An Efficient Iterative Method to General-Form Tikhonov Regularization

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Abstract

Tikhonov regularization is a popular method for linear discrete ill-posed problems. This paper is concerned with the iterative method based on a partial range restricted Arnoldi decomposition of the given matrix. Theoretical analysis and numerical examples are presented to illustrate the benefit of the proposed method.

Keywords: Tikhonov regularization, Arnoldi decomposition, discrepancy principle.

1 Introduction

Consider a linear least squares problem

$$\min_{x\in\mathbb{R}^n} \|Ax-b\|, \quad A\in\mathbb{R}^{m\times n}, \quad m\geq n, \tag{1}$$

where A is severely ill-conditioned. Usually, A has many singular values of different orders of magnitude close to zero and some singular values may vanish. Minimization problems with a matrix of ill-determined rank are often referred to as linear discrete ill-posed problems. They may be obtained by discretizing linear ill-posed problems, such as Fredholm integral equations of the first kind with a smooth kernel. This type integral equations arise in science and engineering when one seeks to determine the cause (the solution) of an observed effect represented by the right-hand side b (the data). Because the entries of b are obtained through observation, they typically are contaminated by measurement errors and also by discretization errors. We denote these errors by $e\in\mathbb{R}^n$ and the unavailable error-free right-hand side associated with b by $\hat{b}\in\mathbb{R}^n$, i.e.,

$$b=\hat{b}+e. \tag{2}$$

We assume that a bound δ for which

$$\|e\|\leq\delta$$

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is available, and the linear system of equations with the unavailable error-free right-hand side

$$Ax = \hat{b} \tag{3}$$

to be consistent. Let $\hat{x}$ denote a desired solution of (3), e.g., the solution of minimal Euclidean norm. We seek to obtain an approximation of $\hat{x}$ by computing an approximate solution of the available linear system of equations (1). Due to the severe ill-conditioning of A and the error e in b, straightforward solution of (1) generally does not yield a meaningful approximation of $\hat{x}$. A common approach to remedy this difficulty is to replace the least-squares problem by a nearby problem that is less sensitive to perturbations. One of the most popular replacement approaches is known as Tikhonov regularization method, which is to solve the minimization problem of the form

$$\min_{x\in\mathbb{R}^n}\{\| Ax - b \|^2 + \frac{1}{\mu} \| Lx \|^2\}, \tag{4}$$

where and throughout this paper, $\| \cdot \|$ denotes the Euclidean vector norm or the associated induced matrix norm. The scalar $\mu > 0$ is the regularization parameter and the matrix $L \in \mathbb{R}^{p\times n}, p \leq n$ is referred to as the regularization matrix. Common regularization matrices L are the identity matrix I and finite difference matrices such as

$$L := \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix} \in \mathbb{R}^{(n-1)\times n}, \tag{5}$$

$$L := \begin{pmatrix} -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \end{pmatrix} \in \mathbb{R}^{(n-2)\times n}. \tag{6}$$

The minimization problem (4) is said to be in *standard form* when $L = I$ and in *general form* otherwise. Many examples of regularization matrices can be found in [1, 2, 9, 21, 23].

The matrix L is assumed to be chosen such that

$$N(A) \cap N(L) = \{0\},$$

where $N(M)$ denotes the null space of the matrix M. Let M^T denote the transpose of the matrix M. Then the Tikhonov minimization problem (4) has the unique solution

$$x_\mu = (A^T A + \frac{1}{\mu} L^T L)^{-1} A^T b.$$

When the matrices A and L are of small to moderate sizes, (4) can be solved with the aid of the Generalized Singular Value Decomposition (GSVD) of the matrix pair $\{A, L\}$, see, e.g., [13, 17, 18] for details. The main drawback of the GSVD is that it is quite expensive to compute for matrices of large sizes.

Kilmer et al. [19] proposed an inner-outer iteration method which computed a partial GSVD of the matrix pair $\{A, L\}$. However, this method required large number of matrix-vector product evaluations with A and A^T. Lewis and Reichel [20] presented an Arnoldi-Tikhonov method based on reducing A by a range-restricted Arnoldi scheme which required L to be a square matrix. Automatic parameter setting for Arnoldi-Tikhonov methods was recently proposed by Gazzola and Novati [14], and this new strategy can work without restrictions on the choice of the regularization matrix. Another method based on reducing both A and L by an Arnoldi-type method was proposed in [22], and this method required both A and L are square matrices. In [12], Dykes and Reichel suggested a simplified GSVD method which described how the standard methods for the computation of the GSVD of a matrix pair can be simplified in the context of Tikhonov regularization.

