Some Mathematical Problems Arising in Biological Models: A Predator-Prey Model Fish-Plankton

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Abstract
The aim of this paper is to develop and analyze a deteministic predator-2 preys models for species living in marine environments. We are interested in the evolution of fish population in office plankton. We first look at the model; this yields a singular system of ordinary differential equations having interesting dynamical futures, such as finite time extinction and persistence of populations. In addition, the mathematical analysis permits to isolate extinction conditions in finite or infinite time. Finally, the numericals simulations permits to establish the effect of the fishing on the evolution of fish population in spite of abundance resource and it permits to know if area is severely exploited or not.

Key words: Dynamic of the populations, mathematical analysis, marine environment, ordinary differential equations, predator-prey systems.
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1 Introduction

We are interested in some deterministic mathematical population dynamics model motivated by biological problems in the marine environment. The problem of the management of bio-diversitie resources in generally and particularly, the resources halieutic management, interest many researchers. Almost 47% of the world fish stocks commercial are maximum exploited [10].
To contribute to resolve the fish resource preservation, here, we develop a deterministic model which govern the dynamic of the fish and her diet (plankton).

In previous works [5,8], a mathematical model has been developed to model the interactions between a native prey and alien predators and preys in insular environments, taking into account the spatial heterogeneity in the habitat. Problems of persistence and extinction of species have been studied in [3], and invasion processes have been analyzed.

In this paper, we develop a mathematical model to better understand interactions between diet (plankton) and fish population in a marine environment, taking into account heterogeneities in the distribution of the species. Some modeling work in this direction concerning unstructured populations has been previously developed in [8,9] for two and three species.
We extended the unstructured model of Courchamp and Sugihara [3] to the case of fish-plankton.

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Along the same line, we derive a predator (fish)-2prey (zooplankton and phytoplankton) model based on Courchamp and al. The models study in the present work use predator-2prey ratio instead of prey density. It's about to develop Fish-zooplankton-phytoplankton deterministic model.

First of all, we consider fish and phytoplankton population dynamic, F-B model. After we consider fish- zooplankton system, and zooplankton-phytoplankton system for to develop F-R and R-B models. In end, combining of these dynamic, we develop the final model B-R-P of the system phytoplankton-zooplankton-fish. One novelty of these models, involving systems of ordinary differential equations, is the possibility for species to go extinct in finite and infinite time; This feature is closely related to a specific mathematical difficulty: the occurrence of singularities i.e vanishing determinators, in the systems of ordinary differential equations (ODE).

The paper is organized as follows. In section 2, we present the different mathematicals models of our problem. Section 3 provides the mathematical analysis of the final model. Computational simulations are performed in section 4 and finally, in the last section, we end with some conclusions remarks and future works.

2 Mathematical models

In this first section, we briefly describe a mathematical models devised in F. Courchamp and Sugihara [3] on the Predator-preys models.

2.1 Fish-Phytoplankton model

If we consider the Fish-Phytoplankton system without the Zooplankton, the dynamic of the system is governed by the following figure 1.

State variables are: B, the phytoplankton density P, the fish density. Let μ_b, the annual individual

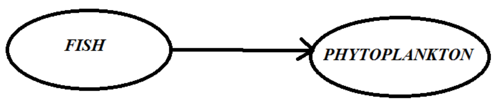

Figure 1: Fish-Phytoplankton model

intake of phytoplankton per fish, q the catch ability coefficient fish, E the fishing effort. The carrying capacity of the introduced fish population is $\dfrac{B}{\mu_b}$, [8] and the fishing mortality rate is qE.

Then according to the figure 1, we obtain the following system of two differential equations singular at $B = 0$.

$$\begin{cases} \dfrac{dB}{dt} = r_b B\left(1 - \dfrac{B}{K_b}\right) - \mu_b P, \quad B(0) = B_0 > 0 \\[2ex] \dfrac{dP}{dt} = r_p P\left(1 - \dfrac{\mu_b P}{B}\right) - q_p P, \quad P(0) = P_0 > 0 \end{cases} \tag{2.1}$$

Where

r_p is the natural growth-rate of the fish population

r_b is the net increase-rate of the phytoplankton quantity.

q_p is the fishing mortality rate

K_b is the carrying capacity of the phytoplankton

2.2 Fish-Zooplankton model

If we consider the Fish-Zooplankton system without the phytoplankton, the dynamic of the system is governed by the following figure 2.

State variables are: R, the zooplankton density and P, the fish density. Let μ_r, the annual individual

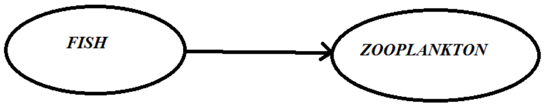

Figure 2: Fish-Zooplankton model

intake of the zooplankton per fish, the carrying capacity of the introduced fish population is $\dfrac{R}{\mu_r}$, [9].
Then according to the figure 2, we obtain the following system of two differential equations singular at $R = 0$.

$$\begin{cases} \dfrac{dR}{dt} = r_r R\left(1 - \dfrac{R}{K_r}\right) - \mu_r P, \quad R(0) = R_0 > 0 \\ \\ \dfrac{dP}{dt} = r_p P\left(1 - \dfrac{\mu_r P}{R}\right) - q_p P, \quad P(0) = P_0 > 0 \end{cases} \tag{2.2}$$

Where
r_p is the natural growth-rate of the fish population
r_r is the growth-rate of the zooplankton population.
q_p is the fishing mortality rate
K_r is the carrying capacity of the zooplankton

2.3 Zooplankton-Phytoplankton model

If we consider the Zooplankton-Phytoplankton system without the fish, the dynamic of the system is governed by the following figure 3.

State variables are: B, the phytoplankton density and R, the zooplankton density. Let δ_b, the

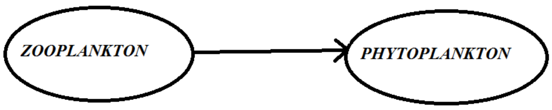

Figure 3: Zooplankton-Phytoplankton model

annual individual intake of phytoplankton per zooplankton, the carrying capacity of the introduced zooplankton population is $\dfrac{B}{\delta_b}$.
Then according to the figure 3, we obtain the following system of two differential equations singular at $B = 0$.

$$\begin{cases} \dfrac{dB}{dt} = r_b B\left(1 - \dfrac{B}{K_b}\right) - \delta_b R, \quad B(0) = B_0 > 0 \\ \\ \dfrac{dR}{dt} = r_r P\left(1 - \dfrac{\delta_b R}{B}\right), \quad R(0) = R_0 > 0 \end{cases} \tag{2.3}$$

Where
r_r is the natural growth-rate of the zooplankton population
r_b is the net increase-rate of the phytoplankton quantity.
K_b is the carrying capacity of the phytoplankton.

2.4 Phytoplankton-Zooplankton-Fish model

If we consider the Phytoplankton-Zooplankton-Fish system, the dynamic of the system is governed by the following figure 4.

Let μ_b and μ_r the annual individual intake of preys phytoplankton and zooplankton per individual

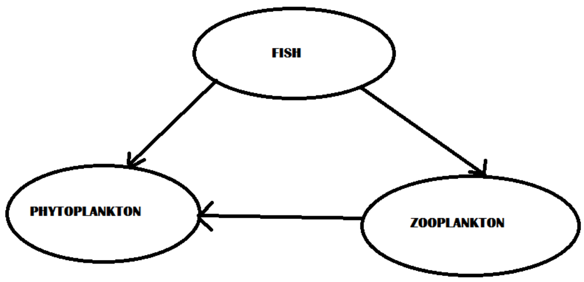

Figure 4: Phytoplankton-Zooplankton-Fish model

fish, so that the carrying capacity of the fish is $\dfrac{B}{\mu_b} + \dfrac{R}{\mu_r}$. Let $\beta > 0$, the preference coefficient for the phytoplankton and zooplankton and $\mu\zeta$ the annual individual intake of prey phytoplankton per individual zooplankton.
Then according to the figure 4, we obtain the following system of three differential equations singular at $B = R = 0$.

$$\begin{cases} \dfrac{dB}{dt} = r_b B\Big(1 - \dfrac{B}{K_b}\Big) - \mu_\zeta RB - \dfrac{\beta B}{\beta B + R}\mu_b P, \quad B(0) = B_0 > 0 \\[2ex] \dfrac{dR}{dt} = r_r R\Big(1 - \dfrac{R}{K_r}\Big) - \dfrac{R}{\beta B + R}\mu_r P, \quad R(0) = R_0 > 0 \\[2ex] \dfrac{dP}{dt} = r_p P\Big(1 - \mu_b\mu_r\dfrac{P}{\mu_r B + \mu_b R}\Big) - q_p P, \quad P(0) = P_0 > 0 \end{cases} \tag{2.4}$$

