Application of Embedded Perturbed Integral Collocation Method for Nonlinear Second-order Multi-point Boundary Value Problems

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Abstract

Many problems in theory of elastic stability and kinetic reactions lead to nonlinear multi-point boundary value problems. Therefore in this paper, we present Embedded Perturbed Chebyshev Integral Collocation Method(EPCICM) for solving nonlinear second-order multi-point boundary value problems. The approaches in this work are of two-fold: First, we employed Newton-Raphson-Kantorovich linearization procedure to linearise the problems before solving them. Second, we solved the nonlinear systems directly without linearization by Newton's method to obtain the unknown coefficients. Our investigations showed that the second approach produced better results than Newton-Raphson-Kantorovich linearization approach.

Keywords: *Chebyshev approximation, Multi-point boundary value problems, Newton's linearization scheme,nonlinear problems.*

1 Introduction

Multi-point boundary value problems play important role in many fields especially in science and engineering. They occur in a wide variety of problems including modeling of railway systems, construction of large bridges with many supports and problems arising from electric power networks. Several numerical methods have been developed and used to approximate the solution of multi-point boundary value problems. Some of these methods are Homotopy Perturbation Method [1], Reproducing Kernel Method [2, 3], Adomain Decomposition Method [4], the Shooting Method [5, 6], Weighted Residual Method [7], Homotopy Perturbation and Variation Iteration Method [8]. The main aim of this paper is to develop a new algorithm for solving second-order multi-point boundary value problems. We employed Newton-Raphson-Kantorovich linearization process to linearise the nonlinear problems after which we replaced all the derivatives and the original function by integrated Chebyshev polynomials. Also, we solved the problems without linearising them and this resulted to system of nonlinear algebraic equations which are solved using Newton's method. In this work, nonlinear second-order multi-point boundary value problem

$$u''(x) + g(u, u') = f(x), 0 \leq x \leq 1, \tag{1}$$

$$u(0) = \alpha, \qquad u(1) = \sum_{i=1}^{m} \alpha_i u(\eta_i) + \gamma, \tag{2}$$

(see [3]) will be investigated by using EPCICM where $\eta_i \in (0, 1)$, $i = 0, 1, \cdots, m$, α and γ are constants.
The structure of this paper is as follows: In the next section, we give some relevant properties of Chebyshev polynomials of the first kind. In section 3, we describe the construction process of EPCICM . The numerical examples are presented in section 4 to show the effectiveness, applicability and validity of the method. Some concluding remarks are given in section 5.

2 Chebyshev Polynomials

The Chebyshev Polynomials of the first kind are polynomials in x of degree n, defined by the relation:

$$T_n(x) = \cos(n\theta), \text{when } x = \cos\theta. \tag{3}$$

The Chebyshev polynomials can be determined with the aid of the following recurrence formula:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), n = 1, 2, \cdots \tag{4}$$

with the initial conditions

$$T_0(x) = 1, \qquad T_1(x) = x. \tag{5}$$

In order to use these polynomials on the interval [0,1], we define shifted Chebyshev polynomials by introducing the change of variable $x = 2x - 1$. The shifted Chebyshev polynomial is denoted by $T_n^*(x)$ and $T_n^*(x) = T_n(2x-1)$. Thus, we have

$$T_0^*(x) = 1, \qquad T_1^*(x) = 2x - 1, \tag{6}$$

and the recurrence relation for shifted Chebyshev polynomials in [0,1] is given by

$$T_{n+1}^*(x) = 2(2x-1)T_n^*(x) - T_{n-1}^*(x), n = 1, 2, \cdots \tag{7}$$

3 Description of EPCICM

To solve problem (1) with the boundary conditions (2), the second-order derivative is sought in truncated Chebyshev series form with perturbation term added and then integrated twice to obtain expressions for first-order derivative and the function u itself. Thus, the process is described as follows:

$$\frac{d^2u(x)}{dx^2} = \sum_{n=0}^{N} a_n T_n(x) + \chi_v H_N(x). \tag{8}$$

Integrating (8) successively, we obtain

$$\begin{aligned}\frac{du(x)}{dx} &= \sum_{n=0}^{N} a_n \int T_n(x)dx + \chi_v \int H_N(x)dx + c_1 \\ &= \sum_{n=0}^{N+1} \delta_{n,1}\phi_n^{[1]}(x) + \chi_v \psi^{[1]}(x)\end{aligned} \tag{9}$$

$$\begin{aligned}u(x) &= \sum_{i=n}^{N} a_n \int \phi_n^{[1]}(x)dx + \chi_v \int \psi^{[1]}(x)dx + c_1x + c_2 \\ &= \sum_{n=0}^{N+2} \delta_{n,0}\phi_n^{[0]}(x) + \chi_v \psi^{[0]}(x),\end{aligned} \tag{10}$$

where $\chi_v = \left\{\begin{array}{ll} 1, & v = 2 \\ 0, & v \neq 2 \end{array}\right.$, and $H_N(x) = \tau_1 T_N(x) + \tau_2 T_{N-1}(x)$