Approximations of the solution x_μ of problem (4) in standard form can be computed by partial Lanczos bidiagonalization of A based on the Krylov subspace

$$K_k(A^TA, A^Tb) = span\{A^Tb, (A^TA)A^Tb, \cdots, (A^TA)^{k-1}A^Tb\}, \tag{7}$$

which is independent of the regularization matrix L. This method carries out k steps of Lanczos bidiagonalization of the matrix A to determine the decompositions

$$A\tilde{V}_k = \tilde{U}_{k+1}\tilde{C}_k, A^T\tilde{U}_k = \tilde{V}_k C_k^T,$$

for a suitable $k > 0$. Here, the matrices $\tilde{U}_{k+1} \in \mathbb{R}^{m\times(k+1)}$ and $\tilde{V}_k \in \mathbb{R}^{n\times k}$ have orthonormal columns, $\tilde{U}_k$ consists of the first k columns of $\tilde{U}_{k+1}$, and $\tilde{C}_k \in \mathbb{R}^{(k+1)\times k}$ is lower bidiagonal. See, e.g., [4, 5, 7, 8] for several solution methods based on this approach.

When $L \neq I$, Hochstenbach and Reichel [15] proposed a iterative method by first computing a partial Lanczos bidiagonalization of the matrix A, and then projecting L onto the space.

The method in this paper differs from [15] in that the space in which we determine an approximate solution of (4) is the Krylov subspace

$$K_k(A, Ab) = span\{Ab, A^2b, \cdots, A^kb\}. \tag{8}$$

This method requires A to be a square matrix, which can be satisfied by zero-padding if necessary.

Our interest in the space (8) stems from the fact that for many linear discrete ill-posed problems the spaces (7) and (8) can be chosen to be of about the

same dimension, and the computation of an orthonormal basis for the space (8) requires fewer matrix-vector product evaluations than for the space (7). The main reason is that each iteration with the space (7) demands the evaluation of one matrix-vector product with the matrix A and one matrix-vector product with A^T, while each iteration with the space (8) requires the evaluation of only one matrix-vector product with A. These evaluations typically constitute the dominant computational effort required. Moreover, the matrix-vector product with A is easier to evaluate than that with A^T.

This paper is organized as follows. Section 2 discusses the proposed iterative method. The determination of the regularization parameter μ and Arnoldi steps k are presented in Section 3. Numerical examples are described in Section 4.

2 The projected Range-Restricted Arnoldi iterative method

We reduce the problem (4) to a problem of smaller size by application of k steps of the Arnoldi process to A with the initial vector $u_1 = Ab/ \parallel Ab \parallel$. This yields the decomposition

$$AU_k = U_{k+1}\bar{H}_k, \tag{9}$$

where $U_{k+1} = [u_1, u_2, \cdots, u_{k+1}] \in \mathbb{R}^{m\times(k+1)}$ has orthonormal columns, which span the Krylov subspace (8). We call this decomposition as range restricted Arnoldi decomposition. The matrix $U_k \in \mathbb{R}^{m\times k}$ consists of the first k columns of U_{k+1}. We assume that k is chosen sufficiently small so that $\bar{H}_k \in \mathbb{R}^{(k+1)\times k}$ is a upper Hessenberg matrix with nonvanishing subdiagonal entries. Then $\bar{H}_k$ is of rank k.

We use the QR factorization

$$LU_k = Q_k R_k, \tag{10}$$

where $Q_k \in \mathbb{C}^{p\times k}$ has orthonormal columns and $R_k \in \mathbb{C}^{k\times k}$ is upper triangular.

The computation of the decomposition (9) requires the evaluation of $k+1$ matrix-vector products with the matrix A. Since the matrix L generally is very sparse, the computational effort needed to evaluate LU_k typically is much smaller than for the evaluation of $k+1$ matrix-vector products with A.

