Variational assimilation method for solving an optimal control problem to identify initial height of dune in an aquatic environment

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Abstract

This paper concerns the estimation of the optimal initial dune height which can favor the evolution of the dune in the depth of an aquatic environment in two space dimension. The problem is formulated as an optimal control problem, governed by nonlinear equations describing the dune height formation. The control is done in a discrete framework, where the model solution gap to the data is minimized using a quasi-Newton method. The functional gradient is computed by the backward time integration of the corresponding discrete adjoint model. Numerical tests are provided in order to discuss the efficiency and the effectiveness of our approach.

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1 Introduction

In the recent decades, the silting phenomenon threatens recurrently some rivers in many African countries : including Tchad lake, Congo river, Niger river, etc. This phenomena is of a major concern and topical subject.
Several studies are increasingly interested to find solutions to this problem [10, 16, 17]. However, due to the complexity of this problem, much remains to be done in this field. The complexity is related to the fact that in most cases some physical or hydro-geological parameters involved in the dunes formation or movement are poorly known or poorly estimated.
The subject of our study is to estimate the optimal initial dune height which can favor its

formation in the depth of incompressible flows in two space dimension. To achieve this we consider an optimal control problem governed by nonlinear equations describing the dune height formation [14, 16]. the control function will play the role of an uncertainty on the initial data of the dune height, and allow to estimate the optimal value of this one which can favor its formation.
Several numerical approaches are used to solve a class of optimal control problems [4, 5, 6, 7, 14, 15, 20, 21]. In this paper, the control is done in a discrete framework, where the model solution gap to the data is minimized using a quasi-Newton method. The functional gradient is computed by the backward time integration of the corresponding discrete adjoint model. To discretize the constraints equations, we use a combinaison of the Crank-Nicholson scheme and a Chebyshev spectral method $\mathbb{P}_{N,M}$ type [2, 16]. This spectral approach aim to approximate the functions and their derivatives by Chebyshev polynomials of degree at most N according to the variable x and at most M according to the variable y, at the Chebyshev-Gauss-Lobatto collocation points [2, 9, 16].
The paper is structured as follows : the second section is devoted to the mathematical formulation of the problem, in the third section we present the discrete adjoint method and the discrete model. The fourth section is dedicated to numerical simulations which wil focus on the estimation of the optimal initial dune heigth which will favor the evolution of the dune. We end this work by conclusion and perspectives.

2 Problem Formulation

The objective in this section is to formulate an optimal control problem gouverned by nonlinear equations describing the dune formation.
Let Ω be a regular bounded domain of $\mathbb{R}^2$ in which involves an incompressible flow.
We consider the presence of a dune in the depth of this flow, occupying a domain $\Gamma =]-1,1[^2$ in Ω. The problem that we are concerned is to estimate the optimal initial dune height h_0 which can favor the evolution of the dune. For this we consider the following optimal control problem with constraints given by :

$$\min J(v) = \frac{1}{2}\int_0^T \| h(t,x,y) - h^{obs}(t,x,y) \|^2_{L^2(\Gamma)}\, dt + \frac{\alpha}{2} \| v(x,y) \|^2_{L^2(\Gamma)}, \tag{1}$$

subject to :

$$\begin{cases} \dfrac{\partial h}{\partial t} = \nabla.(m\nabla h) + \Phi(t,x,y) \quad \text{in }]0,T[\times\Gamma & (2)\\ \| \nabla h \| \leq 1,\ m(\| \nabla h \| -1) = 0 \quad \text{in }]0,T[\times\Gamma & (3)\\ h(0,x,y) = h_0(x,y) + v(x,y) \quad \text{on } \Gamma & (4)\\ h(t,x,y) = 0 \quad \text{in } \partial\Gamma, & (5) \end{cases}$$

where
• h is the dune height;
• h^{obs} is an observation data;

- $\Phi(t,x,y)$ is a source term;
- $m(t,x,y)$ is the mass density of the sand grains carried by the flows;
- $v(x,y)$ is the control variable;
- α denotes a real coefficient of regularization.

We solve this problem using the variational data assimilation method [1, 3, 8] via the discrete adjoint method [19] and a quasi-Newton method.
This method (The variational data assimilation method) aim at combining measured observations of the system with the dynamical model associated in order to improve the estimation of some parameters [3, 13]. In this study, the parameter that we are concerned is the initial condition h_0.

