K in X.

**Theorem 5.3.2 :-** (Dhage ). Let  be an order interval in the real Banach algebra X & let  be positive & non-decreasing operators such that

i) A is Lipschitz with a Lipschitz constant ,

ii) B is compact & continuous, &

iii) The elements  with .

Satisfy .

Further if the cone K is normal, then the operator  has a least & a greatest positive solution in , whenever , where .

**Theorem 5.3.3 :-** Dhage  let k be a positive cone in a real Banach algebra X

and let  be non-decreasing operators such that

i) A is Lipschitz with the Lipschitz constant 

ii) B is bounded, &

iii) There exist elements  such that  satisfying  & .

Further, if the cone k is normal then the operator equation  has a least & greatest positive solution in , whenever  where



**5.4 Existence Result :-**

We need the following definition.

**Definition 5.4.1 :-** A functions  is called Caratheodory if

i)  is -measurable for each  &

ii) The function  is continuous almost everywhere  on.

A Caratheodary function  on  is called  Caratheodary if

iii) For each real number  there exists a functions  such that



For all  with .

A function  is called sub multiplicative if  for all real number . Let  denote the class of function  satisfying the following properties:

i)  is continuous

ii)  is non-decreasing &

iii)  is sub multiplicative

A member  is called a D – function on . These do exist D – function, in fact, the function  defined by ,  is a D – function on  we consider the following set of assumptions:

 for any , the -algebra is compact with respect to the topology generated by the Pseudometric d defined on  by



 The function  is bounded with 

 The function is continuous & these exists a bounded function  with bound  such that



For all 

 q is continuous on  with respect to the pseudo-metric d defined in .

 The function  is -integrable & satisfies



For all 

 The function  is Caratheodary

 There exists a function  such that  on  & D – function  such that



For all 

We frequently use the following estimate of the function g in the subsequent part of the paper. For any , one has



**Theorem 5.4.1 :-** Suppose that the assumptions  holds. Suppose that there exists a real number  such that



Where 

& . Then the AMIGDE (5.2.5) – (5.2.6) has a solution on .

**Proof :-** Consider on open ball  in  centered at the origin 0 and of radius r. Where r satisfies the inequalities in (5.4.1). Define two operators.



by



&



We show that the operators A & B satisfy all the condition of corollary 5.3.1 on .

**Step – I :-** First, we show that A is a Lipschitz on . Let  be arbitrary.

Then by assumption ,



For all . Hence by definition of the norm in  one has



For all . As a result. A is a Lipschitz operator  with the Lipschitz constant .

**Step – II :-** We show that B is continuous on . Let  be sequence of vector measure in  converging to a vector measure. Then by dominated convergence theorem,



For all . Similarly if  then

 and so B is a continuous operator on .

**Step – III :-** Next, we show that B is a totally bounded operator on . Let  be a sequence an . Then we have  for all . We show that the set  is uniformly bounded & equicontinuous set in . In this step, we first show that  is uniformly bounded.

Let . Then there exists two subsets  &  such that



Hence by definition of B,



For all .

From (3.3) it follows that



For all . Hence the sequence  is uniformly bonded in 

**Step – IV :-** Next we show that  is equicontinuous set in .

Let . Then there exist



&  such that

 with 

&  with .

We know the identities



There fore, we have



since g is Caratheodory & satisfies 

We have that



Assume that .

Then we have . As a result  & . As q is continuous on compact , it is uniformly continuous and so



This shows that  is a equicontinuous set in . Now an application of the Arzela – Ascolli theorem yields that B is a totally bounded operator on . Now B is continuous & totally bounded operator on , it is completely continuous operator on .

**Step V :-** Finally we show that hypothesis (iii) of corollary 5.3.1. The Lipschitz constant of A is . Here the number M in the hypothesis (iii) is given by



Now let . Then there are sets  &  such that



From the definition of B is follows that



There fore,



Hence from (4.4.6) it follows that



For all . As a result are have



& so, hypothesis (iii) of corollary 5.3.1 is satisfied.

Now an application of corollary 5.3.1 yields that either the operator  has a solution, or there exist  such that  satisfying  for some . We show that this letter assertion does not hold. Assume the contrary. Then we have



For some .

If , then these sets  &  such that . Then we have





Hence



which further implies that



Substituting  in the above inequality yields



Which is a contradiction to the first inequality in (4.41). In consequence, the operator equation  has a solution  in  with . This further implies that the AMIGDE (5.2.5) – (5.2.6) has a solution on . This completes the proof.