We seek to determine an approximate solution of (1) in the Krylov subspace (8). Substituting $x = U_k y, y \in \mathbb{R}^k$ into (4) and using (10), we get the

reduced minimization problem

$$\begin{aligned} & \min_{x\in\mathbb{R}^n}\{\| Ax-b\|^2+\frac{1}{\mu}\| Lx\|^2\} \\ = & \min_{y\in\mathbb{R}^k}\{\| AU_ky-b\|^2+\frac{1}{\mu}\| LU_ky\|^2\} \\ = & \min_{y\in\mathbb{R}^k}\{\| U_{k+1}\bar{H}_ky-b\|^2+\frac{1}{\mu}\| Q_kR_ky\|^2\} \\ = & \min_{y\in\mathbb{R}^k}\left\| \begin{pmatrix} \bar{H}_k \\ \frac{1}{\sqrt{\mu}}R_k \end{pmatrix} y-\begin{pmatrix} U_{k+1}^Tb \\ 0 \end{pmatrix}\right\|^2. \end{aligned} \tag{11}$$

Since the subspace dimension k is quite small, we evaluate the solution $y_k^{(\mu)}$ of (11) by first transforming the matrix $\begin{pmatrix} \bar{H}_k \\ \frac{1}{\sqrt{\mu}}R_k \end{pmatrix}$ into upper triangular form by application of a judiciously chosen sequence of Givens rotations, and then followed by back substitution. Having determined the solution $y_k^{(\mu)}$, we obtain an associated approximate solution $x_k^{(\mu)}=U_ky_k^{(\mu)}$ of (1).

We will apply the discrepancy principle to determine a suitable value of μ. By simple calculation, we have

$$\|Ax_k^{(\mu)}-b\|^2=\|AU_ky_k^{(\mu)}-b\|^2=\|\bar{H}_ky_k^{(\mu)}-U_{k+1}^Tb\|^2+\|(I-U_{k+1}U_{k+1}^T)b\|^2.$$

The norm of the residual error is a function of both the residual error norm for the reduced problem and the projection error norm $\|(I_n-U_{k+1}U_{k+1}^T)b\|$.

Similar to [22], we use the projected discrepancy principle to determine μ, so that

$$\|\bar{H}_ky_k^{(\mu)}-U_{k+1}^Tb\|=\eta\delta, \tag{12}$$

where $\eta\geq 1$ is a user-specified constant independent of δ and is usually fairly close to unity, and $y_k^{(\mu)}$ solves (11). Let μ_k denote the solution of (12), and clearly $\|Ax_k^{(\mu)}-b\|\geq\eta\delta$.

When the number of Arnoldi step k increases, the QR factorization of LU_k has to be updated. Formulas for updating a QR factorization are described by Daniel et al. [10]; see also [13]. Note that only the upper triangular matrices $R_k, k=1,2,\cdots$, are required, but not the associated matrices Q_k with orthonormal columns.

3 Determining the regularization parameter and iterative step

In this section, we discuss the computation of $\mu=\mu_k$ and the Arnoldi step k so that $y_k=y_k^{(\mu_k)}$ and k statisfies (12) and $\|Ax_k-b\|\leq\eta\delta$ respectively.

Introduce the function

$$\phi_k(\mu) = \|\bar{H}_k y_k - U_{k+1}^T b\|^2, \tag{13}$$

where $y_k = y_k^{(\mu)}$ is the solution of (11). Then equation (13) can be expressed as

$$\phi_k(\mu) = \eta^2 \delta^2. \tag{14}$$

The QR factorization of $\bar{H}_k$ is

$$\bar{H}_k = \tilde{Q}_k \tilde{R}_k,$$

where $\tilde{Q}_k \in \mathbb{C}^{(k+1)\times k}$ has orthonormal columns and $\tilde{R}_k \in \mathbb{C}^{k\times k}$ is upper triangular.

The following theorem discusses some properties of the equation (14).

Theorem 3.1. *Assume that the matrix R_k in (10) is nonsingular. Let $\hat{R} = \tilde{R}_k R_k^{-1}$. Then the function (13) can be expressed as*

$$\phi_k(\mu) = \|(\mu \hat{R}\hat{R}^T + I)^{-1} \tilde{Q}_k^T U_{k+1}^T b\|^2 + \|(I - \tilde{Q}_k \tilde{Q}_k^T) U_{k+1}^T b\|^2. \tag{15}$$

Consequently, $\phi_k(\mu)$ is strictly decreasing and convex, and equation (14) has a unique solution $0 < \mu_k < \infty$, provided that

$$\|P_{N(\tilde{Q}_k)} U_{k+1}^T b\| < \eta\delta < \|b\|, \tag{16}$$

where $P_{N(\tilde{Q}_k)}$ denotes the orthogonal projector onto $N(\tilde{Q}_k)$.