3 Discrete Adjoint Method

The discrete adjoint method aim to calculate the gradient of the discrete cost function using the adjoint model of the discrete problem (the discrete adjoint model) [3, 13, 19].
To discretize the dynamic model (2)-(5) we use the Crank-Nicholson method and the spectral approach $\mathbb{P}_{N,M}$ type.
So we note by $\mathbb{P}_{N,M}(\Gamma)$ the set of Chebyshev polynomials defined on Γ, of degree at most N according to the variable x and at most M according to the variable y.
Let $H_0^1(\Gamma)$ the space of the test functions defined on Γ and zero at the bound.
We note $Q = \mathbb{P}_{N,M}(\Gamma) \cap H_0^1(\Gamma)$ and Consider $\mathcal{U}_{ad} = \{v \in L^2(Q) \ : \| \nabla v \| \leq 1\}$ the admissible controls set.
We choose Φ and m in $L^2(0,T;\mathbb{P}_{N,M}(\Gamma))$, the dune height h and observation data h^{obs} in $L^2(0,T;Q)$.

3.1 Discrete Model

In this section we present the numerical discretization of the model equation in section 2. Then, we start with the time discretization.
For a given positive integer r, we define the knots of the interval $[0,T]$ by $t_n = n\Delta t$, $n = 0,\cdots,r$, with $\Delta t = \dfrac{T}{r}$, a time step ($T > 0$).
For $n = 0,\cdots,r$, we consider :

$$\begin{aligned} h(t_n,x,y) &\approx h^n(x,y), \\ m(t_n,x,y) &\approx m^n(x,y), \\ \phi(t_n,x,y) &\approx \phi^n(x,y). \end{aligned}$$

A two order Crank-Nicholson scheme applied to equation (2) at knots t_n, leads to :

$$\frac{h^{n+1}-h^n}{\Delta t} = \frac{1}{2}(\nabla.(m^{n+1}\nabla h^{n+1}) + \Phi^{n+1}) + \frac{1}{2}(\nabla.(m^n\nabla h^n) + \Phi^n). \tag{6}$$

For a given positive integers N, M consider the Chebyshev-Gauss-Lobatto collocations points defined by :
$x_i = \cos(\frac{i\pi}{N}),\ i = 0, \cdots, N$ and $y_j = \cos(\frac{j\pi}{M}),\ j = 0, \cdots, M$.
For $n = 0, \cdots, r$, $i = 0, \cdots, N$, $j = 0, \cdots, M$, we consider :

$$\begin{aligned} h(t_n, x_i, y_j) &\approx h_{i,j}^n, \\ m(t_n, x_i, y_j) &\approx m_{i,j}^n, \\ \phi(t_n, x_i, y_j) &\approx \phi_{i,j}^n. \end{aligned}$$

For a given function $\psi = h, m$, the $\mathbb{P}_{N,M}$ spectral approach leads to the following spatial approximations :

$$\frac{\partial\psi(t_n, x_i, y_j)}{\partial x} \approx \sum_{k=0}^{N} d_{i,k}^{N,1} \psi_{k,j}^n, \quad \frac{\partial\psi(t_n, x_i, y_j)}{\partial y} \approx \sum_{l=0}^{M} d_{j,l}^{M,1} \psi_{i,l}^n; \tag{7}$$

$$\frac{\partial^2\psi(t_n, x_i, y_j)}{\partial x^2} \approx \sum_{k=0}^{N} d_{i,k}^{N,2} \psi_{k,j}^n, \quad \frac{\partial^2\psi(t_n, x_i, y_j)}{\partial y^2} \approx \sum_{l=0}^{M} d_{j,l}^{M,2} \psi_{i,l}^n, \tag{8}$$

where $d_{.,.}^{s,1}$ and $d_{.,.}^{s,2}$ denote respectively the coefficients of Chebyshev differentiation matrices of one order D_s and two order D_s^2, for $s = N, M$, [9, 11, 18].
For the simplification reasons, we consider the case $N = M$.
The approximations (7)-(8) applied to the equation (6) lead to :

$$\begin{aligned} \frac{h_{i,j}^{n+1} - h_{i,j}^n}{\Delta t} = \frac{1}{2}[(\sum_{k=0}^{N} d_{i,k}^{N,1} m_{k,j}^{n+1})(\sum_{k=0}^{N} d_{i,k}^{N,1} h_{k,j}^{n+1}) + (\sum_{l=0}^{N} d_{j,l}^{N,1} m_{i,l}^{n+1})(\sum_{l=0}^{N} d_{j,l}^{N,1} h_{i,l}^{n+1}) \\ + m_{i,j}^{n+1} \sum_{k=0}^{N} d_{i,k}^{N,2} h_{k,j}^{n+1} + m_{i,j}^{n+1} \sum_{l=0}^{N} d_{j,l}^{N,2} h_{i,l}^{n+1}] \qquad (9) \\ + \frac{1}{2}[(\sum_{k=0}^{N} d_{i,k}^{N,1} m_{k,j}^{n})(\sum_{k=0}^{N} d_{i,k}^{N,1} h_{k,j}^{n}) + (\sum_{l=0}^{N} d_{j,l}^{N,1} m_{i,l}^{n})(\sum_{l=0}^{N} d_{j,l}^{N,1} h_{i,l}^{n}) \\ + m_{i,j}^{n} \sum_{k=0}^{N} d_{i,k}^{N,2} h_{k,j}^{n} - m_{i,j}^{n} \sum_{l=0}^{N} d_{j,l}^{N,2} h_{i,l}^{n}] + \frac{1}{2}\phi_{i,j}^{n+1} + \frac{1}{2}\phi_{i,j}^{n}, \end{aligned}$$

