Detecting Arbitrage opportunities in Asymmetric Game

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Abstract. In this not, an asymmetric coin game is studied. The winning scheme of Skeptic is given. Also, the mixed strategy and arbitrage opportunity is considered.

Keywords: Arbitrage opportunity; Asymmetric game, Mixed strategy; Protocol of game; Reality; Skeptic; Tendency ratio

1 Introduction. Shafer and Vovk (2001) used the game-theoretic framework to present fundamental theorems in probability theory, randomness and mathematical finance. The main advantage of this approach is to present theorems without the use of confusing assumptions. They claimed that two ideas (i) dynamic hedging pricing and (ii) the hypothesis of the impossibility of a gambling system (which is equivalent to the efficient market hypothesis) are enough to both probability and finance.
They studied the Strong Law of Large Numbers, the law of the Iterated Logarithm and pricing options in discrete time and continuous Time. Let Γ_i be the Skeptic capital at stage $i = 1, \dots, n$ and $S_i = \sum_{k=1}^{i} x_k$. Following Shafer and Vovk (2001), the finite horizon fair coin game protocol is:

Parameters: N and $\varepsilon, \alpha, a, b > 0$.
Players: Reality, Skeptic
Protocol:
$\Gamma_0 = \alpha$.
For $i = 1, \dots, n$:
Skeptic announces $M_i \in R$.
Reality announces $x_i \in \{-a, b\}$.
$\Gamma_i = \Gamma_{i-1} + M_i x_i$.
Let $a = b$.
Winning: Skeptic wins if Γ_i is never negative or either $\Gamma_n \geq 1$ or $\left|\frac{S_n}{n}\right| < \varepsilon$.

This game is symmetric because $a = b$. In this note, an asymmetric game is considered and it is supposed that $a \neq b$. Let

$$z_i = \frac{2x_i - (b-a)}{b+a},$$

and $T_i = \sum_{k=1}^{i} z_k$. Then, $T_i = \frac{2S_i - (b-a)i}{b+a}$. Notice that

$$\left|\frac{S_n}{n}\right| \leq 0.5\left((b+a)\left|\frac{T_n}{n}\right| + |b-a|\right).$$

Proposition 1. The Skeptic wins if and only if

$$\left|\frac{T_n}{n}\right| < \frac{\varepsilon}{a+b} \text{ and } 0.5(b-a) < \varepsilon.$$

For example, suppose that $\varepsilon = \varepsilon_n$ such that $n\varepsilon_n$ goes to zero as n goes to infinity and let $a = b = \frac{1}{\sqrt{n}}$.
There are many good references about the game theoretic probability. Some of them are listed as follows.Takemura and Suzuki (2005) considered the game theoretic derivation of discrete distributions and discrete pricing formulas. Takeuchi *et al*. (2007) proposed a formulation of asset trading games in continuous time. Kumon and Takemura (2008) gave a simple strategy for strong law of large numbers in the bounded forecasting game. Kumon *et al*. (2008) which surveyed the optimality properties of a Bayesian Skeptic in coin-tossing games. Takeuchi *et al*. (2008) who considered the multistep Bayesian strategy in coin-tossing games and its application to asset trading

games in continuous time. Kumon *et al*. (2009) considered the sequential optimizing strategy in multi-dimensional bounded forecasting games. Vovk (2009) considered the game theory strategies in continuous time trading. The rest of this note is considered as follows. In the next section the mixed strategies are studied in this class of games. The arbitrage free conditions are extracted. The delta risk neutral games are considered in Section 3.