Proof. The representation (15) follows from

$$\begin{aligned}
\phi_k(\mu) &= \|\bar{H}_k y_k - U_{k+1}^T b\|^2 \\
&= \|\tilde{Q}_k \tilde{R}_k y_k - U_{k+1}^T b\|^2 \\
&= \|\tilde{R}_k y_k - \tilde{Q}_k^T U_{k+1}^T b\|^2 + \|(I - \tilde{Q}_k \tilde{Q}_k^T) U_{k+1}^T b\|^2 \\
&= \|\tilde{R}_k (\bar{H}_k^T \bar{H}_k + \frac{1}{\mu} R_k^T R_k)^{-1} \bar{H}_k^T U_{k+1}^T b - \tilde{Q}_k^T U_{k+1}^T b\|^2 + \|(I - \tilde{Q}_k \tilde{Q}_k^T) U_{k+1}^T b\|^2 \\
&= \|[\tilde{R}_k (\tilde{R}_k^T \tilde{R}_k + \frac{1}{\mu} R_k^T R_k)^{-1} \tilde{R}_k^T - I] \tilde{Q}_k^T U_{k+1}^T b\|^2 + \|(I - \tilde{Q}_k \tilde{Q}_k^T) U_{k+1}^T b\|^2 \\
&= \|\{\tilde{R}_k [R_k^T (\hat{R}^T \hat{R} + \frac{1}{\mu} I) R_k]^{-1} \tilde{R}_k^T - I\} \tilde{Q}_k^T U_{k+1}^T b\|^2 + \|(I - \tilde{Q}_k \tilde{Q}_k^T) U_{k+1}^T b\|^2 \\
&= \|(\mu \hat{R}\hat{R}^T + I)^{-1} \tilde{Q}_k^T U_{k+1}^T b\|^2 + \|(I - \tilde{Q}_k \tilde{Q}_k^T) U_{k+1}^T b\|^2,
\end{aligned} \tag{17}$$

where we have used the expression

$$y = (\bar{H}_k^T \bar{H}_k + \frac{1}{\mu} R_k^T R_k)^{-1} \bar{H}_k^T U_{k+1}^T b$$

for the forth equality and the formula

$$\hat{R} = \tilde{R}_k R_k^{-1}, \quad \hat{R}(\hat{R}^T\hat{R} + \frac{1}{\mu}I)^{-1}\hat{R}^T = I - (\mu\hat{R}\hat{R}^T + I)^{-1}$$

for the last equality.

Define the spectral factorization

$$\hat{R}\hat{R}^T = W\Lambda W^T,$$

where $\Lambda = diag[\lambda_1, \lambda_1, \cdots, \lambda_k]$ and $W \in \mathbb{R}^{k\times k}$ is orthonormal. Then (17) is equal to

$$\tilde{b}^T(\mu\Lambda + I)^{-2}\tilde{b} + \|(I - \tilde{Q}_k\tilde{Q}_k^T)U_{k+1}^T b\|^2,$$

where $\tilde{b} = W^{-2}\tilde{Q}_k^T U_{k+1}^T$. Thus ϕ_k is decreasing and convex. Moreover, we obtain from (15) that

$$\lim_{\mu\to 0}\phi_k(\mu) = \|b\|^2, \quad \lim_{\mu\to\infty}\phi_k(\mu) = \|P_{N(\tilde{Q}_k)}U_{k+1}^T b\|^2.$$

Therefore, when the bounds (16) hold, the equation (14) has a unique bounded solution. □

The stopping index k for the Arnoldi process can be determined by the discrepancy principle, i.e., the iterations are terminated as soon as an approximate solution x_k satisfies

$$\|Ax_k - b\| \le \eta\delta, \tag{18}$$

where $\eta \ge 1$ is a user-specified constant independent of δ. See [6] for a validity of this stopping criterion.

4 Numerical experiments

We consider five linear discrete ill-posed problems that arise from the discretization of Fredholm integral equation of the first kind with a smooth kernal. We use these numerical examples to illustrate the performance of our method described in Section 2. The error-free $\hat{b}$ is available by

$$\hat{b} = A\hat{x}.$$

The error vector e has normally distributed entries with zero mean and is scaled so that the contaminated b, defined by (2), has a specified noise level relative error

$$\epsilon = \|e\|/\|\hat{b}\|.$$

We let $\epsilon = 1 \cdot 10^{-3}$ and determine the regularization parameter μ by (12) in all examples.