for $n = 0, \cdots, r-1$, $i = 1, \cdots, N-1$; $j = 1, \cdots, M-1$,
with boundaries conditions :

$$h_{i,j}^n = 0, \text{ for } n = 0, \cdots, r,\ i = 0, N \text{ and } j = 0, M. \tag{10}$$

For $n = 0, \cdots, r-1$ note by :

$$H^{n+1} = (h_{1,1}^{n+1}, \ldots, h_{N-1,1}^{n+1}, h_{1,2}^{n+1}, \ldots, h_{N-1,2}^{n+1}, \ldots, h_{1,N-1}^{n+1}, \ldots, h_{N-1,N-1}^{n+1})^t$$

an $(N-1)^2$-dimensional coordinate vector,

$\mathbf{m}^{n+1} = (m_{1,1}^{n+1}, \ldots, m_{N-1,1}^{n+1}, m_{1,2}^{n+1}, \ldots, m_{N-1,2}^{n+1}, \ldots, m_{1,N-1}^{n+1}, \ldots, m_{N-1,N-1}^{n+1})^t$

an $(N-1)^2$-dimensional coordinate vector,

$\mathbf{m}_1^{n+1} = (m_{0,0}^{n+1}, \ldots, m_{N,0}^{n+1}, m_{0,1}^{n+1}, \ldots, m_{N,1}^{n+1}, \ldots, m_{0,N}^{n+1}, \ldots, m_{N,N}^{n+1})^t$

an $(N-1)^2$-dimensional coordinate vector,

$\mathbf{m}_2^{n+1} = (m_{i,j}^{n+1})_{0\le i\le N,\, 0\le j\le N}$ an $(N+1)\times(N+1)$ matrix,

$\mathbf{m}_3^{n+1} = [\mathbf{m}_2^{n+1}(2:N,1); \mathbf{m}^{n+1}; \mathbf{m}_2^{n+1}(2:N,N+1)]$ an $(N-1)^2$-dimensional

coordinate vector written under Matlab environment,

$\mathbf{I}$ an $(N-1)\times(N-1)$ identity matrix,

$\bar{D}_N$ an $(N-1)\times(N-1)$ matrix obtained by deleting the first row and

column and the last row and column of the matrix D_N,

$\bar{D^2}_N$ an $(N-1)\times(N-1)$ matrix obtained by deleting the first row and

column and the last row and column of the matrix D_N^2,

$\mathbf{O}$ an $(N-1)\times(N-1)$ zero-matrix,

$Q = [D_N(2:N,1), \mathbf{O}, D_N(2:N,N+1)]$ an $(N-1)\times(N+1)$ matrix written under Matlab environment,

$\Phi^{n+1} = (\phi_{1,1}^{n+1}, \ldots, \phi_{N-1,1}^{n+1}, \phi_{1,2}^{n+1}, \ldots, \phi_{N-1,2}^{n+1}, \ldots, \phi_{1,N-1}^{n+1}, \ldots, \phi_{N-1,N-1}^{n+1})^t$

an $(N-1)^2$-dimensional coordinate vector.

Let $K_1^{n+1},\ K_2^{n+1},\ K_3^{n+1},\ K_4^{n+1},\ K_5^{n+1},\ K_6^{n+1}$ be the $(N-1)^2\times(N-1)^2$

matrices defined by :