3 Mathematical analysis

3.1 Two species model

We study a mathematical problem arising in modeling Fish-Phytoplankton interaction in marine environment. The dynamic is presented by the following system:

$$\begin{cases} \dfrac{dB}{dt} = r_b B\left(1 - \dfrac{B}{K_b}\right) - \mu_b P, \quad B(0) = B_0 > 0 \\[2ex] \dfrac{dP}{dt} = r_p P\left(1 - \dfrac{\mu_b P}{B}\right) - q_p P, \quad P(0) = P_0 > 0 \end{cases} \tag{3.1}$$

Herein B is the phytoplankton species density with growth-rate, r_b, and carrying capacity, K_b. P is the fish density with growth-rate r_p, while, μ_b, is the annual intake of phytoplankton per individual

fish.
Hypothesis 1*:* All parameters in the system (3.1) are positives constants.
The mathematical analysis simplifies upon introducing a new state variable $F = \dfrac{P}{B}$, yielding a non singular locally Lipschitz continuous system of ODEs.
Sure enough, we have

$$\begin{cases} \dfrac{dB}{dt} = \left[r_b\left(1 - \dfrac{B}{K_b}\right) - \mu_b F\right]B, \quad B(0) = B_0 > 0 \\ \\ \dfrac{dF}{dt} = \left[r_p - r_b + r_b\dfrac{B}{K_b} - \mu_b\left(r_p - 1 - q_p\right)F\right]F, \quad F(0) = F_0 > 0 \end{cases} \tag{3.2}$$

System (3.1) has a solution if and only if system (3.2) has a solution.

Proposition 3.1 *System (3.2) has unique componentwise positive solution* (B, F)*, defined on a maximal existence interval,* $\left[0,\ T_{max}(B_0, F_0)\right[$.

Proof: Sure enough, the system (3.2) can be written by the following form

$$\frac{dB}{dt} = \varphi_1(B, F)B, \quad \frac{dF}{dt} = \varphi_2(B, F)F.$$

The functions φ_1 and φ_2 are C^∞, so according to the Cauchy-Lipschitz, system (3.2) has an unique componentwise positive solution, (B, F), defined on a maximal existence interval, $\left[0, T_{max}\right[$ if $B_0 > 0$ and $F_0 > 0$.

Proposition 3.2 $(0, 0)$*(unstable),* $\Big(K_b, 0\Big)$*(unstable) and* $\left(0, \dfrac{1}{\mu_b}\dfrac{r_p - r_b}{r_p - q_p - 1}\right)$*(stable) if and only if* $(r_p - r_b)(r_p - q_p - 1) > 0$ *are the stationaries states of the System (3.2).*
Therfore we have a third on with positive components $\left(B^* = \dfrac{r_b - 1}{r_b}K_b, F^* = \dfrac{1}{\mu_b}\right)$ *if and only if* $r_b - 1 > 0$.

Proof: A direct calculating gave the four states.
Concerning to the stability and instability, we consider the Jacobian matrix of the system.
Sure enough, we have

$$J(B, F) = \begin{pmatrix} r_b - 2\dfrac{r_b B}{K_b} - \mu_b F & -\mu_b B \\ \\ r_b\dfrac{F}{K_b} & r_p - r_b + \dfrac{r_b B}{K_b} - 2\mu_b(r_p - q_p - 1)F \end{pmatrix}$$

Let *specter(J)* the set of eigenvalues of the Jacobian matrix.

- The Jacobian matrix at $(0, 0)$ and at $(K_b, 0)$ show that, these states are always unstable.
- The Jacobian matrix at $\left(0, \dfrac{1}{\mu_b}\dfrac{r_p - r_b}{r_p - q_p - 1}\right)$ if $1 > r_p$, is

$$\begin{pmatrix} \dfrac{r_p\Big(1 - r_b\Big)}{1 - r_p} & 0 \\ \\ \dfrac{r_b\Big(r_b - r_p + q_p\Big)}{K_b\mu_b\Big(1 - r_p\Big)} & r_b - r_p + q_p \end{pmatrix}$$

So, this state is asymptotically stable if $(r_p - r_b)(r_p - q_p - 1) > 0$ ie $r_b < 1 < r_p - q_p$

- The Jacobian matrix at state $\Big(B^* = \dfrac{r_b - 1}{r_b}K_b, F^* = \dfrac{1}{\mu_b}\Big)$ is

$$\begin{pmatrix} 1 - r_b & -\mu_b K_b \dfrac{r_b - 1}{r_b} \\ \\ \dfrac{r_b}{K_b \mu_b} & 1 - r_p + q_p \end{pmatrix}$$

  Equally, the state is stable if $1 > r_b$

Concerning to the global solution existence, we assume the following assumption:
Hypothesis 2: $r_p - q_p > 1$.

Proposition 3.3 *According to the assumption 2, system (3.2) has global solution (B, F), defined on interval, $\Big[0, +\infty\Big)$.*

Proof: We have

$$\frac{dF}{dt} \leq \Big(r_p - q_p + r_b max\Big(\frac{B_0}{K_b}, 1\Big) - \mu_b(r_p - q_p - 1)F\Big)F$$

for all $t \in \Big[0, T_{max}\Big]$ and then, according to the hypothesis 2 and by integrating, we obtain

$$0 \leq F(t) \leq max\left(F_0, \frac{r_p - q_p r_b + r_b max\Big(\dfrac{B_0}{K_b}, 1\Big)}{\mu_b(r_p - q_p - 1)}\right), \quad so \ \ T_{max} = +\infty$$

3.2 Three species model

We are interested of the mathematical analysis of Fish-Phytoplankton-Zooplankton model. The mathematical model associate is the following differential system singular at $B = R = 0$.

$$\begin{cases} \dfrac{dB}{dt} = r_b B\Big(1 - \dfrac{B}{K_b}\Big) - \mu_\zeta RB - \dfrac{\beta B}{\beta B + R}\mu_b P, \quad B(0) = B_0 > 0 \\ \\ \dfrac{dR}{dt} = r_r R\Big(1 - \dfrac{R}{K_r}\Big) - \dfrac{R}{\beta B + R}\mu_r P, \quad R(0) = R_0 > 0 \\ \\ \dfrac{dP}{dt} = r_p P\Big(1 - \mu_b \mu_r \dfrac{P}{\mu_r B + \mu_b R}\Big) - q_p P, \quad P(0) = P_0 > 0 \end{cases} \tag{3.3}$$

3.2.1 Global analysis

Concerning to the analysis of the system (3.3), we rescale the state variables.
Indeed, let take $\widehat{B} = \beta B$, $\widehat{K_b} = \beta K_b$, $\widehat{\mu_b} = \beta \mu_b$ and assume $\beta = 1$. Then, using $H = R + B$ the total resource,

- $\phi = \dfrac{R}{B + R}$ the proportion of zooplankton within the total resource quantity,

- $Q = \dfrac{P}{R + B}$ the ratio of fish/resources .

as new states variables one gets a non-singular system.

Our aim is to reduce the nonsingular system in the following form

$$\begin{cases} \dfrac{dH}{dt} = g_1(H,\phi,Q), \\ \\ \dfrac{d\phi}{dt} = g_2(H,\phi,Q), \\ \\ \dfrac{dQ}{dt} = g_3(H,\phi,Q), \end{cases} \tag{3.4}$$

Under the hypothesis $0 \leq \phi \leq 1$, we obtain

$$\frac{dH}{dt} = \frac{dR}{dt} + \frac{dB}{dt} = r_b B\Big(1 - \frac{B}{K_b}\Big) - \mu_\zeta RB - \frac{\beta B}{\beta B + R}\mu_b P + r_r R\Big(1 - \frac{R}{K_r}\Big) - \frac{R}{\beta B + R}\mu_r P.$$

When we develop and we replace H, ϕ and Q by their values, we get that

$$\frac{dH}{dt} = \Big[\big(r_b - \mu_b Q\big)(1-\phi) + (r_r - \mu_r Q)\phi - r_b\frac{(1-\phi)^2}{K_b} - r_r\frac{\phi^2}{K_r} + \mu_\zeta \Phi(1-\phi)\Big]H.$$