**5.5 Existence of Extremal Solutions :-**

In this section we shall prove the existence of a minimal and a maximal solutions for the AMIGDE (5.2.5) – (5.2.6) on  under Carathedory as well as discontinuous case of non-lineality g involved in it.

**5.5.1 Carathedory Case :-**

We need following definitions

**Definition 5.5.1 :-** A vector measure  is called a lower solution of the AMIGDE (5.2.5) – (5.2.6) if



Similarly a vector measure  is called an upper solution to AMIGDE (5.2.5) – (5.2.6) if



A vector measure  is a solution to AMIGDE (5.2.5) – (5.2.6) it is upper as well as lower solution to AMIGDE (5.2.5) – (5.2.6) on .

**Definition 5.5.2 :-** A solution  is called as maximal solution to AMIGDE (5.2.5) – (5.2.6) if for any other solution  for the AMIGDE (5.2.5) – (5.2.6) we have that



Similarly a minimal solution  of AIGDE (5.2.5) – (5.2.6) is defined on .

We consider the following assumptions :

  define the functions



 The functions  are non-decreasing in  for each .

 The AMIGDE (5.2.6) – (5.2.6) has a lower solution u and an upper solution v such that  on .

 The function  is  Caratheodary.

**Theorem 5.5.1 :-** Suppose that the assumptions

 holds. Further suppose that



Where . Then the AMIGDE (5.2.5) . (5.2.6) has a minimal and maximal solution defined on .

**Proof :-** AMIGDE (5.2.5) - (5.2.6) is equivalent to the abstract measures integral equation . Define the operators  by (5.4.2) & (5.4.3) respectively. Then the AMIGDE (5.2.5) – (5.2.6) is equivalent to the operator equation



We shall show that the operator A & B satisfy all the conditions of theorem 5.3.2 on  since  is a positive measure, from assumption  if follows that A & B on positive operators on . To show this let  be such that  on . From  it follows that



For all  &



For . Hence A is non-decreasing on 

Similarly, we have



For all 

Again if , then



There fore the operator B is also non-decreasing on . Now it can be shown that as the proof of theorem 5.3.1 that A is Lipschitz operator on  with the Lipschitz constant . Since the cone K is normal in X, the order interval  is norm-bounded.

Hence there is a real number  such that  for all . As g is  Caratheodary, there is a function  such that  on  for all . Now proceeding with the arguments as in the proof of theorem 5.4.1 with  and , it can be proved that B is compact & continuous operator on . Since u is lower solution of AMIGDE (5.2.5) – (5.2.6) we have



From the above inequality in the gives



& so . Similarly, since  is an upper solution of AMIGDE (5.2.5) – (5.2.6) it can be proved that  for all  & consequently  on . Thus hypothesis (iii) of theorem 5.3.2

Are satisfied. Now definition of norm, it follows that



For any partition  of  such that 

Let . Then, for any  one has



Therefore for any , there are sets  &  such that .

Hence, we obtain



As . Thus the operator A & B satisfy all the conditions of theorem 5.3.2 & so an application of it yields that the operator equation  has a maximal & a minimal solution in . Thus further implies that AMIGDE (5.2.5) – (5.2.6) has a maximal & a minimal solution on . This completes the proof.

**5.5.2 Discontinuous Case :-**

In the following we obtain an existence result for external solution for the AMIGDE (5.2.5) – (5.2.6) when the nonlinearity g is a discontinuous function in all its three variables.

We consider the following assumptions :

 The function  defined by



Is -integrable on .

**Remark 5.5.1 :-** Assume that the hypothesis  hold. Then



All .

**Theorem 5.5.2. :**- Suppose that the assumptions  and  hold. Further suppose that



Then the AMIGDE (5.2.5) – (5.2.6) has a minimal and a maximal solution defined on .

Proof :- The proof is similar to Theorem 5.5.2 with appropriate modifications. Here, the function h plays the role of  . Now the desired conclusion follows by an application of theorem 5.5.3.

Notice that we do not need any type of continuity of the nonlinear function g in above theorem 5.5.2 for guaranteeing the existence of extermal solutions for the AMIGDE (5.2.5) - (5.2.6) on  instead we assumed the monotonicity condition on it.

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