$$\begin{aligned}
K_1^{n+1} &= \Big(diag((\mathbf{I}\otimes Q)\mathbf{m}_1^{n+1}(N+2:N(N+1)))\Big)\Big(\mathbf{I}\otimes \bar{D_N}\Big),\\
K_2^{n+1} &= \Big(diag((\mathbf{I}\otimes \bar{D}_N)\mathbf{m}^{n+1})\Big)\Big(\mathbf{I}\otimes \bar{D}_N\Big),\\
K_3^{n+1} &= \Big(diag((Q\otimes \mathbf{I})\mathbf{m}_3^{n+1})\Big)\Big(\bar{D}_N\otimes \mathbf{I}\Big),\\
K_4^{n+1} &= \Big(diag((\bar{D}_N\otimes \mathbf{I})\mathbf{m}^{n+1})\Big)\Big(\bar{D}_N\otimes \mathbf{I}\Big),\\
K_5^{n+1} &= \Big(diag(\mathbf{m}^{n+1})\Big)\Big(\mathbf{I}\otimes \bar{D^2}_N\Big),\\
K_6^{n+1} &= \Big(diag(\mathbf{m}^{n+1})\Big)\Big(\bar{D^2}_N\otimes \mathbf{I}\Big),
\end{aligned}$$

where $\otimes$ denotes Kronecker product [12] and $diag(X)$ is an $L\times L$ matrix defined from $X=(X_1,X_2,\ldots,X_L)^t$ by :

$$diag(X)=\begin{pmatrix} X_1 & 0 & & & \\ 0 & X_2 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & X_{L-1} & 0 \\ & & & 0 & X_L \end{pmatrix},\quad L=(N-1)^2. \tag{11}$$

So we get the discrete model as follow :

$$\begin{cases} \dfrac{H^{n+1}-H^n}{\Delta t} - A^{n+1}H^{n+1} - A^nH^n = \dfrac{1}{2}\Phi^{n+1}+\dfrac{1}{2}\Phi^n,\ n=0,\cdots,r-1 & (12)\\ H^0 \text{ given,} & (13)\end{cases}$$

where $A^{n+1}=\frac{1}{2}(K_1^{n+1}+K_2^{n+1}+K_3^{n+1}+K_4^{n+1}+K_5^{n+1}+K_6^{n+1})$

3.2 Discrete Adjoint Model

Using the rectangles integration method, we get the discrete form of the cost function J in the following form :

$$J(V)=\frac{1}{2}\sum_{n=1}^{r}\Delta t\ (H^n-H^n_{obs})(H^n-H^n_{obs})^t+\frac{\alpha}{2}VV^t, \tag{14}$$

where $V=(v_{1,1},\ldots,v_{N-1,1},v_{1,2},\ldots,v_{N-1,2},\ldots,v_{1,N-1},\ldots,v_{N-1,N-1})^t$.
We calculate the directional derivative of this function as follow :

$$\begin{aligned}
\hat{J}(V)[\delta V] &= \lim_{\beta\to 0}\frac{J(V+\beta\delta V)-J(V)}{\beta}\\
&= \sum_{n=1}^{r}\Delta t\ \hat{H}^n(H^n-H^n_{obs})^t+\alpha(\delta V)V^t\\
&= \sum_{n=0}^{r-1}\Delta t\ \hat{H}^{n+1}(H^{n+1}-H^{n+1}_{obs})^t+\alpha(\delta V)V^t, \qquad (15)
\end{aligned}$$

where $\hat{H}=\lim_{\beta\to 0}\dfrac{H(V+\beta\delta V)-H(V)}{\beta}$ denotes the solution of the linear model tangent to the model (12)-(13) given by :

$$\begin{cases}
\dfrac{\hat{H}^{n+1}-\hat{H}^n}{\Delta t}-A^{n+1}\hat{H}^{n+1}-A^n\hat{H}^n=0,\ n=0,\cdots,r-1 & (16)\\
\hat{H}^0=\delta V \text{ given.} & (17)
\end{cases}$$

For $n=0,\cdots,r$ note the adjoint variable by λ^n, an $(N-1)^2$-dimensional coordinate vector.
Multiply the equation (16) by λ^n and integrate in time by parts in discrete form :

$$\begin{aligned}
0 &= \sum_{n=0}^{r-1}\Delta t\Big(\frac{\hat{H}^{n+1}-\hat{H}^n}{\Delta t}-A^{n+1}\hat{H}^{n+1}-A^n\hat{H}^n\Big)(\lambda^n)^t\\
&= \sum_{n=0}^{r-1}\hat{H}^{n+1}(\lambda^n)^t-\sum_{n=0}^{r-1}\hat{H}^{n}(\lambda^n)^t-\sum_{n=0}^{r-1}\Delta t\ A^{n+1}\hat{H}^{n+1}(\lambda^n)^t-\sum_{n=0}^{r-1}\Delta t\ A^{n}\hat{H}^{n}(\lambda^n)^t\\
&= \sum_{n=0}^{r-1}\hat{H}^{n+1}(\lambda^n)^t-\sum_{n=0}^{r-2}\hat{H}^{n+1}(\lambda^{n+1})^t-\hat{H}^0(\lambda^0)^t-\sum_{n=0}^{r-1}\Delta t\ \hat{H}^{n+1}\big((A^{n+1})^t\lambda^n\big)^t\\
&\quad -\sum_{n=0}^{r-2}\Delta t\ \hat{H}^{n+1}\big((A^{n+1})^t\lambda^{n+1}\big)^t-\Delta t\ \hat{H}^0\big((A^0)^t\lambda^0\big)^t\\
&= \sum_{n=0}^{r-2}\hat{H}^{n+1}\big(\lambda^n-\lambda^{n+1}-\Delta t(A^{n+1})^t\lambda^n-\Delta t(A^{n+1})^t\lambda^{n+1}\big)^t+\hat{H}^r\big(\lambda^{r-1}-\Delta t(A^r)^t\lambda^{r-1}\big)^t\\
&\quad -\hat{H}^0\big(\lambda^0+\Delta t(A^0)^t\lambda^0\big)^t. \qquad (18)
\end{aligned}$$