Likewise, we have:

$$\frac{d\phi}{dt} = \phi\Big(r_r - \frac{r_r\phi H}{K_r} - \mu_r Q + r_p Q - r_b - \mu_\zeta \phi H\Big)(1-\phi) + (1-\phi)^2\phi H\frac{r_b}{K_b}$$

and

$$\begin{aligned}\frac{dQ}{dt} = \Big[&r_p - r_r\phi - r_b(1-\phi) + \frac{r_b}{K_b}(1-\phi)^2 H + \mu_\zeta\phi(1-\phi)H + \frac{r_r}{K_r}\phi^2 H - \\ &\frac{r_p\mu_b\mu_r Q - \Big(\mu_r\phi + \mu_b(1-\phi)\Big)\Big(\mu_r(1-\phi) + \mu_b\phi\Big)Q}{\mu_r(1-\Phi) + \mu_b\phi} - q_p\Big]Q.\end{aligned}$$

By setting :

$$\begin{cases} G_1(\phi,Q) = \Big(r_b - \mu_b Q\Big)\Big(1-\phi\Big) + \Big(r_r - \mu_r Q)\phi\Big), \\ G_2(\phi) = r_b\dfrac{\Big(1-\phi\Big)^2}{K_b} + r_r\dfrac{\phi^2}{K_r} + \mu_\zeta\phi\Big(1-\phi\Big), \\ L(\phi) = \Big(1-\phi\Big)^2\phi\dfrac{r_b}{K_b}, \\ V_1(\phi) = r_p - r_r\phi - r_b\Big(1-\phi\Big), \\ V_2(\phi) = \dfrac{r_b}{K_b}\Big(1-\phi\Big)^2 + \mu_\zeta\phi(1-\phi) + \dfrac{r_r\phi^2}{K_r}, \\ V_3(\phi) = \dfrac{r_p\mu_b\mu_r - (\mu_r\phi + \mu_b(1-\phi))(\mu_r(1-\phi) + \mu_b\phi)}{\mu_r(1-\phi) + \mu_b\phi}. \end{cases}$$

we obtain the following system regular if $0 \leq \phi \leq 1$.

$$\begin{cases} \dfrac{dH}{dt} = \Big[G_1(\phi,Q) - G_2(\phi)H\Big]H = g_1\big(H,\phi,Q\big), \\ \\ \dfrac{d\phi}{dt} = \Big[r_r\Big(1 - \dfrac{\phi H}{K_r}\Big) - \big(\mu_r\mu_b\big)Q - r_b\mu_\zeta\phi H\Big]\phi(1-\phi) + HL(\phi) = g_2\big(H,\phi,Q\big), \\ \\ \dfrac{dQ}{dt} = \Big[V_1(\phi) + V_2(\phi)H - V_3(\phi)Q\Big]Q = g_3\big(H,\phi,Q\big), \end{cases} \tag{3.5}$$

Solutions of model system (3.3) exist if and only if the solutions of model system (3.5) exist.

Proposition 3.4 *The system (3.5) admit one maximal unique solution* $\big(H(t), \phi(t), Q(t)\big)$ *definite on a* $[0, T_{max}[$. *More, the set* $\{H \geq 0,\ \ 0 \leq \phi \leq 1,\ \ Q \geq 0\}$ *is positively invariant for the system(3.5).*

Proof: The local existence of system (3.5) solution result from the fact that this system is locally Lipschitz. According to the system (3.5), we obtain

$$\begin{cases} g_1\big(0, \phi, Q\big) = 0, \quad for \quad 0 \leq \phi \leq 1 \quad and \quad Q \geq 0, \\ g_2\big(H, \phi = 0, Q\big) = 0, \quad for \quad H \geq 0 \quad and \quad Q \geq 0, \\ g_2\big(H, \phi = 1, Q\big) = 0, \quad for \quad H \geq 0 \quad and \quad Q \geq 0, \\ g_3\big(0, \phi, Q\big) = 0, \quad for \quad 0 \leq \phi \leq 1 \quad and \quad H \geq 0. \end{cases}$$

we conclude that $\{H \geq 0,\ 0 \leq \phi \leq 1,\ Q \geq 0\}$ is positively invariant for the system (3.5).

Proposition 3.5 *A state variable* Q *of system (3.5) explode in finite time under hypothesis:* $1 > r_p > max(r_r, r_b) > 0$, *i.e that exist* $T_{max} < +\infty$ *such as* $Q(t) \longrightarrow +\infty$ *when* $t \longrightarrow +\infty$. *The Cauchy problem admit one global unique solution at time* $\big(H > 0,\ 0 \leq \phi \leq 1,\ Q > 0\big)$ *i.e* $T_{max} = +\infty$ *under hypothesis* $\dfrac{\mu_b^2 + 2\mu_b\mu_r + \mu_r^2}{4\mu_b\mu_r} < r_p$.

Proof: We have

- $V_2 > 0$ for $0 \leq \phi \leq 1$.
- again, we obtain $V_3(\phi) < 0$ when $r_p < 1$
- and $V_3(\phi) \geq 0$ when $r_b < r_p$ and $r_r < r_p$.

In this case $Q(t) \longrightarrow +\infty$ when $t \longrightarrow T_{max}(T_{max} < +\infty)$, i.e that Q explodes in the finite time if $1 > r_p > max(r_r, r_b) > 0$. Inversely, always under hypothesis $0 \leq \phi \leq 1$, we have $V_2 > 0$. On the other hand $V_3(\phi) > 0$ means that

$$\frac{r_p\mu_b\mu_r - \Big(\mu_r\phi + \mu_b(1-\phi)\Big)\Big(\mu_r(1-\phi) + \mu_b\phi\Big)}{\mu_r(1-\phi) + \mu_b\phi} > 0$$

If we set or define

$$\lambda(\phi) = \frac{\mu_b\mu_r - \Big(\mu_r\phi + \mu_b(1-\phi)\Big)\Big(\mu_r(1-\phi) + \mu_b\phi\Big)}{\mu_r\mu_b},$$

we obtain $V_3(\phi) > 0$ if $r_p > \lambda(\phi)$.
Under condition $0 \leq \phi \leq 1$, the function $\lambda(\phi)$ admit a minimum in

$$\phi = \frac{1}{2} \qquad \text{and} \qquad \lambda\left(\frac{1}{2}\right) = \frac{\mu_b^2 + 2\mu_b\mu_r + \mu_r^2}{4\mu_b\mu_r}.$$

so $V_3(\phi) > 0$ if

$$r_p > \lambda\left(\frac{1}{2}\right) = \frac{\mu_b^2 + 2\mu_b\mu_r + \mu_r^2}{4\mu_b\mu_r}.$$

If we proceed in the same way, we can said that $Q(t)$ can not tighten to infinity when $t \longrightarrow T_{max}(T_{max} < +\infty)$ with the condition $\dfrac{\mu_b^2 + 2\mu_b\mu_r + \mu_r^2}{4\mu_b\mu_r} < r_p$.
When, $0 \leq \phi \leq 1$ we necessary obtain $V_2 \geq 0$. It result of this last result and expression of V_2 that state variable H is narrow.

3.2.2 Equilibrium stability analysis

3.2.2.1 Stationaries states

Proposition 3.6 *The stationaries states of system (3.3) are:*
(E_1) : $\Big(B = K_b, R = 0, P = 0\Big)$,
(E_2) : $\Big(B = 0, R = K_r, P = 0\Big)$,
(E_3) : $\Big(B = \frac{r_b - \mu_\zeta K_r}{r_b} K_b, R = K_r, P = 0\Big)$ *eligible if only if* $r_b > \mu_\zeta K_r$,
(E_4) : $\Big(B = 0, R = \frac{r_p r_r - r_p + q_p}{r_p r_r} K_r, P = \frac{r_r r_p - r_p + q_p}{r_r r_p^2 \mu_r} K_r (r_p - q_p)\Big)$ *eligible if only if* $r_r > \frac{r_p - q_p}{r_p}$,
(E_5) : $\Big(B = \frac{r_p r_b - r_p + q_p}{r_p r_b} K_b, R = 0, P = \frac{r_b r_p - r_p + q_p}{r_b r_p^2 \mu_b} K_b (r_p - q_p)\Big)$ *feasible if only if* $r_b > \frac{r_p - q_p}{r_p}$.