We denote the projected Lanczos bidiagonalization-Tikhonov iterative method in [15] as PLBDT and our projected range-restricted Arnoldi-Tikhonov iterative method as PRRAT respectively. In all examples, we compare the relative errors $\|x-\hat{x}\|/\|\hat{x}\|$, iteration steps and the CPU time of the two methods.

Example 4.1. *The Fredholm integral equation of the first kind*

$$\int_0^{\pi} K(s,t)x(t)dt = g(s), \quad 0 \le s \le \frac{\pi}{2}, \tag{19}$$

with the kernel and solution given by

$$k(s,t) := exp(scos(t)),$$

$$x(t) := sin(t).$$

The right-hand side function $g(s)$ is defined by (19). This integral equation is discussed by baart [3].

We discretize the integral equation by the Galerkin method with orthonormal box functions as test and trial functions using the MATLAB code **baart** from [16] and obtain the matrix $A \in \mathbb{R}^{1000\times 1000}$ and the discretized solution $\tilde{x}$ of the error-free linear system (3). The associated contaminated vector b in (1) is obtained by adding 0.1% normally distributed zero mean "noise" e to $\hat{b}$; cf.(2).

Method	Regularization matrix	Relative error	Iterative step	CPU(s)
PLBDT	I	$1.14 \cdot 10^{-1}$	4	0.163s
PRRAT	I	$3.58 \cdot 10^{-2}$	3	0.097
PLBDT	(5)	$1.14 \cdot 10^{-1}$	4	0.136
PRRAT	(5)	$3.88 \cdot 10^{-2}$	4	0.131
PLBDT	(6)	$9.89 \cdot 10^{-2}$	38	5.711
PRRAT	(6)	$3.39 \cdot 10^{-2}$	3	0.087

Table 1: Relative errors, iterative steps and CPU time of the numerical solutions for example 4.1.

Example 4.2. *Consider the Fredholm integral equation of the first kind*

$$\int_0^{1} K(s,t)x(t)dt = g(s), \quad 0 \le s \le 1, \tag{20}$$

with the kernel and solution given by

$$k(s,t) := \begin{cases} s(t-1), & s < t, \\ t(s-1), & s \ge t, \end{cases}$$

$$x(t) := t.$$

The right-hand side function $g(s)$ is defined by (20). This integral equation is discussed by Delves and Mohamed [11].

We discretize the integral equation by the Galerkin method with orthonormal box functions as test and trial functions using the MATLAB code **deriv2** from [16] and obtain the matrix $A \in \mathbb{R}^{1000\times 1000}$ and the discretized solution $\tilde{x}$ of the error-free linear system (3). The associated contaminated vector b in (1) is obtained by adding 0.1% normally distributed zero mean "noise" e to $\hat{b}$; cf.(2).

Method	Regularization matrix	Relative error	Iterative step	CPU(s)
PLBDT	I	$1.37 \cdot 10^{-1}$	21	2.338
PRRAT	I	$1.35 \cdot 10^{-1}$	12	0.526
PLBDT	(5)	$1.30 \cdot 10^{-1}$	110	44.005
PRRAT	(5)	$1.35 \cdot 10^{-1}$	12	0.459
PLBDT	(6)	$1.31 \cdot 10^{-1}$	43	7.353
PRRAT	(6)	$1.37 \cdot 10^{-1}$	13	0.638

Table 2: Relative errors, iterative steps and CPU time of the numerical solutions for example 4.2.

Example 4.3. *The Fredholm integral equation of the first kind*

$$\int_0^\infty K(s,t)x(t)dt = g(s), \quad s \geq 0, \tag{21}$$

with the kernel and solution given by

$$k(s,t) := exp(-st),$$

$$x(t) := exp(-t/2).$$

The right-hand side function $g(s)$ is defined by (21). This integral equation is discussed by Varah [25].

We discretize the integral equation by the Galerkin method with orthonormal box functions as test and trial functions using the MATLAB code **ilaplace** from [16] and obtain the matrix $A \in \mathbb{R}^{1000\times 1000}$ and the discretized solution $\tilde{x}$ of the error-free linear system (3). The associated contaminated vector b in (1) is obtained by adding 0.1% normally distributed zero mean "noise" e to $\hat{b}$; cf.(2).