Identifying this equation with equation (15), we obtain the discrete adjoint model as follow :

$$\begin{cases}
\dfrac{\lambda^n-\lambda^{n+1}}{\Delta t}-(A^{n+1})^t\lambda^n-(A^{n+1})^t\lambda^{n+1}=H^{n+1}-H^{n+1}_{obs},\ n=0,\cdots,r-1 & (19)\\
\lambda^r=\hat{0}. & (20)
\end{cases}$$

Using equations (15), (18) and (19)-(20) we get :

$$\begin{aligned}
\hat{J}(V)[\delta V] &= \sum_{n=0}^{r-1} \Delta t\ \hat{H}^{n+1}(H^{n+1} - H_{obs}^{n+1})^t + \alpha(\delta V)V^t \\
&= \sum_{n=0}^{r-1} \Delta t\ \hat{H}^{n+1}\Big(\frac{\lambda^n - \lambda^{n+1}}{\Delta t} - \big(A^{n+1}\big)^t\lambda^n - \big(A^{n+1}\big)^t\lambda^{n+1}\Big)^t + \alpha(\delta V)V^t \\
&= \sum_{n=0}^{r-1} \hat{H}^{n+1}\Big(\lambda^n - \lambda^{n+1} - \Delta t\big(A^{n+1}\big)^t\lambda^n - \big(A^{n+1}\big)^t\lambda^{n+1} + \alpha(\delta V)V^t \\
&= \hat{H}^0\big(\lambda^0 + \Delta t\big(A^0\big)^t\lambda^0\big)^t + \alpha(\delta V)V^t \qquad (21) \\
&= \delta V\big(\lambda^0 + \Delta t\big(A^0\big)^t\lambda^0 + \alpha V\big)^t \\
&= \delta V\big(\nabla J(V)\big)^t.
\end{aligned}$$

We deduce :

$$\nabla J(V) = \lambda^0 + \Delta t\big(A^0\big)^t\lambda^0 + \alpha V. \qquad (22)$$

4 Numerical Result

To test our approach we realize twin experiments, considering the following data :

$h(0, x, y) = \frac{1}{\pi^2}(1 - x^2)(1 - y^2)$,

$m(t, x, y) = (1 - x^2y^2)\exp(-t)$,

$\phi(t, x, y) = \frac{1}{\pi^2}\big[- (1 - x^2)(1 - y^2)\exp(-t) + 2(2 - x^2 - y^2)(1 - 3x^2y^2)\exp(-2t)\big]$,

$\alpha = 1,\ T = 1.$

The observation data is constructed noising the approximate solution of the model (2)-(5).
We note by $E(t, \Delta t)$ the normalized error calculated on the $(N-1)\times(N-1)$ collocation inner points of Γ, given by :

$$\begin{aligned}
E(t, \Delta t) &= \|h_{opt}(t) - h_{ex}(t)\|_{L^2(\Gamma)}^N \\
&= \frac{1}{N-1}\Big(\sum_{i,j=1}^{N-1}\big(h^{opt}(t, x_i, y_j) - h^{ex}(t, x_i, y_j)\big)^2\Big)^{\frac{1}{2}}.
\end{aligned}$$

We pose $\Delta t = dt$.
The numerical results on the estimate of the optimal initial dune height for 361 collocation inner points of Γ, are given in the following table :

dt	5.10^{-3}	4.10^{-3}	3.10^{-3}	2.10^{-3}	10^{-3}
$\|h_0^{opt} - h_0\|_{L^2(\Gamma)}^N$	$2,514.10^{-4}$	$1,621.10^{-4}$	$1,595.10^{-4}$	$1,178.10^{-4}$	$7,64.10^{-5}$