Proof: We search the equilibrium states with positive or zero components. For this, we must solve the following system:

$$\begin{cases} r_b B\Big(1 - \dfrac{B}{K_b}\Big) - \mu_\zeta RB - \dfrac{\beta B}{\beta B + R}\mu_b P = 0, \\ \\ r_r R\Big(1 - \dfrac{R}{K_r}\Big) - \dfrac{R}{\beta B + R}\mu_r P = 0, \\ \\ r_p P\Big(1 - \mu_b \mu_r \dfrac{P}{\mu_r B + \mu_b R}\Big) - q_p P = 0, \end{cases} \tag{3.6}$$

So we get:

- if $P = R = 0$ and $B \neq 0$, we obtain $B = K_b$
- if $P = B = 0$ and $R \neq 0$, we have $R = K_r$

Now, we consider the states with one nil component. So we have if $B = 0$, then R and P verify the following equation:

$$r_r R\Big(1 - \frac{R}{K_r}\Big) - \mu_r P = 0 \text{ and } r_p P\Big(1 - \frac{\mu_r P}{R}\Big) - q_p P = 0.$$

As we search $P \neq 0$, pulling the value of R in the second equation of system (3.6) we find $R = \frac{r_p \mu_r P}{r_p - q_p}$ and we obtain

$$R = \frac{r_p r_r - r_p + q_p}{r_p r_r} \mu_r K_r \text{ and } P = \frac{r_r r_p - r_p + q_p}{r_r r_p^2} K_r (r_p - q_p)$$

which exist if and only if $r_r > \frac{r_p - q_p}{r_p}$.
In the similar way, supposing that $R = 0$ we get

$$B = \frac{r_p r_b - r_p + q_p}{r_p r_b} \mu_b K_b \text{ and } P = \frac{r_b r_p - r_p + q_p}{r_b r_p^2} K_b (r_p - q_p)$$

which exist if and only if $r_b > \dfrac{r_p - q_p}{r_p}$.

Finally, for $C = 0$ the states variable R and B must verify the following equation

$$r_b B\Big(1 - \frac{B}{K_b}\Big) - \mu_\zeta BR = 0 \ \text{ and } \ r_r R\Big(1 - \frac{R}{K_R}\Big) = 0.$$

Since we are looking for variables B and R strictly positive, we have $r_b - r_b\dfrac{B}{K_b} - \mu_\zeta R = 0$ and $R = K_r$. As a result there will $B = \dfrac{r_b - \mu_\zeta K_r}{r_b}K_b$ which exist if and only if $r_b > \mu_\zeta K_r$

Proposition 3.7 *The states E_1, E_2 and E_3 are always stable according to the Lyapunov theorem. Equally, according to the Lyapunov theorem, state E_4 is localy asymptoticaly stable if and only if*

$$r_b - \frac{\mu_\zeta K_r}{r_p r_r}\Big(r_p r_r - r_p + q_p\Big) - \frac{\beta\mu_b}{r_p\mu_r}(r_p - q_p) < 0 \ , r_p + r_r + q_p(\frac{2}{r_p} - 1) > 2 \ \textit{and} \ r_p(r_r - 1) + q_p > 0$$

and state E_5 localy asymptoticaly stable if and only if

$$r_r - \frac{-\mu_r}{\beta\mu_b r_p(r_p - q_p)} < 0, \ r_p + r_b + q_p(\frac{2}{r_p} - 1) > 2 \ \textit{et} \ r_p(r_b - 1) + q_p > 0$$

Proof: Let $J(B, R, P)$, the Jacobian matrix of the system (3.3), we have:

$$J(B,R,P) = \begin{pmatrix} r_b - 2\dfrac{r_b}{K_b}B - \mu_\zeta R - \dfrac{\beta\mu_b RP}{(\beta B + R)^2} & -\mu_\zeta B - \dfrac{\beta\mu_b BP}{(\beta B + R)^2} & -\dfrac{\beta\mu_b B}{\beta B + R} \\ \dfrac{\beta\mu_r RP}{(\beta B + R)^2} & r_r - 2\dfrac{r_r}{K_r}R - \dfrac{\beta\mu_r BP}{(\beta B + R)^2} & -\dfrac{\mu_r R}{\beta B + R} \\ \dfrac{r_p\mu_r^2\mu_b P^2}{(\mu_r B + \mu_b R)^2} & \dfrac{r_p\mu_b^2\mu_r P^2}{(\mu_r B + \mu_b R)^2} & r_p - \dfrac{2r_p\mu_r\mu_b P}{\mu_r B + \mu_b R} - q_p \end{pmatrix}$$

Let *specter(J)* set of the eigenvalues of the Jacobian matrix:

- we have $\mathit{specter}(J(E_1)) = \{-r_b,\ r_r,\ r_p - q_p\}$, so, the state (E_1) is unstable.
- we obtain also $\mathit{specter}(J(E_2)) = \{-r_r,\ r_b - \mu_\zeta K_r,\ r_p - q_p\}$, so this state is unstable.
- concerning to the state (E_3), we have $\mathit{specter}(J(E_3)) = \{-r_r,\ r_b - 2\mu_\zeta K_r,\ r_p - q_p\}$, this state is also unstable.
- for to the state (E_4), the eigenvalues of $J(E_4)$ are:

$$\gamma_1 = r_b - \frac{\mu_\zeta K_r}{r_p r_r}\Big(r_p r_r - r_p + q_p\Big) - \frac{\beta\mu_b}{r_p\mu_r}\Big(r_p - q_p\Big),\ \gamma_2 \text{ and } \gamma_3 \text{ satisfactory}$$
$$\gamma_2 + \gamma_3 = -\Big(\frac{r_p r_r - 2r_p + 2q_p + r_p^2 - r_p q_p}{r_p}\Big) \text{ and } \gamma_2\gamma_3 = \frac{-r_p r_r + r_p - q_p}{r_p}\Big(q_p - r_p\Big).$$

  So state (E_4) is localy asymptoticaly stable if and only if $\gamma_1 < 0$,$\gamma_2 + \gamma_3 < 0$ and $\gamma_2\gamma_3 > 0$, i.e $r_b - \dfrac{\mu_\zeta K_r}{r_p r_r}(r_p r_r - r_p + q_p) - \dfrac{\beta\mu_b}{r_p\mu_r}(r_p - q_p) < 0$, $r_p + r_r + q_p(\dfrac{2}{r_p} - 1) > 2$ and $r_p(r_r - 1) + q_p > 0$

- in the end, the eigenvalues of state $J(E_5)$ are:

$$\gamma_1 = r_r - \frac{-\mu_r}{\beta\mu_b r_p(r_p - q_p)},\ \gamma_2 \text{ and } \gamma_3 \text{ satisfactory}$$
$$\gamma_2 + \gamma_3 = \frac{-r_p r_b + 2r_p - 2q_p - r_p^2 + r_p q_p}{r_p} \text{ and } \gamma_2\gamma_3 = \frac{-r_p r_b + r_p - q_p}{r_p}(q_p - r_p).$$

So state (E_5) is localy asymptotically stable if and only if $\gamma_1 < 0$, $\gamma_2 + \gamma_3 < 0$ and $\gamma_2\gamma_3 > 0$, i.e

$$r_r - \frac{-\mu_r}{\beta\mu_b r_p(r_p - q_p)} < 0,\ r_p + r_b + q_p(\frac{2}{r_p} - 1) > 2 \text{ and } r_p(r_b - 1) + q_p > 0.$$

3.2.2.2 Component strictly equilibrium

Consider the functions π at ϕ define by:

$$\pi(\phi) = a_2\phi^2 + a_1\phi + a_0, \tag{3.7}$$

with

$$\begin{aligned}
a_0 &= \frac{r_b K_r}{\beta K_b}\Big(\frac{\mu_r}{\beta\mu_b} - 1\Big)\frac{\Big(r_p - q_p\Big)}{r_p} \\
a_1 &= \frac{\Big(r_p - q_p\Big)}{r_p}\Big[r_b\Big(1 - \frac{r_b K_r}{\beta K_b}\Big) - K_r\Big(\frac{r_b}{\beta K_b}\Big)\Big(1 - \frac{\mu_r}{r_r\beta\mu_b}\Big) - 1\Big] \\
a_2 &= \frac{\mu_r}{\beta\mu_b}\Big[\frac{K_r}{r_r}\Big(\frac{r_b}{\beta K_b} - \mu_\zeta\Big) + \frac{\beta\mu_b}{\mu_r}\Big]\frac{(r_p - q_p)}{r_p}
\end{aligned} \tag{3.8}$$

Consider the affine functions h and l define at $[0,1]$ to $\mathbb{R}$ by:

$$\begin{aligned}
h(\phi) &= r_r - \frac{(r_p - q_p)}{r_p}\Big(\phi + (1-\phi)\frac{\mu_r}{\beta\mu_b}\Big) \\
l(\phi) &= r_b - \frac{(r_p - q_p)}{r_p}\Big(\frac{\beta\mu_b}{\mu_r} + (1-\phi)\Big)
\end{aligned}$$

The following proposition gives us the necessary and sufficient conditions for the existence of strictly positive equilibrium:

Proposition 3.8 *There is a strictly positive component to equilibrium* $(B > 0,\ R > 0,\ P > 0)$ *of system* (3.3) *exist if and only if there exists a solution* $\phi \in]0,1[$ *of* $\pi(\phi) = 0$ *equation as one of the following assertions holds:*

(i) $h(\phi) > 0$,
(ii) $l(\phi) > 0$.