Substituting equations (8)- (10) into equation (1), we have

$$\begin{aligned}&\sum_{n=0}^{N} a_n T_n(x) + \chi_v H_N(x) \\ &+ g\left(\left(\sum_{n=0}^{N+2} \delta_{n,0}\phi_n^{[0]}(x) + \chi_v \psi^{[0]}(x)\right)\left(\sum_{n=0}^{N+1} \delta_{n,1}\phi_n^{[1]}(x) + \chi_v \psi^{[1]}(x)\right)\right) = f(x).\end{aligned} \tag{11}$$

Thus collocating equation (11) at point $x = x_j$, we have

$$\begin{aligned}&\sum_{n=0}^{N} a_n T_n(x_j) + \chi_v H_N(x_j) \\ &+ g\left(\left(\sum_{n=0}^{N+2} \delta_{n,0}\phi_n^{[0]}(x_j) + \chi_v \psi^{[0]}(x_j)\right)\left(\sum_{n=0}^{N+1} \delta_{n,1}\phi_n^{[1]}(x_j) + \chi_v \psi^{[1]}(x_j)\right)\right) = f(x_j),\end{aligned} \tag{12}$$

where

$$x_j = a + \frac{(b-a)j}{N+4}, j = 1, 2, \cdots, N+3. \tag{13}$$

Thus, equation (12) gives a system of $(N+3)$ linear or nonlinear algebraic equations in $(N+5)$ unknown constants. Extra two equations are obtained from the boundary conditions. Altogether, we have a system of $(N+5)$ linear or nonlinear algebraic equations. These $(N+5)$ algebraic equations are solved by using Guassian elimination method for linear case while Newton's method is employed for nonlinear case to obtain the unknown coefficients. These coefficients are then substituted into equation (10) to obtain the approximate solution.

4 Numerical Examples

In this section, to show the effectiveness, applicability and validity of our proposed method, we consider three examples.

Example 1: Consider the nonlinear multi-point boundary value problem [3]

$$u''(x) + \frac{x^2(1-x)}{2}u'(x) + u^2(x) = f(x) \tag{14}$$

$$u(0) = 0, \quad u(1) = \sum_{i=0}^{4} \left(\frac{1}{1+i}\right) u\left(\frac{i}{5}\right) + 0.708667 \tag{15}$$

with the exact solution $u(x) = x^2$, when $f(x) = x^3 + 2$.

Method 1: Linearisation Approach

The nonlinear multi-point boundary value problem (14) is linearised by the Newton-Raphson-Kantorovich technique to obtain:

$$u''_{k+1}(x) + 2u_k(x)u_{k+1} + \frac{1}{2}x^2(1-x)u'_k(x) - (u_k(x))^2 = x^3 + 2, k = 0, 1, \cdots \tag{16}$$

subject to the boundary conditions:

$$u_{k+1}(0) = 0, \quad u_{k+1}(1) = \sum_{i=0}^{4} \left(\frac{1}{1+i}\right) u_{k+1}\left(\frac{i}{5}\right) + 0.708667 \tag{17}$$

Using the initial approximation

$$u_0(x) = -0.099793138x + x^2 + \frac{1}{20}x^5$$

and after four iterations(k=3) we obtain the following approximate solution for case $N = 4$:

$$u(x) = 0.0000009816368107x + 0.9999970673x^2 + 0.00003165830867x^3$$
$$-0.00008025332007x^4 + 0.00007761704736x^50 - 00002606288578x^6$$

Method 2: Nonlinearisation Approach
In this case, we solved Problem (14) together with its boundary conditions (15) directly by using our proposed method which eventually resulted to a system of nonlinear algebraic equations. These equations are solved by using Newton's method to obtain the unknown coefficients. Thus, for the case $N = 2$ we obtain $u(x) = x^2$ which is the exact solution.
Table 1 shows comparison of absolute errors in numerical results by Method 1 and [3].

Table 1: Comparison of Absolute Errors for Example 1

x	Exact Solution	Das et al [3]	Method I
0	0.0000	0.0000	0.0000
0.1	0.0100	-	9.3220E-08
0.2	0.0400	2.0000E-07	2.2705E-07
0.3	0.0900	-	4.0488E-07
0.4	0.1600	4.0000E-07	5.8310E-07
0.5	0.2500	-	7.1740E-07
0.6	0.3600	6.0000E-07	7.9010E-07
0.7	0.4900	-	8.1890E-07
0.8	0.6400	8.0000E-07	8.4710E-07
0.9	0.8100	-	9.1390E-07
1.0	1.0000	1.0000E-06	1.0090E-06