Method	Regularization matrix	Relative error	Iterative step	CPU(s)
PLBDT	I	$7.61 \cdot 10^{-1}$	22	2.019
PRRAT	I	$4.22 \cdot 10^{-1}$	8	0.444
PLBDT	(5)	$7.58 \cdot 10^{-1}$	30	3.095
PRRAT	(5)	$4.22 \cdot 10^{-1}$	8	0.391
PLBDT	(6)	$7.69 \cdot 10^{-1}$	> 200	100.851
PRRAT	(6)	$4.22 \cdot 10^{-1}$	8	0.368

Table 3: Relative errors, iterative steps and CPU time of the numerical solutions for example 4.3.

Example 4.4. *The Fredholm integral equation of the first kind*

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K(s,t)x(t)dt = g(s), \quad -\frac{\pi}{2} \leq s \leq \frac{\pi}{2}, \tag{22}$$

with the kernel and solution given by

$$k(s,t) := (cos(s) + cos(t))(\frac{sin(u)}{u})^2, \quad u = \pi(sin(s) + sin(t)).$$

$$x(t) := sin(t).$$

The right-hand side function $g(s)$ is defined by (22). This integral equation is discussed by shaw [24].

We discretize the integral equation by the Galerkin method with orthonormal box functions as test and trial functions using the MATLAB code **shaw** from [16] and obtain the matrix $A \in \mathbb{R}^{1000\times1000}$ and the discretized solution $\tilde{x}$ of the error-free linear system (3). The associated contaminated vector b in (1) is obtained by adding 0.1% normally distributed zero mean "noise" e to $\hat{b}$; cf.(2).

Method	Regularization matrix	Relative error	Iterative step	CPU(s)
PLBDT	I	$4.73 \cdot 10^{-2}$	8	0.468
PRRAT	I	$4.75 \cdot 10^{-2}$	7	0.362
PLBDT	(5)	$4.63 \cdot 10^{-2}$	12	0.793
PRRAT	(5)	$4.59 \cdot 10^{-2}$	8	0.365
PLBDT	(6)	$5.96 \cdot 10^{-1}$	> 200	119.566
PRRAT	(6)	$3.46 \cdot 10^{-2}$	8	0.460

Table 4: Relative errors, iterative steps and CPU time of the numerical solutions for example 4.4.

Example 4.5. *The Fredholm integral equation of the first kind*

$$\int_0^1 K(s,t)x(t)dt = g(s), \quad 0 \le s \le 1, \tag{23}$$

with the kernel and solution given by

$$k(s,t) := d(d^2 + (s-t)^2)^{-3/2}, \quad d = 0.25.$$

$$x(t) := sin(\pi t) + \frac{1}{2} sin(2\pi t).$$

The right-hand side function $g(s)$ is defined by (23). This integral equation is discussed by Wing [26].

We discretize the integral equation by the Galerkin method with orthonormal box functions as test and trial functions using the MATLAB code **gravity** from [16] and obtain the matrix $A \in \mathbb{R}^{1000\times 1000}$ and the discretized solution $\tilde{x}$ of the error-free linear system (3). The associated contaminated vector b in (1) is obtained by adding 0.1% normally distributed zero mean "noise" e to $\hat{b}$; cf.(2).

Method	Regularization matrix	Relative error	Iterative step	CPU(s)
PLBDT	I	$1.04 \cdot 10^{-2}$	11	0.752
PRRAT	I	$9.20 \cdot 10^{-3}$	9	0.546
PLBDT	(5)	$4.92 \cdot 10^{-1}$	> 200	121.801
PRRAT	(5)	$9.60 \cdot 10^{-3}$	9	0.455
PLBDT	(6)	$9.50 \cdot 10^{-3}$	> 200	116.906
PRRAT	(6)	$9.80 \cdot 10^{-3}$	10	0.534

Table 5: Relative errors, iterative steps and CPU time of the numerical solutions for example 4.5.

We can see that in some cases the relative error by these two methods is about the same (such as examples 4.2, 4.3 and 4.4), however, the computational time required by the proposed method is less than that required by PLBDT. On the other hand, the results from examples 4.1 and 4.5 are superior to that by PLBDT in terms of the relative error, iterative step and CPU time. It is clear that PRRAT is more efficient (in iterations and computational times) than PLBDT. Furthermore, we can see that the change of the regularization matrix has greater influences on PLBDT than on PRRAT in terms of iterative step and CPU time.

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