Incremental Sharpe and other performance ratios

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Abstract

We present a new methodology of computing incremental contribution for performance ratios for portfolio like Sharpe, Treynor, Calmar or Sterling ratios. Using Euler's homogeneous function theorem, we are able to decompose these performance ratios as a linear combination of individual modified performance ratios. This allows understanding the drivers of these performance ratios as well as deriving a condition for a new asset to provide incremental performance for the portfolio. We provide various numerical examples of this performance ratio decomposition.

JEL classification: C12, G11.

Keywords: Sharpe, Treynor, recovery, incremental Sharpe ratio, portfolio diversification

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1.   Introduction

When facing choices to invest in various funds (whether mutual or hedge funds), it is quite common to compare their Sharpe ratio, or other performance ratios like Treynor or recovery ratio in order to rank funds. These ratios aim at measuring performance for a given risk. They achieve two important things: they measure performance taking into account risk. They allow constructing the optimal performance as the result of an optimization program.

The usual performance metric is the eponymous Sharpe ratio established in Sharpe (1966). It is a simple number easy to derive and intuitive to understand as it computes the ratio of the excess return over the strategy standard deviation. It has various limitations that have been widely emphasized by various authors (Pilotte and Sterbenz (2006), Sharpe (1998), Nielsen and Vassalou (2004)) leading to other performance ratios like Treynor ratio (see Treynor and Black (1973)), but also Calmar (see Young (1991)), Sterling (see McCafferty (2003)) or Burke rqtio (see Burke (1994)). Other authors have also tried to provide additional constraints to the Sharpe as in Bertrand (2008) or more recently in Darolle, Gourieroux, and Jay (2012) or to use option implied volatility and skewness as in DeMiguel, Plyakha, Uppal, and Vilkov (2013). There have been also numerous empirical work on Sharpe ratio as for the most recent ones in Giannotti and Mattarocci (2013), Anderson, Bianchi, and Goldberg (2014)

An important feature that has been noted in Darolle et al. (2012) or Steiner (2011) but only for the Sharpe ratio is the fact that most of the performance ratios are so called 0 Euler homogeneous with respect to the portfolio weights. In financial terms, there are not sensitive to the leverage of the portfolio.

The contribution of our paper is to exploit this mathematical property and re-derive well known results on the Sharpe ratio in a new manner. As a consequence, we obtain the condition for a new asset to increase the overall Sharpe of a portfolio. We also extend the incremental performance marginal sensitivity to all performance ratios that are 0 Euler homogeneous with respect to the portfolio weights. This allows in particular to understand the performance ratios drivers. We finally show how to decompose performance ratios between a benchmark and the individual portfolio constituents.

2. Euler homogeneous functions and its application to performance ratios

2.1. Euler's theorem

In mathematics, one call a homogeneous function one that has a multiplicative scaling behaviour. If we multiply all its arguments by a constant factor, then its value is multiplied by some power of this factor. If we denote by $f\colon \mathbb{R}^n \to \mathbb{R}$ a multidimensional function from $\mathbb{R}^n$ to $\mathbb{R}$, then the function f is said to be homogeneous of degree k if the following holds:

$$f(\alpha v) = \alpha^k f(v) \tag{1}$$

for all positive $\alpha > 0$ and $v \in \mathbb{R}^n$. If the function is continuously differentiable (and this generalized also to almost surely continuously differentiable function), the Euler's homogeneous function theorem [1] states that the function is homogeneous if and only if

$$x \cdot \nabla f(x) = k f(x) \tag{2}$$

where $\nabla f(x)$ stands for the gradient of f. This theorem (shown in various book like for instance Lewis (1969)) gives in particular a nice decomposition of any homogeneous function provided we can compute the gradient function as it says that the function is a linear combination of partial derivatives as follows:

$$f(x) = \frac{1}{k} \sum_{i=1..n} x_i \frac{\partial f}{\partial x_i}$$

2.2. Intuition with Sharpe ratio

Let us see how this can be applied to any homogeneous performance ratio. In order to build our intuition, we will start by the Sharpe ratio as this is a simple and well know ratio. We assume we have a portfolio of n assets with weights w_i. We denote by R_f the risk free rate and R_p the portfolio return. The Sharpe ratio is defined as the fraction of the portfolio excess return r_p over the portfolio volatility σ_p and given by

$$S_p = \frac{R_p - R_f}{\sigma_p} = \frac{r_p}{\sigma_p} \tag{3}$$

If we decompose the portfolio excess return r_p as the convex combination of its assets ex-