Proof: With the previous results we have the following equations:

$$r_b - r_b\frac{B}{K_b} - \mu_\zeta R - \frac{\beta}{\beta B + R}\mu_b P = 0, \tag{3.9}$$

$$r_r - r_r\frac{R}{K_r} - \frac{1}{\beta B + R}\mu_r P = 0, \tag{3.10}$$

$$r_p\beta\mu_b\mu_r P = (r_p - q_p)\beta(\mu_r B + \mu_b R), \tag{3.11}$$

By performing the scale change of variable:

$$\overline{\mu_b} = \beta\mu_b,\ \overline{B} = \beta B \text{ and } \overline{K_b} = \beta K_b,$$

we obtain the following equation:

$$r_b - r_b\frac{B}{K_b} - \mu_\zeta R - \frac{1}{\beta B + R}\mu_b P = 0, \tag{3.12}$$

$$r_r - r_r\frac{R}{K_r} - \frac{1}{B+R}\mu_r P = 0, \tag{3.13}$$

$$r_p\mu_b\mu_r P = (r_p - q_p)(\mu_r B + \mu_b R), \tag{3.14}$$

Then, we consider the states variable

- $H = R + B$ the total resource,
- $\phi = \dfrac{R}{B+R}$ the proportion of zooplankton within the total resource quantity,
- $Q = \dfrac{P}{R+B}$ the ratio fish/ total resource.

When, we use them expressions
$B = H(1-\phi)$, $R = \phi H$, $P = QH$, the above equations become:

$$r_b - H\Big(\frac{r_b}{K_b}(1-\phi) + \mu_\zeta\phi\Big) - \mu_b Q = 0, \tag{3.15}$$

$$r_r - \frac{r_r}{K_r}\phi H - \mu_r Q = 0, \tag{3.16}$$

$$r_p\mu_b\mu_r Q = \big(r_p - q_p\big)\Big(\mu_r\big(1-\phi\big) + \mu_b\phi\Big), \tag{3.17}$$

Consequently, through these equations we have the expressions of H and Q

$$Q = \frac{\big(r_p - q_p\big)}{r_p\mu_b\mu_r}\Big[\mu_r\big(1-\phi\big) + \mu_b\phi\Big] \quad \textit{and} \quad H = \frac{K_r}{r_r\phi}\Big[r_r - \frac{\big(r_p - q_p\big)}{r_p}\Big(\frac{\mu_r}{\mu_b}\big(1-\phi\big) + \phi\Big)\Big] \tag{3.18}$$

After a substitution in (3.17), we obtain:

$$r_b - \frac{K_r}{r_r\phi}\Big[r_r - \frac{\big(r_p - q_p\big)}{r_p}\big(\frac{\mu_r}{\mu_b}\big(1-\phi\big) + \phi\big)\Big]\big(\frac{r_b}{K_b}\big(1-\phi\big) + \mu_\zeta\phi)\big) - \frac{\big(r_p - q_p\big)}{r_p\mu_r}\Big[\mu_r\big(1-\phi\big) + \mu_b\phi\big)\Big] = 0 \tag{3.19}$$

We get the following equation by multiplying (3.19) by ϕ

$$a_2\phi^2 + a_1\phi + a_0 = 0 \tag{3.20}$$

with

$$\begin{aligned} a_0 &= \frac{r_b K_r}{\beta K_b}\Big(\frac{\mu_r}{\beta\mu_b} - 1\Big)\frac{\big(r_p - q_p\big)}{r_p} \\ a_1 &= \frac{\big(r_p - q_p\big)}{r_p}\Big[r_b\Big(1 - \frac{r_b K_r}{\beta K_b}\Big) - K_r\big(\frac{r_b}{\beta K_b}\big)\Big(1 - \frac{\mu_r}{r_r\beta\mu_b}\Big) - 1\Big] \\ a_2 &= \frac{\mu_r}{\beta\mu_b}\Big(\frac{K_r}{r_r}\big(\frac{r_b}{\beta K_b} - \mu_\zeta\big) + \frac{\beta\mu_b}{\mu_r}\Big)\frac{\big(r_p - q_p\big)}{r_p}. \end{aligned} \tag{3.21}$$

We search the triplets (H, ϕ, Q) verifying $0 < \phi < 1$, $H > 0$ and $Q > 0$.

Consequently, it's about to find the root $0 < \phi < 1$ of function polynomial π.
Existence of this root $0 < \phi < 1$ give us the existence of equilibrium state at component strictly positive of the system (3.5). Indeed, suppose that ϕ has a root $0 < \phi < 1$. According to (3.18) we have $Q > 0$ and H which write

$$H = \frac{K_r}{r_r \phi} h(\phi) \quad with \quad h(\phi) = r_r - \frac{\big(r_p - q_p\big)}{r_p}\Big(\frac{\mu_r}{\mu_b}(1-\phi) + \phi\Big). \tag{3.22}$$

In the same manner using the equations (3.15) and (3.16) we can write H in the following style.

$$H = \frac{1}{\Big(\dfrac{r_b}{K_b}(1-\phi) + \mu_\varsigma \phi\Big)} l(\phi) \quad with \quad l(\phi) = r_b - \Big(\frac{\big(r_p - q_p\big)}{r_p}\Big)\Big(\frac{\mu_b}{\mu_r}\phi + (1-\phi)\Big) \tag{3.23}$$

As we have $0 < \phi < 1$, according to (3.22) $H > 0$ if and only if $h(\phi) > 0$. Likewise $H > 0$ in (3.21) if and only if $h(\phi) > 0$

Remarque 1 *In practice, it is sufficient to check whether at least one of the following conditions is satisfied:*
(i) $h(0) \geq 0$ *and* $h(1) > 0$
(ii) $h(0) > 0$ *and* $h(1) \geq 0$
(iii) $h(0) > 0$ *and* $h(1) < 0$
(iv) $h(0) < 0$ *and* $h(1) > 0$

Proposition 3.9 *The polynomial (3.7) has a single positive root strictly if and only if one of the following is true:*
(a) $-signe(a_0) = signe(a_1) = signe(a_2)$
(b) $a_1^2 = 4a_0a_2$ *and* $signe(a_0) = signe(a_2) = -signe(a_1)$
(c) $signe(a_0) = -signe(a_1)$ *and* $a_2 = 0$
(d) $signe(a_0) = -signe(a_1)$ *and* $a_2 = 0$
(e) $signe(a_1) = -signe(a_2)$ *and* $a_0 = 0$
In fact, the polynomial (3.7) *admit two strictly positive root if and only if* $a_1^2 - 4a_0a_2 > 0$ *and* $signe(a_0) = signe(a_2) = -signe(a_1)$

Proof: First, we consider the case $a_0 \neq 0, \quad a_2 \neq 0$ and $a_1 \neq 0$. Let $\triangle = a_1^2 - 4a_0a_2$.

(i) When $\triangle > 0$ we have a following case:
a) Two roots $\chi_1 > 0$ and $\chi_2 > 0$ if and only if $\dfrac{a_0}{a_2} > 0$ and $\dfrac{a_1}{a_2} < 0$ i.e $-signe(a_0) = signe(a_1) = signe(a_2)$
b) One strictly positive root $\chi_0 > 0$ if and only if $\dfrac{a_0}{a_2} < 0$ and $\dfrac{a_1}{a_2} > 0$ i.e $-signe(a_0) = signe(a_1) = signe(a_2)$

(ii) If $\triangle = 0$, we have an unique root $\chi = \dfrac{-a_1}{2a_2} > 0$ if and only if $-signe(a_0) = signe(a_1) = signe(a_2)$

We look at now of the particular case:

- first, if $a_0 \neq 0$, $a_2 = 0$ and $a_1 \neq 0$, then we have only root $\chi = \dfrac{-a_0}{a_1} > 0$ if and only if $-signe(a_0) = signe(a_1)$.