Example 2: Consider the following nonlinear multi-point boundary value problem [3]

$$u''(x) + xu(x)u'(x) - 2u(x) = f(x) \tag{18}$$

$$u(0) = 0, \quad u(1) = \sum_{i=0}^{4} \left(\frac{1}{1+i}\right) u\left(\frac{i}{5}\right) + 0.252 \tag{19}$$

with exact solution $u(x) = x(1 - x)$, when $f(x) = x^3 - x^2 + 2$

Method 1: Linearisation Approach
The linearised form of equation (18) is given as:

$$u''_{k+1}(x) + 2\left(u_k(x)\right)^2 + x\left(u'_{k+1}u_k(x) + u_{k+1}(x)u'_k(x) - u'_k(x)u_k(x)\right)$$
$$-4u_{k+1}(x)u_k(x) = x^3 - x^2 + 2 \tag{20}$$

subject to the boundary conditions:

$$u_{k+1}(0) = 0, \quad u_{k+1}(1) = \sum_{i=0}^{4}\left(\frac{1}{1+i}\right)u_{k+1}\left(\frac{i}{5}\right) + 0.252 \tag{21}$$

Similarly, using the initial approximation

$$u_0(x) = -0.9398765936x + x^2 - \frac{1}{12}x^4 + \frac{1}{20}x^4 + \frac{1}{20}x^5$$

and iterating five times(k=4), we obtain the following approximate solution for case $N = 4$.

$$u_5(x) = -1.000000114x + 0.9999995128x^2 + 0.000003160165775x^3$$
$$-0.000004743845332x^4 + 0.000002528617272x^5 - 0.0000003597136402x^6$$

Method 2: Nonlinearisation Approach

On solving Problem (18) together with its boundary conditions (19) for case $N = 2$ using our proposed method, we obtain a system of 5 nonlinear algebraic equations. Extra 2 equations are obtained from the boundary conditions and altogether we solved 7 nonlinear algebraic equations to obtain the unknown coefficients by using Newton's method. Thus, the unknown values are obtained and substituting these values into (10), we obtain $u(x) = x(x-1)$ which is the exact solution to this problem.

Table 2: Comparison of Absolute Errors for Example 2

x	Exact Solution	Das et al [3]	Method I
0	0.0000	0.0000	0.0000
0.1	-0.0900	-	1.3560E-08
0.2	-0.1600	2.0000E-07	2.3800E-08
0.3	-0.2100	-	2.5300E-08
0.4	-0.2400	4.0000E-07	1.8300E-08
0.5	-0.2500	-	6.9000E-09
0.6	-0.2400	6.0000E-07	3.8000E-09
0.7	-0.2100	-	9.1000E-09
0.8	-0.1600	8.0000E-07	6.2000E-09
0.9	-0.0900	-	3.9000E-09
1.0	0.0000	0.0000	1.6800E-08

Example 3: Consider the nonlinear multi-point boundary value problem [3]

$$u''(x) + u(x)u'(x) = f(x) \tag{22}$$

$$u(0) = 0, \quad u(1) = \sum_{i=0}^{4} \left(\frac{1}{1+i}\right) u\left(\frac{i}{5}\right) + 0.3277 \tag{23}$$

with exact solution $u(x) = \sin x$, when $f(x) = (\cos x - 1)\sin x$

Using the same procedures discussed in Examples 1 and 2, we obtain the following approximate solutions for linearised approach when $N = 6$, $k = 1$ and nonlinearised approach when $N = 6$, respectively

$$u_2(x) = 1.000000533x + 0.0000194625x^2 - 0.1668681082x^3 + 0.00053786012x^4 \\ +0.007784932282x^5 + 0.0001943941854x^6 - 0.0001964096722x^7$$

and

$$u(x) = x - 0.000000842679x^2 - 0.166657638920005x^3 - 0.0000399488764x^4 \\ +0.008387154885641x^5 - 0.0000153843780x^6 - 0.0002210499975x^7 + 0.00001706990995x^8$$

Table 3: Comparison of Absolute Errors for Example 3

x	Exact Solution	Das et al [3]	Method I	Method II
0	0.00000000000	0.0000	0.0000	0.0000
0.1	0.09983341665	-	6.3123E-07	6.2957E-07
0.2	0.1986693308	0.00005	1.2595E-06	1.2555E-06
0.3	0.2955202067	-	1.8654E-06	1.8573E-06
0.4	0.3894183423	0.00010	2.4269E-06	2.4115E-06
0.5	0.4794255386	-	2.9339E-06	2.9092E-06
0.6	0.5646424734	0.00015	3.3908E-06	3.3566E-06
0.7	0.6442176872	-	3.8054E-06	3.7638E-06
0.8	0.7173560909	0.00020	4.1777E-06	4.1314E-06
0.9	0.7833269096	-	4.4964E-06	4.4462E-06
1.0	0.8414709848	0.00025	4.7538E-06	4.6995E-06

5 Conclusion

In this paper, an algorithm for obtaining numerical solution of nonlinear second-order multi-point boundary value problems is presented. The derivation of our proposed method is essentially based on Chebyshev integral collocation and the accuracy and applicability of the method were investigated by considering three examples. The numerical results showed that the accuracy of the obtained solutions is satisfactory and it is also observed that nonlinearised approach produced better results compared to linearized approach.

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