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[1]This theorem is trivially proved by differentiating $f(\alpha v) = \alpha^k f(v)$ with respect to α for the implication condition and by integrating the differential equation $x \cdot \nabla f(x) = k f(x)$ for the reverse condition

cess returns with percentage weights w_i, we get that the Sharpe ratio is a convex combination of the modified Sharpe

$$S_p = \frac{\sum_{i=1}^n w_i r_i}{\sigma_p} = \sum_{i=1}^n w_i \frac{r_i}{\sigma_p} \tag{4}$$

This says that if we were looking at a modified Sharpe ratio of each portfolio constituent where the volatility of the constituent is modified into the one of the portfolio, then the Sharpe ratio of the portfolio is simply the convex combination of these modified Sharpe ratio. This is nice from a theoretical point of view but not very useful as this forces us to compute the volatility of the portfolio and does not give any hindsight about asset i volatility. This is where Euler homogeneous formula comes at the rescue. The Sharpe ratio like many other performance ratio has the particularity that it is the fraction of two homogeneous function of degree 1. The decomposition for the excess return in terms of a linear combination of the portfolio weight is obvious.

More subtle is the fact that the volatility of the portfolio can also be decomposed as the convex sum of individual volatility contributions. Indeed, the volatility is an homogeneous function of degree 1 of the portfolio weights as scaling the weights by a factor increases the portfolio by the same factor. In the sequel, we will denote for asset i, $\rho_{i,p}$ its correlation with portfolio P, σ_i its volatility and S_i its Sharpe ratio. Thanks to Euler's homogeneous function theorem, we know that the portfolio volatility can be written as follows

Proposition 1. *The weighted marginal contributions to volatility sum up to portfolio volatility as follows:*

$$\sigma_p = \sum_{i=1}^n w_i \frac{\partial \sigma_p}{\partial w_i} = \sum_{i=1}^n w_i \rho_{i,p} \sigma_i \tag{5}$$

Proof. trivial consequence of Euler's homogeneous function theorem and given in A.0.1. □

This results was first derived in Maillard, Roncalli, and Teiletche (2010) but in the case of equally weighted portfolio and was also noted in Darolle et al. (2012). Using this first property, it is now easy to derive a convex combination for the portfolio Sharpe ratio as follows.

Proposition 2. *The portfolio Sharpe ratio is a convex combination of individual Sharpe ratios weighted by the inverse of the asset i correlation with portfolio P, $\rho_{i,p}$:*

$$S_p = \sum_{i=1}^n \theta_i \frac{1}{\rho_{i,p}} S_i \tag{6}$$

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The risk weights $(\theta_i)_{i=1..n}$ sums to one and are given by

$$\theta_i = \frac{w_i \rho_{i,p} \sigma_i}{\sigma_p} \tag{7}$$

The coefficient $1/\rho_{i,p}$ measures the diversification effect. It increases Sharpe ratio for low correlation.

Proof. given in A.0.2. $\square$

As a byproduct, we get the condition for a new asset to improve the overall portfolio Sharpe summarized below

Proposition 3. *It is optimal to include an asset i in a portfolio if and only if*

$$S_i \geq \rho_{i,p} S_p \tag{8}$$

Proof. See A.0.3. $\square$

This result is complementary from the standard mean variance approach as presented in Lobo, Fazel, and Boyd (2007) and generalized in Goto and Xu (2015), which investigates about the optimal weights in a mean variance framework and states that the optimal weights are the result of a normal equation.

2.3. General case

A large number of performance ratios like Sharpe, Treynor, Sortino, Calmar, Sterling, information ratios or M2 write as the fraction of an excess return or a return or a return over a benchmark over some risk measure. The numerator and the denominator are homogeneous functions of degree 1. This leads to a performance ratio that is an homogeneous function of degree 0. In financial terms, the performance ratio is insensitive to leverage.