- then, if $a_0 \neq 0$, $a_2 \neq 0$ and $a_1 = 0$, so we have one root $\chi = \sqrt{\frac{-a_0}{a_2}} > 0$ if and only if $signe(a_0) = -signe(a_2)$.
- at last, if $a_0 = 0$, $a_2 \neq 0$ and $a_1 \neq 0$, so we have one root $\chi = \dfrac{-a_1}{a_2} > 0$ if and only if $signe(a_1) = -signe(a_2)$.

4 Numerical experiments and results

In this section, we present a synthesis of results observing during our numerical simulations of these different mathematical models.

4.1 Numerical simulations of two species system

For ours simulations, we use the following parameters.

Parameters	B_0	R_0	μ_r	μ_b	P_0	q
Value used	100000	10000	180	180	300	1

4.1.1 Fish-Phytoplankton(B-P)

In this simulation, we take $r_b = 1,52$; $r_p = 0,53$ and $E = 0$. We consider that the system evolve in no exploited area i.e there is no fishing in this area.
The figure 5 show the existence of the orbits of solution. As overall result, we have the existence and dynamic stability in long time for the system fish-phytoplankton(B-P). So we can say that if the area is not exploited the dynamic of Fish-Phytoplankton system remains stable.

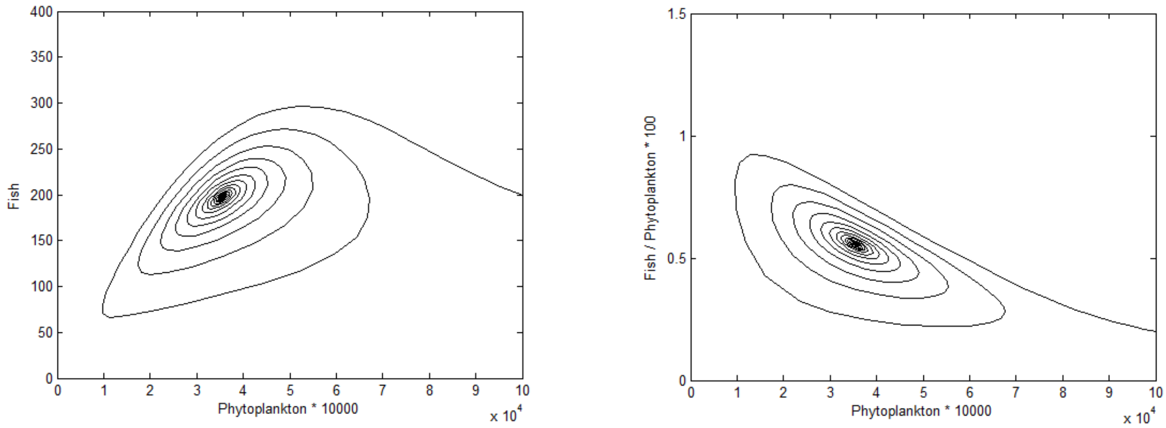

Figure 5: Fish-Phytoplankton(B-P) model with $E = 0$

Then, we consider that the system Fish-Phytoplankton evolve in exploited area i.e that fishing is practiced in this area. In this simulation, we take $r_b = 1,45$; $r_p = 0,5$ and $E = 0,5$.
The figure 6 show the existence of the orbits of solution. As overall result, we have the existence and dynamic stability in long time for the system fish-phytoplankton(B-P). We observe a slight perturbation of this dynamic. We can say that if area is exploited with a fishing effort $E = 0,5$, the dynamic of Fish-Phytoplankton is stable. In this case, we speak about area normally exploited.

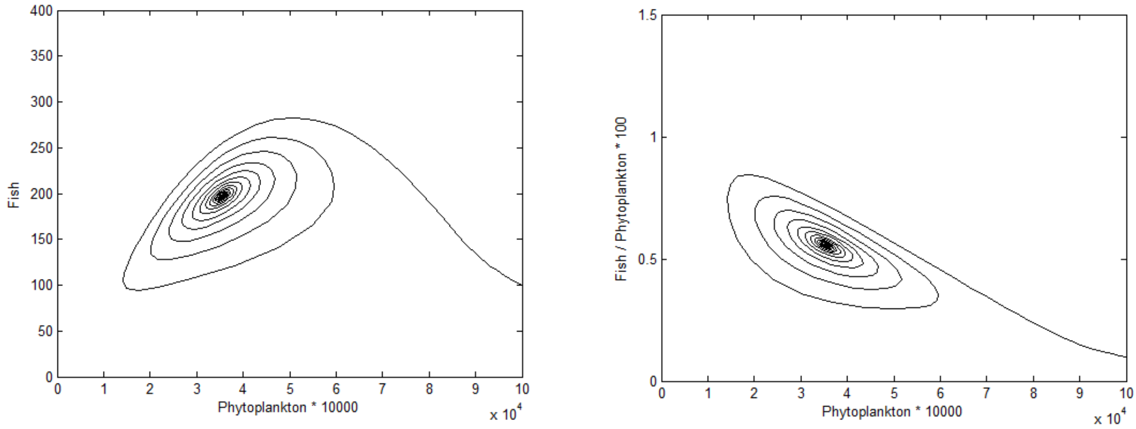

Figure 6: Fish-Phytoplankton(B-P) model with $E = 0,5$

We continue the computational simulation and we consider that the system Fish-Phytoplankton evolve in exploited area i.e that fishing is practiced in this area. In this simulation, we take $r_b = 1,42$; $r_p = 0,58$ and $E = 2,3$.
The figure 7 show the existence of the orbits of solution. As a consequence global we have extinction with large time of this dynamic. We observe a real disturbance of this dynamic. We can say that if area is exploited with a fishing effort $E = 2,3$ the dynamic of Fish-Phytoplankton is unstable. In this case, we speak about the severely exploitation of the area.

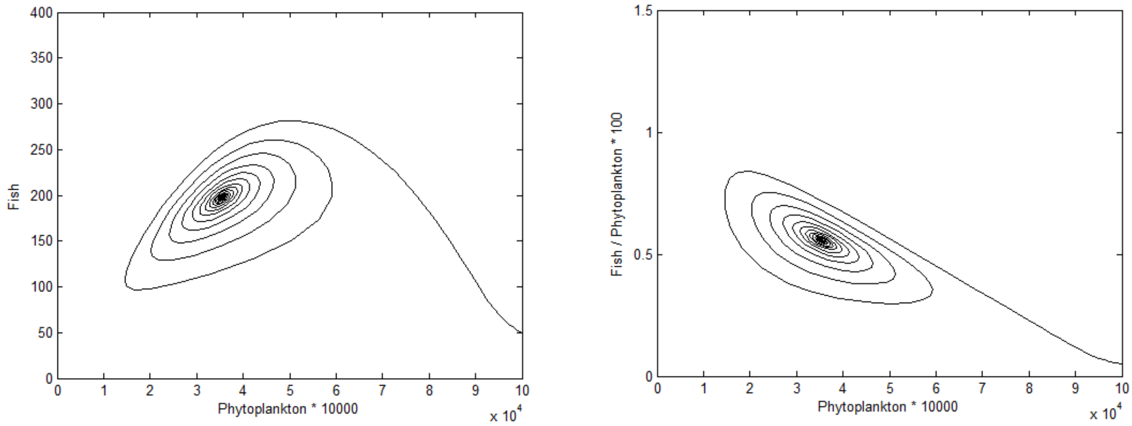

Figure 7: Fish-Phytoplankton(B-P) model with $E = 2,3$

4.1.2 Fish-Zooplankton(R-P)

Here we take $r_r = 1,52$; $r_p = 0,53$ and $E = 0$ for the simulation. We consider that the system evolve in no exploited area i.e there is no fishing in this area.
The figure 8 show the existence of the orbits of solution. As a consequence global existence with large time stabilization of this dynamic is observed. We can say that if area is not exploited, the dynamic of Fish-Zoooplankton(R-P) is stable.