2 Mixed strategies and arbitrage opportunity. Here, to define the mixed strategies, assume that the p_a is the tendency ratio of Skeptic prior belief to state a, and p_b is defined analogously. Then the subjective expectation μ_i of capital attained by Skeptic is

$$\mu_i = \mu_{i-1} + M_i(bp_b - ap_a),$$

or equivalently (assuming $\mu_0 = 0$) $\mu_i = M^*(bp_b - ap_a)$, where $M^* = \sum_{k=1}^{i} M_k$. Therefore, $\mu_n = 0$ if and only if $p_b = 1 - p_a = \frac{a}{a+b}$. If $p_b > \frac{a}{a+b}$ (or $p_b < \frac{a}{a+b}$), then $\mu_n > 0$ ($\mu_n < 0$). The following proposition summarizes the above discussion:

Proposition 2. The subjective expectation of Skeptic capital, under the tendency ratio of him p_a to the state of a, is zero if and only if

$$p_b = 1 - p_a = \frac{a}{a+b}.$$

Next, suppose that the tendency ratio at each stage differs and it is $p_a^i, i = 1, \dots, n$. Then,

$$|\mu_n| \leq nM_*|b\bar{p}_b - a\bar{p}_a|.$$

Remark 1. Under the tendency ratio of $p_a^i, i = 1, \dots, n,$ for each stage, then $\mu_n = 0$ if and only if

$$\bar{p}_b = 1 - \bar{p}_a = \frac{a}{a+b}.$$

Hereafter, the arbitareg free conditions are surveyed in a asymmetric coin game. Suppose that the risk free rate is $r, (0 < r < b)$. Then,

$$\frac{\Gamma_i}{(1+r)^i} = \frac{\Gamma_{i-1}}{(1+r)^{i-1}} + \frac{M_i x_i - r\Gamma_{i-1}}{(1+r)^i}.$$

The conditional subjective expectation of $\frac{\Gamma_i}{(1+r)^i}$ given information up to time $i-1$, is $\frac{\Gamma_{i-1}}{(1+r)^{i-1}}$ if and only if the expectation of $M_i x_i - r\Gamma_{i-1}$ is zero. Necessary conditions to this end are $\Gamma_{i-1} = M_i$ and $bp_b - ap_a = r$. The following proposition summarizes.

Proposition 3. The asymmetric game is arbitrage free if $\Gamma_{i-1} = M_i$ and $p_b = \frac{r+a}{a+b}$.

Remark 2. The asymmetric coin game may be generalized by replacing $\{-a, b\}$ to $(-a, b)$. Again the no arbitrage assumption is hold, by assuming a uniform tendency measure for Skeptic on $(-a, b)$ is $\Gamma_{i-1} = M_i$ and $b = a + 2r$.

Remark 3. Shafer and Vovk (2001) studied the protocol of predictably unbounded forecasting game for proving the iterated logarithm law at which the Forecaster announces $m_i \in R, c_i \geq 0$. While the Skeptic announces $M_i \in R$ and Reality announces $x_n \in [-c_n, d_n]$ and the capital at stage i is

$$\Gamma_i = \Gamma_{i-1} + M_i(x_i - m_i).$$

Then the no arbitrage conditions are $\Gamma_{i-1} = M_i$ and $d_i - c_i = 2(m_i - r)$.

3 Risk neutral game. Here, suppose that there are two Skeptics in an asymmetric coin game with capital Γ_i^1 and Γ_i^2, i=1,…,n. Then, assume that the first Skeptic takes the long position and the second takes the short position in γ units of his capital. Therefore, the self-financed portfolio of both skeptics has the capital $\Gamma_i^1 - \gamma\Gamma_i^2$, which is

$$\Gamma_i^1 - \gamma\Gamma_i^2 = \left(\Gamma_{i-1}^1 - \gamma\Gamma_{i-1}^2\right) + M_{i,1}x_{i,1} - \gamma M_{i,2}x_{i,2}$$

$$= \left(\Gamma_{i-1}^1 - \gamma\Gamma_{i-1}^2\right) + (M_{i,1}\sigma_1 - M_{i,2}\sigma_2)\epsilon_i.$$

Here, σ_k, k=1,2, are discrepancy measures. To make this portfolio risk neutral, it is enough to take

$$\gamma = \frac{M_{i,1}\sigma_1}{M_{i,2}\sigma_2}.$$

The following proposition gives the result.

Proposition 4. A self-financed portfolio including long position for the first Skeptic and and the short position in γ units of his capital for second Skeptic is delta neutral if and only if at each stage i,

$$\gamma = \frac{M_{i,1}\sigma_1}{M_{i,2}\sigma_2}.$$

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