We will write therefore any portfolio leverage-insensitive ratio $PR(p)$ as the fraction of an portfolio homogeneous return R_p over an homogeneous function of degree 1, $f(p)$ as follows:

$$PR(p) = \frac{R_p}{f(p)} \tag{9}$$

We will also denote for the asset i by $f(i)$ its denominator function and $PR(i)$ its corespond-ing performance ratio. Since the general return is an homogeneous functions of degree 1, it

can be written as the convex combination of individual asset general returns:

$$R_p = \sum_{i=1}^{n} w_i R_i \tag{10}$$

Since the denominator is an homogeneous functions of degree 1, it can be written as a convex combination of individual asset contribution thanks to the Euler's homogeneous function theorem:

$$f(p) = \sum_{i=1}^{n} w_i \frac{\partial f}{\partial w_i} \tag{11}$$

Combining equations 10 and 11 leads to a decomposition of the leverage-insensitive ratio PR into individual or incremental performance ratio for each asset i

$$PR(p) = \sum_{i=1}^{n} \frac{w_i \frac{\partial f}{\partial w_i}}{f(p)} \times \frac{f(i)}{\frac{\partial f}{\partial w_i}} \times \frac{R_i}{f(i)} = \sum_{i=1}^{n} \theta_i \times D_i \times PR(i) \tag{12}$$

The risk factor θ_i and the diversification factor D_i are given respectively by

$$\theta_i = \frac{w_i \frac{\partial f}{\partial w_i}}{f(p)}, \quad D_i = \frac{f(i)}{\frac{\partial f}{\partial w_i}} \tag{13}$$

As in the case of the Sharpe, it is then easy to derive a condition for a new asset to improve the overall portfolio performance ratio summarized below

Proposition 4. *It is optimal to include an asset i in a portfolio in order to maximize the performance ratio PR(p) if and only if*

$$PR(i) \geq \frac{PR(p)}{D_i} \Leftrightarrow PR(i) \geq \frac{\frac{\partial f}{\partial w_i}}{f(i)} \times PR(p) \tag{14}$$

Proof. See A.0.4. □

The real work at this stage is to compute the derivative function of the denominator with respect to its weight $\frac{\partial f}{\partial w_i}$. We provide results for various performance ratios in the table below

performance ratio	definition $= \frac{R_p}{f(p)}$	performance marginal sensitivity $= \frac{\partial f}{\partial w_i}$	diversification factor$= f(i)/\frac{\partial f}{\partial w_i}$
Sharpe	$S_p = \frac{r_p - r_f}{\sigma_p}$	$\rho_{i,p}\sigma_i$	$1/\rho_{i,p}$
Sortino	$Sor_p = \frac{r_p - r_f}{TSD_p}$	$\rho_{i,p}TSD_i$	$1/\rho_{i,p}$
Information	$IR_p = \frac{r_p - r_b}{\sigma_{p-b}}$	$\rho_{i,p-b}\sigma_i$	$1/\rho_{i,p-b} \times \sigma_{i-b}/\sigma_i$
Treynor	$T_p = \frac{r_p - r_f}{\beta_p}$	β_i	1
Recovery	$Rec_p = \frac{r_p - r_f}{MDD_p}$	$\widetilde{MDD}_i$	$MDD_i/\widetilde{MDD}_i$
Calmar	$Cal_p = \frac{r_p - r_f}{MDD_p^{36m}}$	$\widetilde{MDD}_i^{36m}$	$MDD_i^{36m}/\widetilde{MDD}_i^{36m}$
Sterling	$Ster_p = \frac{r_p - r_f}{ALD_p}$	$\widetilde{ALD}_i$	$ALD_i/\widetilde{ALD}_i$

Table 1: We provide here above the results for the most common performance ratios

In table 1, we have used the following notations:

- TSD stands for target semi deviation(standard deviation of return below target).
- r_b is the benchmark return.
- σ_{p-b} is the standard deviation of the difference between the portfolio and benchmark returns.
- MDD (respectively MDD^{36m}, ALD) stands for the maximum drawdown, the maximum drawdown over 36 months and the annual average maximum drawdown over the entire historical period.