Table : Normalized error on the estimate of the optimal initial dune height calculated on 361 collocation inner points ($N = 20$)

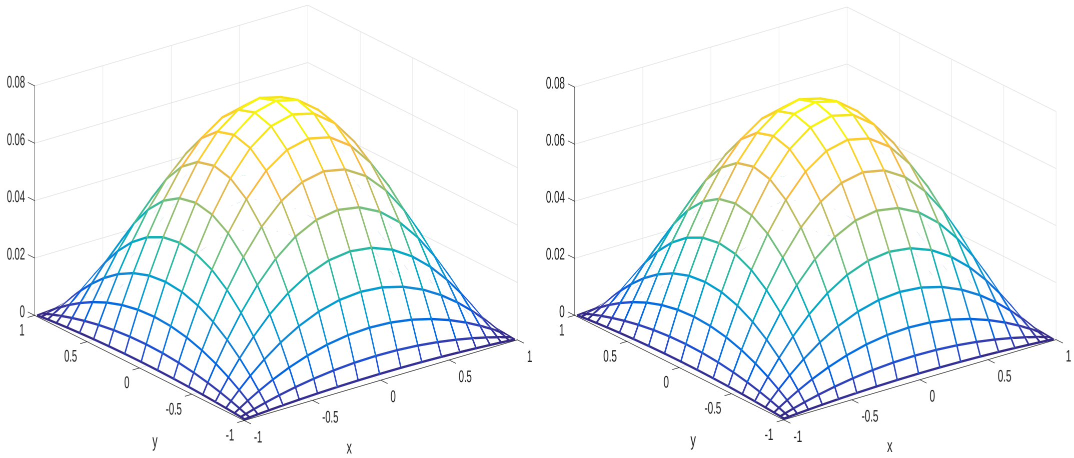

Figure 1: Profile of approached (on left) and optimal dune height (on right) calculated on 361 collocation inner points ($N = 20$) for a time step $\Delta t = 2.10^{-3}$ at $t = 0, 25$.

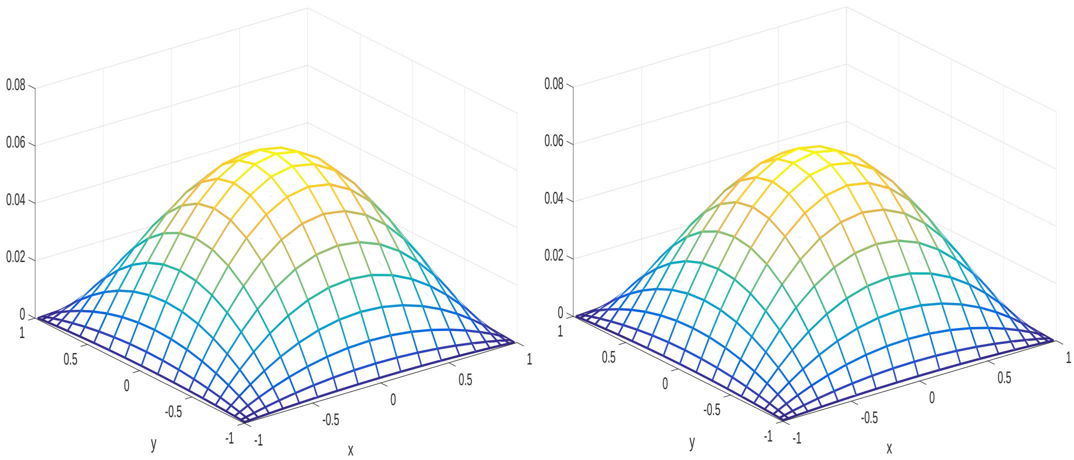

Figure 2: Profile of approached (on left) and optimal dune height (on right) calculated on 361 collocation inner points ($N = 20$) for a time step $\Delta t = 2.10^{-3}$ at $t = 0, 5$.