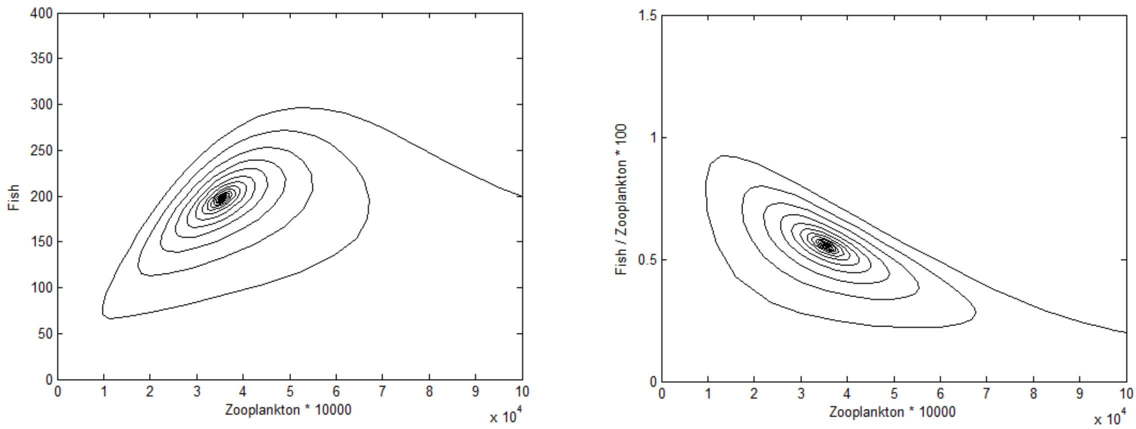

Figure 8: Fish-Zooplankton(P-R) model with $E = 0$

Then, we consider that the system Fish-Zooplankton evolve in exploited area i.e that fishing is practiced in this area. In this simulation, we take $r_r = 1,45$; $r_p = 0,5$ and $E = 0,5$.
The figure 9 shows the existence of the orbits of solution. As a consequence global existence with large time stabilization of this dynamic is observed. We observe also a slight perturbation of this dynamic. We can say that if, area is exploited with a fishing effort $E = 0,5$, the dynamic of Fish-Phytoplankton is stable. In this case, we say that the area is normally exploited.

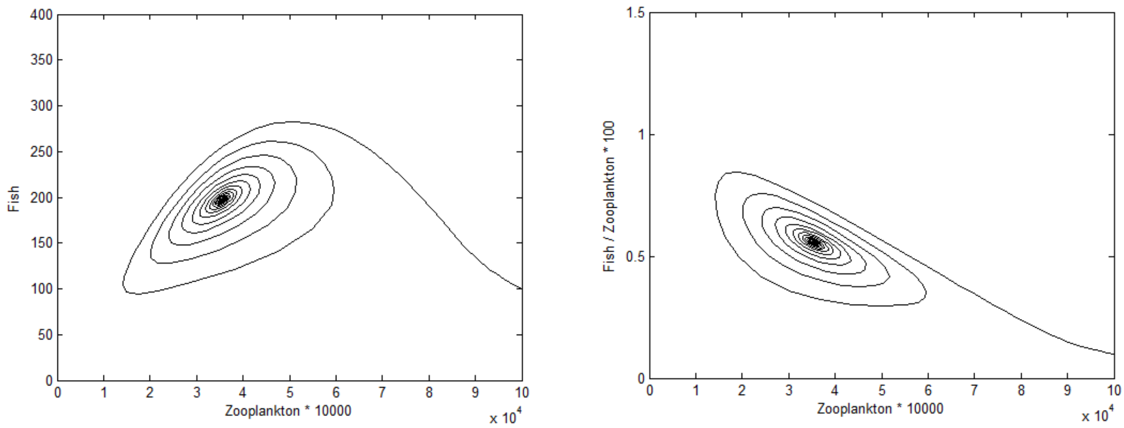

Figure 9: Fish-Zooplankton(P-R) model with $E = 0,5$

We continue the computational simulation and we consider that the system Fish-Zooplankton evolve in exploited area. In this simulation, we take $r_b = 1,42$; $r_p = 0,58$ and $E = 2,3$.
The figure 10 shows the existence of the orbits of solution. We observe a real disturbance of this dynamic. We can say that if area is exploited with a fishing effort $E = 2,3$, the dynamic of Fish-Phytoplankton is unstable. In this case, we say that area severely exploited.

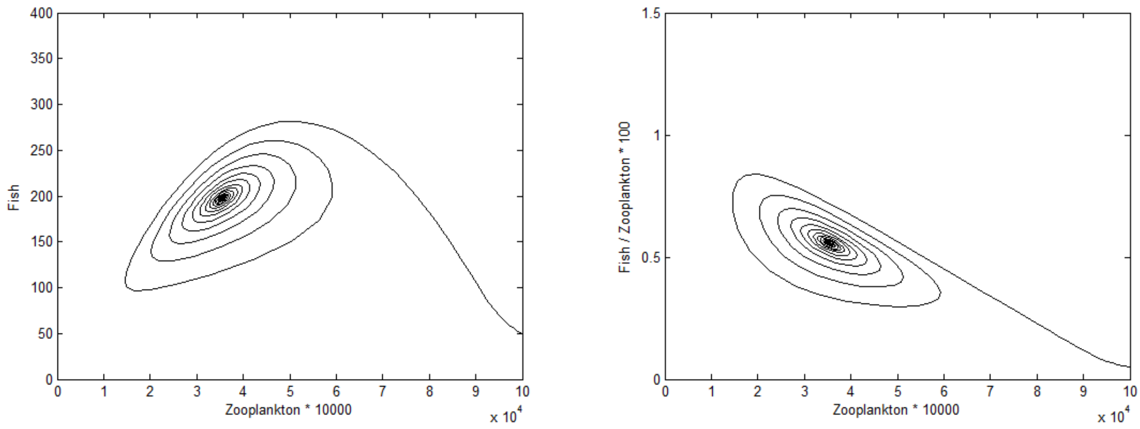

Figure 10: Fish-Zooplankton(P-R) model with $E = 2,3$

4.1.3 Zoooplankton-Phytoplankton(B-R)

The numerical simulation, show that for $r_b = 1,53$; $r_r = 0,53$, we have the existence of orbits of solution. As a consequence global we have the existence with large time stabilization of this dynamic. The figure 11 shows this observation.

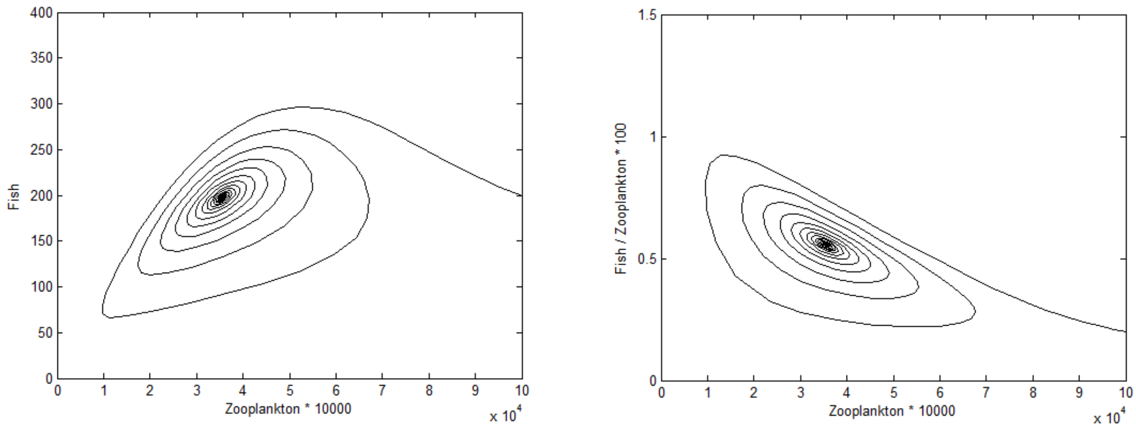

Figure 11: Zoooplankton-Phytoplankton(B-R)

4.2 Numerical simulations of three species system: Phytoplankton-Zooplankton-Fish

In this subsection, for our simulations, the set of the demographic parameters used are given by the following list.

Parameters	B_0	R_0	μ_r	μ_b	P_0	q	K_r	K_b	μ_ζ	β
Value used	50000	2000	180	180	100	1	20000	100000	0,02	1,5

4.2.1 Numerical simulations on the Phytoplankton-Zooplankton-Fish system evolve in no exploited area

First, we look at the Phytoplankton-Zooplankton-Fish system which evolve in no exploited area and so we take the fishing effort $E = 0$.

The figure 12 explains the stability of the different populations. As a consequence global we have the existence with large time stabilization of this dynamic.