Proof. See A.0.5. □

3. Numerical application

3.1. Sharpe ratio

Let us apply the above formulas to a portfolio consisting of three assets with the characteristics described in table 2. The portfolio weights are the optimal ones in terms of the highest Sharpe ratio with the constraints of weights between 0 to 100% (no short selling allowed neither extra leverage). We also provide in the characteristics the correlation between the asset i and the portfolio as this is useful for risk decomposition.

Asset	I	II	III	Total
Weight	34.87%	28.07%	37.06%	100.00%
Expected Return	3.20%	3.50%	4.50%	
Volatility	4.87%	5.63%	5.12%	
Correlation with portfolio	49.87%	37.55%	65.87%	

Table 2: Portfolio characteristics

Once the characteristics established, we can easily compute the portfolio performance ratio as provided in table 3. We compute the portfolio return as the convex combination of the assets returns as well as the portfolio volatility. For the latter, we use the volatility reconstruction formula 1. The portfolio Sharpe is then the fraction of the latter two. We can notice that the resulting portfolio Sharpe (1.4000) is substantially higher than the best asset Sharpe (0.8789). We are benefiting fully from the diversification effect.

	Portfolio
Expected Return	3.77%
Volatility	2.69%
Sharpe Ratio	1.4000

Table 3: Portfolio resulting Sharpe ratio: all these numbers are computed from table 2

The table 4 gives us a nice view of the Portfolio Sharpe decomposition. The asset III has the highest Sharpe ratio (0.8789) but the lowest Sharpe diversification. This results in particular in a the highest risk (46.46%), which is a strong indication that asset III contributes more to the overall portfolio Sharpe ratio. Its risk weight (46.46%) is higher than its portfolio weight (37.06%) indicating that the Sharpe contribution will be over-weighted. In contrast, risk weights for asset I and II (31.48% and 22.06%) are smaller than their corresponding portfolio weights (34.87% and 28.07%). They will contribute less and will be under-weighted in the overall portfolio Sharpe ratio. Thanks to the strong asset III contribution and the diversification, as noted above, the overall portfolio achieved a significant increase in its Sharpe ratio (1.4000). In table 4, the Sharpe ratio relative contribution is defined as the Sharpe ratio contribution divided by the portfolio Sharpe ratio. It sums to 100 %.

Asset	I	II	III	Total
Asset Sharpe Ratio	0.6571	0.6217	0.8789	
Sharpe Diversification	2.0054	2.6632	1.5182	
Component Sharpe Ratio	1.3177	1.6557	1.3344	
Risk Weight	31.48%	22.06%	46.46%	100.00%
Sharpe Ratio Contribution	0.4148	0.3652	0.6200	1.4000
Sharpe Ratio Relative Contribution	29.63%	26.09%	44.29%	100.00 %

Table 4: Portfolio Sharpe decomposition: all these numbers are computed from table 2

3.2. Recovery ratio

Recovery ratio is an important performance ratio in the funds' world as it provides the expected return divided by the maximum drawdown. Maximum drawdown is closely monitored by professional investors as it gives an hint about the maximum potential loss should they invest and dis-invest at the worst time. For the sake of comparison with the previous study in section 3.1, we will first start with the same portfolio with the same percentage weights. For each asset, we provide in table 5 its maximum drawdown as well its performance marginal sensitivity (whose formula is $\frac{\partial f}{\partial w_i}$) as provided in 1.

Asset	I	II	III	Total
Weight	34.87%	28.07%	37.06%	100.00%
Asset MDD	5.71%	6.34%	4.53%	
Performance marginal sensitivity	3.14%	1.90%	4.07%	

Table 5: Portfolio characteristics for recovery ratio

Like for the Sharpe ratio, we can compute the portfolio resulting characteristics in table 6. Expected return is like before computed as the convex combination of the asset returns (and is the same as in table 3). The recovery ratio is then simply the fraction of the latter two.