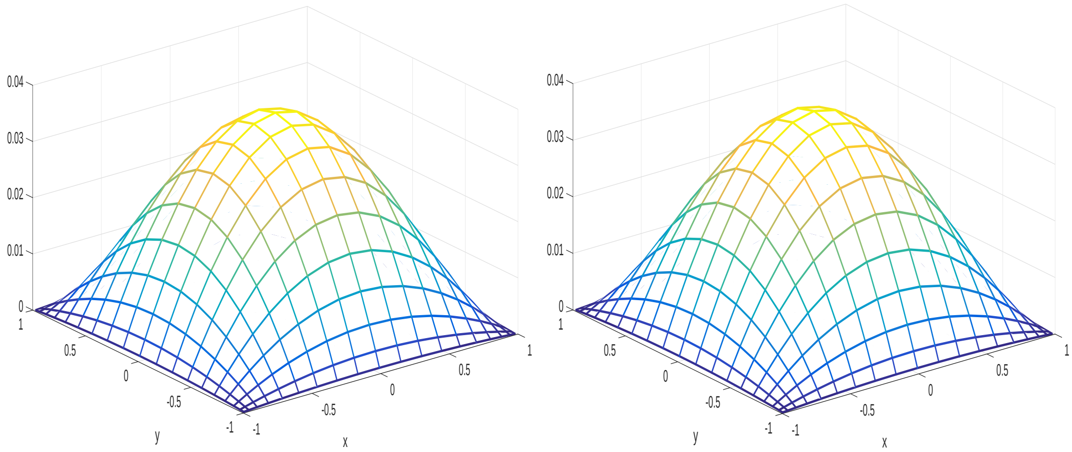

Figure 3: Profile of approached (on left) and optimal dune height (on right) calculated on 361 collocation inner points ($N = 20$) for a time step $\Delta t = 2.10^{-3}$ at $t = 1$.

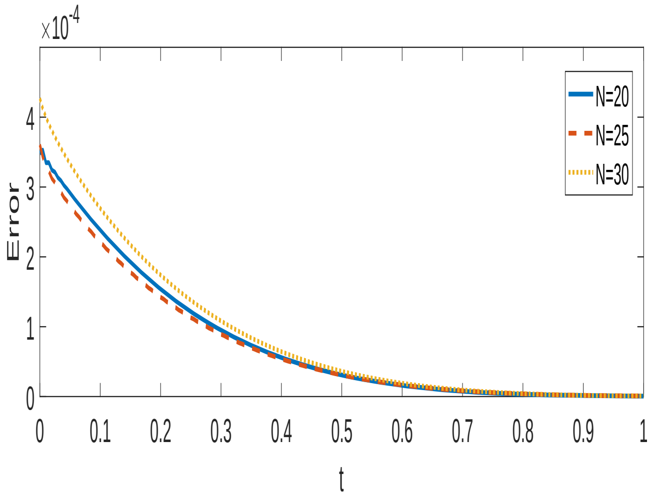

Figure 4: temporal evolution of the error in time over the optimal dune height calculated on 361 ($N = 20$), 576 ($N = 25$), 841 ($N = 30$) collocation inner points, for a time step $dt = 5.10^{-3}$.

The numerical results in this Table show that more the time step is smaller, more the estimate of the initial optimal dune height h_0^{opt} is better, favoring the dune formation. The figures 1-3 illustrates this result.
The figures 4 and 5 describe the temporal evolution of the error in time over the optimal dune height calculated on 361 ($N = 20$), 576 ($N = 25$), 841 ($N = 30$) collocation inner points, for a time step $dt = 5.10^{-3}$ (figure 4); and on 361 ($N = 20$) collocation inner points, for a time step $dt = 5.10^{-3}$, 3.10^{-3}, 2.10^{-3} (figure 5).
These numerical experiments show the efficiency and the effectiveness of our approach.

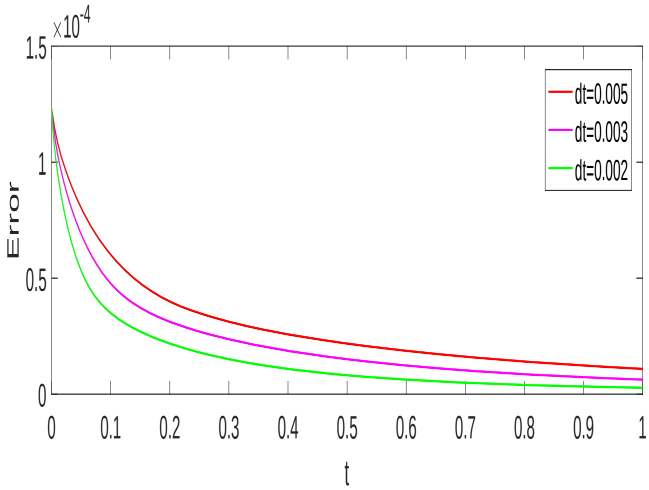

Figure 5: temporal evolution of the error in time over the optimal dune height calculated on 361 ($N = 20$) collocation inner points, for a time step $dt = 5.10^{-3},\ 3.10^{-3},\ 2.10^{-3}$.