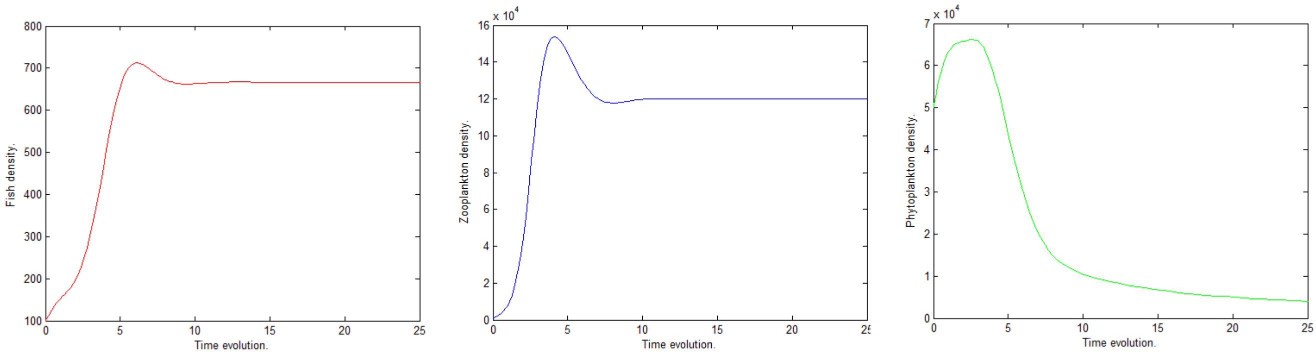

Figure 12: Evolution of three species: Phytoplankton-Zooplankton-Fish system evolve in no exploited area

4.2.2 Numerical simulations of the Phytoplankton-Zooplankton-Fish system evolve in exploited area

We continue our numerical simulations supposing that the Phytoplankton-Zooplankton-Fish system evolve in exploited area with a fishing effort $E = 0,5$.
The figure 13 explains the stability of the Zooplankton and phytoplankton, and an imbalance of the fish populations. As a global consequence we observe the existence with large time of the fish population and his a imbalance. In this case, we speak about to the normally exploited of area. So, if a fish area is exploited with a fishing effort $E = 0,5$ there is not risk for the fish population.

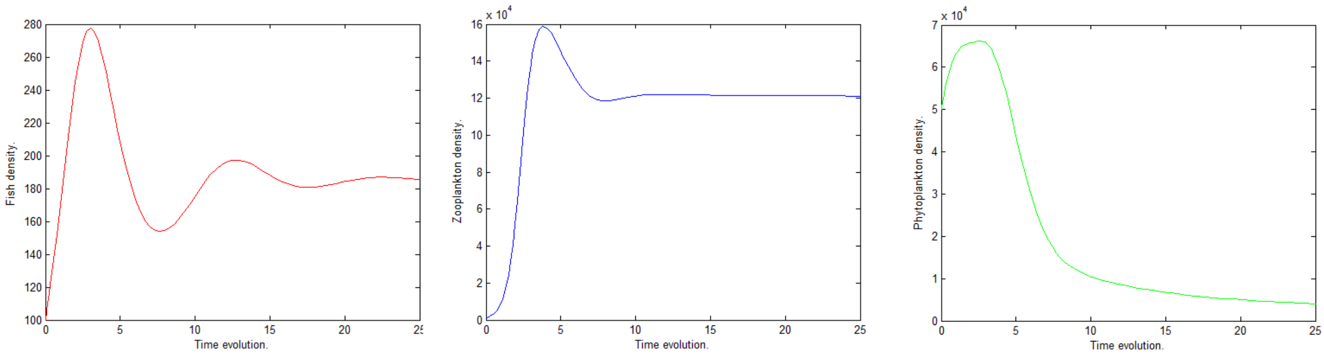

Figure 13: Evolution of the Phytoplankton-Zooplankton-Fish system evolve in normally exploited area

4.2.3 Numerical simulations on the Phytoplankton-Zooplankton-Fish system evolve in severly exploited area

Finally, we consider that the system Phytoplankton-Zooplankton-Fish evolve in exploited area with a fishing effort $E = 2,3$.
The figure 14 shows the stability of the Zooplankton and phytoplankton species, and endangered fish species. So we have the extinction with large time of the fish species. In this case, we speak about the severely exploitation of the area. We can say that, if a fish area is exploited with a fishing effort $E = 2,3$ there is a risk for the fish population and so it's necessary to develop political management for this area, otherwise fish species risk to disappear.

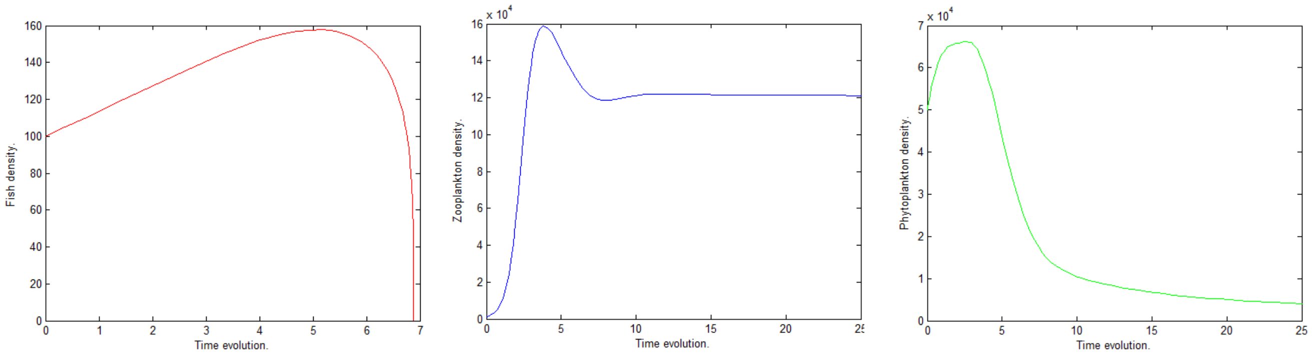

Figure 14: Evolution of the Phytoplankton-Zooplankton-Fish system evolve in severely exploited area

5 Conclusions and future works

In this paper, we are interested in the study of fish population dynamic under a diet on a plankton (phytoplankton and zooplankton). The mathematic model associate to this dynamic is ODE singular system, the denominator of one the reaction terms can be zero. The mathematical analysis permits to isolate extinction condition based on a growth-rate, in finished or persisting time and gives some conditions for persistence or finite time extinction of populations. The asymptotical behaviour depends on additional assumptions. We use numerical experiments to point out the main effects of the fish catching on the dynamic. It is important to note that, for the fish dynamic, we could obtain stale coexistence. Equally, computational simulations, show that if the fishing effort $E = 0,5$ area is exploited normally, while if $E > 0,5$ area is surexploited. In the future, we shall extend this work by taking into account two classes of the fish population: Larva and adult. This approach should give additional numerical results and should help us to determine which will be the more efficient strategy to protect the fish species.

References

[1] B. E. Ainseba, F. Heiser, M. Langlais. 2002. *A mathematical analysis of a predator-prey system in a highly heterogeneous environment.* Differential and Integral Equations, 15, 385-404.

[2] D. Claessen, AM de Roos, and L Persson. *Population dynamic theory of size-dependent canabalism. Proceedings of the Royal society of London.* Serie B: Biological sciences, 271 (1537): 333-340, FEB 22 2004.

[3] F. Courchamp, M. Langlais and G. Sugihara, 2000. *Rabbits Killing birds: modelling the hyperpredation process.* Journal of animal Ecology, 69, 154-164.

[4] G. Kreisselmeir and R. Engel. *Nonlinear observer autonomous Lipschitz continuous systemes, IEEE Trans, on Autotic control*, Vol. 48, pp 451-464, 2003.

[5] J. Raul 2012. *Modélisation mathematique structurée en taille du zooplancton*, Thèse de doctorat, Université de Nice-Sophia Antipolis .

[6] M Kot *Elements of mathematical ecology.* Cambridge Univ. Press, 2001.

[7] M. Rosenzweig and R. MacArthur. *Graphical representation and stability conditions of predator-prey interaction.* American Naturalist, 97: 209-223, 1963.

[8] S. Gaucel, M. Langlais and D. Pontier. 2005. *Invading introduced species in insular heterogeneous environments.* Ecology Modelling, 188, 62-75.

[9] S. Gaucel, *Analyse mathématique et simulation numérique d'un modèle prédateur-proie en milieu insulaire hétérogène.* Thèse de doctorat, Université Bordeau 2, France, 2005.

[10] S. M. SARR, Jean-André Tinkoudgou KABRE et Hamet DIADHIOU2 *Age et croissance de Mugil cephalus (Linneaus, 1758, Mugilidea) dans l'estuaire du fleuve Sénégal* , Mémoire de DEA 2012.

[11] SARR Serigne Modou, 2012, *Age et croissance, paramètres d'exploitation et régime alimentaire des mulets jaunes, Mugil cephalus (Linnaeus, 1758, Mugilidae) dans l'estuaire du Fleuve Sénégal* , Thèse de Doctaurat de l' Université Polytechnique de Bobo-Dioulasso, Burkina Faso.