	Portfolio
Expected Return	3.77%
Portfolio Drawdown	3.14%
Portfolio recovery ratio	1.1999

Table 6: Portfolio resulting recovery ratio

More interestingly is to analyze portfolio recovery decomposition as provided in table 6. Again, thanks to portfolio diversification, we achieve a higher performance ratio (recovery of 1.1999) compared to the highest asset performance ratio (obtained for asset III 0.9939). Like for the Sharpe ratio, the risk weight of asset III (48.12%) is over-weighted compared to its portfolio weight (37.06%). The opposite situation arises for asset II (risk weight of 17.02% compared to a 28.07%). By complete chance, risk and portfolio weight for asset I are equal up to the fourth decimal (risk weight of 34.865% compared to portfolio weight of 34.872%). As in the case of the Sharpe ratio, we can check that the sum of the risk weights are equal to 100%. In table 10, the recovery ratio relative contribution is defined as the recovery ratio contribution divided by the portfolio recovery ratio. It sums to 100 %.

Asset	I	II	III	Total
Asset Recovery ratio	0.5609	0.5518	0.9939	
Recovery Diversification	1.8182	3.3333	1.1111	
Component Recovery	1.0198	1.8393	1.1043	
Risk Weight	34.87%	17.02%	48.12%	100.00%
Recovery Contribution	0.3556	0.3130	0.5314	1.1999
Recovery Relative Contribution	29.63%	26.08%	44.28%	100.00%

Table 7: Portfolio recovery decomposition

A natural question that arises when looking at the recovery ratio for the portfolio is to determine if the optimal weights for the Sharpe ratio are also optimal for the recovery ratio. The answer is no in general. Recovery ratio is substantially different from Sharpe ratio. Hence the optimal portfolio for the recovery ratio has no reason to have the same weights as for the optimal portfolio for the Sharpe ratio. Because the recovery ratio implies a non convex function, namely the maximum drawdown, there is no closed form solution for the optimal portfolio as opposed to the Sharpe ratio settings. Using the CRG (that stands for Generalized Reduced Gradient) method (as presented in Lasdon, Fox, and Ratner (1974)), we can determine the optimal weights for this portfolio as provided in table 8. For this new portfolio, Asset performance marginal sensitivity changes slightly as the portfolio drawdown times are different.

Asset	I	II	III	Total
Weight	5.99%	24.78%	69.24%	
Asset MDD	5.71%	6.34%	4.53%	
Asset performance marginal sensitivity	3.74%	1.95%	3.49%	

Table 8: Optimal Portfolio characteristics for maximum drawdown

We can recompute the new portfolio characteristics as provided in table 9. We achieve a substantially higher portfolio recovery ratio (1.3379 versus 1.1999). This is due both to a higher expected return (4.17% versus 3.77%) and a lower portfolio maximum drawdown (3.12% versus 3.14%).

	Portfolio
Expected Return	4.17%
Portfolio Drawdown	3.12%
Portfolio recovery ratio	1.3379

Table 9: Optimal Portfolio resulting recovery ratio

As for previous studies, we can look at maximum drawdown decomposition as provided in table 10. Compared to the previous portfolio with same weights as the optimal ones for the Sharpe ratio, the risk weight for asset III increases even more (77.38% versus 48.12%). This is quite logical as this optimal portfolio for the maximum drawdown is indeed very much geared towards asset III (asset weight of 69.24% versus 37.06%). Interestingly, thanks to diversification, the recovery contribution for asset III (0.9986) is even higher to the asset recovery ratio (0.9939).

Asset	I	II	III	Total
Asset Recovery ratio	0.5609	0.5518	0.9939	
Recovery Diversification	1.5268	3.2606	1.2984	
Component Recovery	0.8563	1.7991	1.2905	
Risk Weight	7.17%	15.45%	77.38%	100.00%
Recovery Contribution	0.0614	0.2779	0.9986	1.3379

Table 10: Optimal Portfolio maximum drawdown decomposition

4.   Concluding Remarks

We have introduced in this paper a unified framework for deriving asset contribution for performance ratios that are homogeneous function. This allows us finding easily previous results on incremental Sharpe ratio contribution of a new asset as well as extend this to new performance ratios like Sortino, Information, Treynor, Recovery, Calmar or Sterling ratios where this did not exist. We also compare the impact of a new asset to a portfolio performance thanks to these incremental performance marginal sensitivity and show a methodology to analyse asset contribution to a portfolio.