5 Conclusion

We have presented an approach to estimate the optimal initial dune height which can favor the evolution of the dune in the depht of an incompressible flow in two space dimension. This is an approach that combines the discrete adjoint method and the quasi newton method. We have considered an optimal control problem governed by nonlinear equations describing the dune height formation. The control is done in a discrete framework, where the model solution gap to the data is minimized using a quasi-Newton method and The functional gradient is computed by the backward time integration of the corresponding discrete adjoint model. To discretize the constraints equations, we have used a combinaison of the Crank-Nicholson scheme and a Chebyshev spectral method $\mathbb{P}_{N,M}$ type. The twin experiments that we achived and the obtained numerical results ensure the efficiency and the effectiveness of our approach. In prospect, we count approach the dune displacement problem.

References

[1] Anne K. Griffith and Nichols N.K. : *Data assimilation using optimal control theory.* Numerical analysis report, The University of Reading, October 1994.

[2] Botella O. : *Résolution numérique de problèmes de Navier-Stokes singuliers par une méthode de projection Tchebychev*, thèse de doctorat, Université de Nice, 1998;

[3] D'adamo J., Papadakis N., Mémin E. et Artana G., : *Variationnal assimilation of POD Low-order dynamical Systems,*- Journal of Turbulence, Volume 8, 2007.

[4] El-Gindy T. M., El-Hawary H. M., Salim M. S., and El-Kady M. : *A Chebyshev Approximation for Solving Optimal Control Problems,* Comput. Mathemat. with Applic., Vol. 29, no. 6, pp. 35-45, 1995.

[5] EL JAI A. : *Quelques problèmes de contrôle propres aux systèmes distribués*, Annals of University of Craiova, Math. Comp. Sci. Ser. Volume 30, 2003, Pages 137-153.

[6] Edrisi-Tabriz Y. and Lakestani M. : *Direct solution of nonlinear constrained quadratic optimal control problems using B-spline functions,* Kybernetika, Vol. 51 (2015), no. 1, 81-98.

[7] Fahroo F. and Ross I. M. : *Direct trajectory optimization by a Chebyshev pseudospectral method,* J. Guid., Control, and Dynam., Vol. 25, no. 1, pp 160-166, 2002.

[8] Fang F., Pain C.C., Piggott M.D., Gorman G.J., and Goddard A.J.H : *An adaptive mesh adjoint data assimilation method applied to free surface flows.* International Journal for Numerical Methods in Fluids, 47:995–1001, 2005.

[9] Hesthaven J. S., Gottlieb S., Gottlieb D. : *Spectral Methods for Time-Dependent Problems*, Cambridge University Press 2007.

[10] Igbida N. : *Mathematical Models for Sandpile Problems,* XLIM-DMI, UMR-CNRS 6172, Workshop MathEnv, Essaouira, 2012.

[11] Jie S. and Tao T. : *Spectral and High-Order Methods with Applications,* Science Press, 2006.

[12] Lancaster P. : *Theory of Matrices,* Academic Press, New York, 1969.

[13] Maëlle N. : *assimilation of lagarangian data in oceanography,* Inverse Problems, 22(1),2006.

[14] Mahaman Nouri Y. A., Bisso S. and Ali A. M. : *Numerical Approach of an Optimal Control Problem for Sand dune formation in Aquatic Environment.* Journal of Computations Modelling, Vol.7, no.4, 2017, 21-38.

[15] Ma H., Qin T., Zhang W. : *An Efficient Chebyshev Algorithm for the Solution of Optimal Control Problems,* IEEE Trans. Autom. Control, Vol. 56, no. 3, pp. 675-680, March 2011.

[16] Nouri Y. M., and Bisso S. : *Numerical Modelling and Simulation of Sand Dune formation in an Incompressible Out-Flow.* Scientific Research Publishing, Applied Mathematics, Volume 6, p. 864-876, 2015.

[17] Prigozhin L. : *Sandpiles and river networks : Extended systems with nonlocal interactions.* Physical Review E, V49, Number 2, 1994.

[18] Trefethen L. N. : *Spectral Methods in MATLAB*, Siam book, 2000.

[19] van Bloemen Waanders B. G., Bartlett R. A., Collis S. S., Keiter E. R., Ober C. C., Smith T. M., Akcelik V., Ghattas O., Hill J. C., Berggren M., Heinkenschloss M., and Wilcox L. C. : *Sensitivity technologies for large scale simulation.* Technical Report SAND2004-6574, Sandia National Laboratories, January 2005.

[20] Vlassenbroeck J. and Dooren R. V. : *A Chebyshev technique for solving nonlinear optimal control problems,* IEEE Trans. Autom. Control, Vol. 33, no. 4, pp. 333-340, Apr. 1988.

[21] Zhang W. and Ma H. P. : *Chebyshev-Legendre method for discretizing Optimal Control Problems,* J. Shanghai Univ. (English Version), Vol. 13, no. 2, pp. 113-118, 2009.