Appendix A.   Various Proofs

A.0.1.   Proof of Proposition 1

Denoting by $\rho_{i,j}$ the correlation between asset i and j, we can decompose the porfolio variance as a combination of assets' volatility as follows:

$$\sigma_p^2 = \sum_{i=1...n} w_i^2 \sigma_i^2 + 2 \sum_{i,j=1...n, i\neq j} w_i w_j \rho_{i,j} \sigma_i \sigma_j \tag{15}$$

Differentiating the above equation 15 with respect to w_i, we have

$$2\sigma_p \frac{\partial \sigma_p}{\partial w_i} = 2 w_i \sigma_i^2 + 2 \sum_{j=1...n, j\neq i} w_j \rho_{i,j} \sigma_i \sigma_j \tag{16}$$

We can notice that the correlation between asset i and the portfolio p is given by

$$\rho_{i,p} = \frac{w_i \sigma_i^2 + \sum_{j=1...n, j\neq i} w_j \rho_{i,j} \sigma_i \sigma_j}{\sigma_i \sigma_p} \tag{17}$$

which shows that

$$\frac{\partial \sigma_p}{\partial w_i} = \rho_{i,p} \sigma_i \tag{18}$$

Since the portfolio volatility is homogeneous of degree 1, the Euler's homogeneous function theorem states that

$$\sigma_p = \sum_{i=1...n} w_i \frac{\partial \sigma_p}{w_i} = \sum_{i=1...n} w_i \rho_{i,p} \sigma_i \tag{19}$$

$\square$

A.0.2.   Proof of Proposition 2

Dividing and multiplying by $\rho_{i,p}\sigma_i$ in the formula of the portfolio Sharpe ratio and regrouping the terms leads to the final results as follows:

$$S_p = \sum_{i=1}^{n} w_i \frac{r_i}{\sigma_p} = \sum_{i=1}^{n} \frac{w_i \rho_{i,p}\sigma_i}{\sigma_p} \frac{1}{\rho_{i,p}} \frac{r_i}{\sigma_i} = \sum_{i=1}^{n} \theta_i \frac{1}{\rho_{i,p}} S_i \tag{20}$$

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□

A.0.3. Proof of Proposition 3

Let us denote by $P(w_i, i = 1..n)$ the portfolio composed of n assets with percentage weights w_i and n the new asset. The portfolio percentage weights sum to 1: $\sum_{i=1..n} w_i = 1$. The optimization program writes as follows:

$$\text{maximize} \quad \text{Sharpe Ratio}(P(w_i, i = 1..n)), \tag{21}$$

$$\text{subject to} \quad \sum_{i=1..n} w_i = 1 \tag{22}$$

Using proposition 2 and multiplying and dividing by $1 - \theta_n$ (with the additional constraint that $\theta_n \neq 1$ [2]), the optimal solution is also the solution of this program

$$\text{maximize} \quad (1 - \theta_n) \sum_{i=1..n-1} \frac{\theta_i}{1 - \theta_n} \frac{1}{\rho_{i,P}} S_i + \theta_n \frac{1}{\rho_{n,P}} S_n \tag{23}$$

$$\text{subject to} \quad \sum_{i=1..n} \theta_i = 1, \quad \theta_n \neq 1 \tag{24}$$

Fixing θ_n and noticing that the weights $\frac{\theta_i}{1-\theta_n}$ for $i = 1..n-1$ sum to 1, the optimization program is indeed a two steps program where we can optimize first in terms of the weights $\frac{\theta_i}{1-\theta_n}$ and then in terms of θ_n. As the $n-1$ terms are indeed the percentage weights of an $n-1$ portfolio composed of $n-1$ assets, the first optimization is exactly the same as the optimization of the optimal portfolio with $n-1$ assets in terms of its Sharpe ratio. The first step therefore leads to the optimal portfolio without asset n for the Sharpe ratio. We will denote this portfolio by $\widetilde{P}$. The maximization program is then equivalent to

$$\text{maximize} \quad (1 - \theta_n) S_{\widetilde{P}} + \theta_n \frac{1}{\rho_{n,P}} S_n \tag{25}$$

$$\text{subject to} \quad 0 \leq \theta_n \leq 1 \tag{26}$$

This optimization program is a linear function whose optimal solution θ_n is not equal to zero if and only if the slope coefficient is positive (which proves the result):

$$\frac{1}{\rho_{n,P}} S_n - S_{\widetilde{P}} \geq 0 \Leftrightarrow S_n \geq \rho_{n,P} S_{\widetilde{P}} \tag{27}$$

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[2] the particular case of $\theta_n = 1$ can be handled easily by taking the left limit for $\theta_n \uparrow 1$

□

A.0.4. Proof of Proposition 4

The proof is exactly the same as in A.0.3 and leads at the end to solve the following linear maximization program where like in A.0.3 , we denote by $\widetilde{P}$ the optimal portfolio with $n-1$ assets in terms of the performance ratio.

$$\text{maximize} \quad (1-\theta_n)PR(\widetilde{P}) + \theta_n D_n PR(n) \tag{28}$$

$$\text{subject to} \quad 0 \leq \theta_n \leq 1 \tag{29}$$

This optimization program is a linear function whose optimal solution θ_n is not equal to zero if and only if the slope coefficient is positive (which proves the result):

$$D_n PR(n) - PR(\widetilde{P}) \geq 0 \Leftrightarrow \quad PR(n) \geq \tfrac{PR(\widetilde{P})}{D_n} \tag{30}$$

□

A.0.5. Proof of Table 1 results

Results for Sharpe are already proved in A.0.1.

Denoting by r_M the return of the market asset, the beta in the Treynor ratio is given by

$$\beta_p = \frac{Cov(r_p, r_M)}{\sigma_M^2} = \frac{Cov(\sum_{i=1}^n w_i r_i, r_M)}{\sigma_M^2} = \sum_{i=1}^n \frac{Cov(r_i, r_M)}{\sigma_M^2} \tag{31}$$

A straight derivation leads to the results $\frac{\partial f}{\partial w_i} = \beta_i$

The target semi deviation is quite similar to the standard deviation with the additional constraint that we only use returns that are below its mean. The proof is therefore similar to the one of the portfolio in the Sharpe ratio with the additional constraint to use only down returns, leading to a target standard deviation for the assets' performance marginal sensitivity:

$$\frac{\partial f}{\partial w_i} = \rho_{i,p} TSD_i \tag{32}$$

The proof for the recovery, Calmar and Sterling ratio are similar and we will detail only the first one as the other derivation are just an averaging or windowing of the first proof. The

maximum drawdown measures the largest peak-to-trough decline in the value of a portfolio. Denoting by 0 to T the historical times at which we observe the portfolio return and by cr_p^j the cumulative return of the portfolio from time 0 to j with the convention that the return at time 0 is null, the maximum drawdown can be written mathematically as the maximum of the following discrete optimization program

$$\text{MDD} \equiv \min_{j=0..T,\ k=j..T} \frac{1 + cr_p^k}{1 + cr_p^j} - 1 \tag{33}$$

The portfolio contains n assets with percentage weights w_i. Assuming no re-balancing, the portfolio cumulative returns writes as the convex combination of the cumulative asset i return, leading to the following definition of the maximum drawdown

$$\text{MDD} = \min_{j=0..T,\ k=j..T} \frac{\sum_{i=1}^n w_i(1 + cr_i^k)}{\sum_{i=1}^n w_i(1 + cr_i^j)} - 1 \tag{34}$$

As this is a discrete optimization program, the optimum is attained for j_* and k_*. Deriving the maximum drawdown leads therefore to

$$\frac{\partial}{\partial w_i}\text{MDD} = \left(\frac{1 + cr_i^{k_*}}{1 + cr_p^{j_*}} - 1\right) - \left(\frac{\frac{1 + cr_i^{j_*}}{1 + cr_p^{j_*}} \times (1 + cr_p^{k_*})}{1 + cr_p^{j_*}} - 1\right) \equiv \widetilde{MDD}_i \tag{35}$$

Hence the sensitivity of the maximum drawdown is given by the difference of

- the drawdown between between the portfolio cumulative return at time j_* and the asset at time k_*
- the drawdown between between the portfolio cumulative return at time j_* and the portfolio return at time k_* augmented by the difference of cumulative return between the asset i and portfolio at time j_*

For the Calmar and Sterling ratio, similar formulae exist where the sensitivity of the maximum drawdown is taken over 36 months, respectively as the annual average. □

A.0.6. Correlation matrix for the three assets

For the sake of completeness, we provide below the correlation matrix. This matrix is consistent with asset correlation with portfolio coefficients.

Asset	I	II	III
I	1.00	- 0.20	0.40
II	- 0.20	1.00	0.30
III	0.40	0.30	1.00

Table 11: Asset correlation matrix

From this correlation matrix denoted by Σ and for asset i with the corresponding Kronecker delta vector defined by $\delta_i = (0...1...0)^T$ with one at the ith row and zero elsewhere, it is then straightforward to compute for any asset its correlation with portfolio (whose weight vector is defined as $W = (w_1, ..w_j, .., w_n)^T$ as follows:

$$\rho_{i,p} = \frac{\delta_i \Sigma W}{\sqrt{\delta_i \Sigma \delta_i} \sqrt{W \Sigma W}} \tag{36}$$

References

Anderson, R. M., Bianchi, S. W., Goldberg, L. R., 2014. Determinants of levered portfolio performance. Financial Analysts Journal 70, No. 5, 53–72.

Bertrand, P., 2008. Risk-adjusted performance attribution and portfolio optimizations under tracking-error constraints. GREQAM Discussion Paper .

Burke, G., 1994. A sharper sharpe ratio. The Computerized Trader .

Darolle, S., Gourieroux, C., Jay, E., 2012. Robust portfolio allocation with systematic risk contribution restrictions. ssrn .

DeMiguel, V., Plyakha, Y., Uppal, R., Vilkov, G., 2013. Improving portfolio selection using option-implied volatility and skewness. Journal of Financial and Quantitative Analysis 48, No. 6, 1813–1845.

Giannotti, C., Mattarocci, G., 2013. Risk measurement choice in selecting reits: Evidence from the u.s. market. Journal of Real Estate Portfolio Management 19, No. 2, 137–154.

Goto, S., Xu, Y., 2015. Improving mean variance optimization through sparse hedging restrictions. Journal of Financial and Quantitative Analysis (JFQA) 50, Issue 6, 1415–1441.

Lasdon, L. S., Fox, R. L., Ratner, M. W., 1974. Nonlinear optimization using the generalized reduced gradient method. RAIRO - Operations Research 8, 73–103.

Lewis, J., 1969. Homogeneous Functions and Euler's Theorem Chapter XXII. Palgrave Macmillan, London.

Lobo, M., Fazel, M., Boyd, S., 2007. Portfolio optimization with linear and fixed transaction costs. Annals of Operations Research, special issue on financial optimization 152(1), 341–365.

Maillard, S., Roncalli, T., Teiletche, J., 2010. The properties of equally weighted risk contribution portfolios. The Journal of Portfolio Management 36, 60–70.

McCafferty, T., 2003. The Market is Always Right. McGraw Hill.

Nielsen, L. T., Vassalou, M., 2004. Sharpe ratios and alphas in continuous time. Journal of Financial and Quantitative Analysis 39, No. 1, 103–114.

Pilotte, E. A., Sterbenz, F. P., 2006. Sharpe and treynor ratios on treasury bonds. Journal of Business 79, 149–180.

Sharpe, W. F., 1966. Mutual fund performance. Journal of Business pp. 119–138.

Sharpe, W. F., 1998. Morningstar's risk-adjusted ratings. Financial Analysts Journal 54, 21–33.

Steiner, A., 2011. Sharpe ratio contribution and attribution analysis. ssrn .

Treynor, J. L., Black, F., 1973. How to use security analysis to improve portfolio selection. Journal of Business pp. 66–85.

Young, T., 1991. Calmar ratio: A smoother tool. Futures 20, Nr 1, 40–41.