**An algorithm for a link-based variational inequality model of**

**dynamic user optimal route choice**

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**ABSTRACT**

In this paper, we have presented a new relaxation with Frank-Wolfe (FW) algorithm to solve the link-based variational inequality model of ideal dynamic user optimal route choice problem. It is a modified version of Ran’s algorithm and has corrected some critical mistakes in the original algorithm. The algorithm does not need time-space network expansion and is efficient in solving real problem.

*Key words*: dynamic user optimal route choice, relaxation, Frank-Wolfe algorithm, dynamic shortest path**.**

**1. INTRODUCTION**

Transportation network models can be classified into two categories: static models and dynamic models. The static models, with the User Equilibrium (UE) (Beckmann, 1956) as the most fundamental one, are applicable to long-term transportation planning. LeBlanc et al. (1975) have shown the Frank-Wolfe (FW) algorithm can be used to solve the UE model. However, the resultant link volume of a static model may be several times more than the capacity of a link, which is not consistent with actual situation. In addition, the static models cannot provide the real time traffic volume and travel time and cannot reflect the time-dependent variation of traffic on a road network. Thus, the application of static models on the operation of a transportation network is limited. As the dynamic generalization of static models, the dynamic models can provide the real time link/path traffic volume and link/path travel time, which are the necessary data for the implementation of any travel guidance systems. Thus, dynamic transportation network models are useful in managing the real time operation or assessing the performance of a transportation system. Dynamic models are also applicable to long-term transportation planning.

The Dynamic User Optimal (DUO) route choice model is the dynamic generalization of UE model. It can be described as: Given time-dependent OD demand of each OD pair, determine the flow pattern on the network such that for each OD pair at each instant of time, the actual travel times experienced by travelers departing at the same time are equal and minimal (this state is called ideal or predictive user optimal state), or for any departure flow from each decision node (intersection) to each destination node at each instant of time, the instantaneous travel times equal the minimal instantaneous route travel time ( this state is called reactive or instantaneous user optimal state) (Ran and Boyce, 1996).

The approaches used to model dynamic traffic assignment (DTA) can be classified into two types: simulation-based approach and analytical approach. Each has its characteristics. The most distinguishable characteristics of analytical models is solution properties such as existence and uniqueness are provable and the convergence is guaranteed (Ran, 2002). The analytical approach includes mathematical programming, optimal control, and variational inequality. The variational inequality (VI) method can overcome the limitations of mathematical programming (Ziliaskopoulos and Peeta, 2001) and optimal control (Boyce, Lee and Ran, 2001) and has been a useful tool to model dynamic transportation networks.

Ran and Boyce (1996) have proved that the FW algorithm is appropriate to solve the dynamic traffic assignment problem if a time-space network is considered. Many VI models and solution algorithms have been presented for dynamic user optimal route choice problem (Friesz et al., 1993; Smith and Wisten, 1995; BYUNG-WOOK et al. 1995; Ran and Boyce, 1996a, 1996c; Chen and Hsueh, 1998; Wie, 2002; Jang, Ran, et al. 2005; Bellei et al. 2005; Michiel et al. 2000; Ran et al. 2002; Akamatsu, 2001; Kim and Jayakrishnan, 2006; Mahut et al. 2008; Ramadurai et al. 2008; Liu et al. 2003; Ran et al. 2002; Lo et al. 2002; YOUNES et al. 2004; Deren et al. 2004). However, no solution algorithm has been perfect and solved all the critical issues. Further effort on developing new algorithms is still needed to efficiently solve dynamic transportation network models.

In this paper, a new relaxation algorithm is proposed for the link-based variational inequality model of ideal dynamic user optimal route choice problem. It is a modified version of Ran’s algorithm and has corrected some critical mistakes in the original algorithm. The algorithm does not need time-space network expansion and is thus efficient in solving real problem. The notations used in this paper are consistent with those used by Ran and Boyce (1996) and Ran (2002).

The rest of the paper is organized as follows: Section 2 covers the literature review. Section 3 presents the link-based VI model of DUO. Section 4 proposes the relaxation algorithm for the link-based VI DUO model. Section 5 introduces dynamic shortest path algorithm. Section 6 presents a numerical example. Section 7 concludes the paper.

**2. LITERATURE REVIEW**

Friesz et al. (1993) formulated a continuous time, infinite-dimensional VI model for the departure time/route choice problem but did not provide solution to the model.

Smith and Wisten (1995) introduced a smooth day-to-day dynamic user-equilibrium assignment VI model. BYUNG-WOOK et al. (1995) formulated the dynamic network user equilibrium problem as a variational inequality problem in discrete time in terms of unit path cost functions. A heuristic algorithm is presented to solve the model. Ran and Boyce (1996a) propose a link-based discretized VI formulation for the ideal DUO problem with fixed departure times. In the paper, the traffic network constraints and link-based DUO route choice conditions are presented, and the necessity and sufficiency of the VI is proved. Ran and Boyce (1996b) used the FW algorithm to solve the dynamic traffic assignment problem on a time-space network. Chen and Hsueh (1998) propose a link-based VI formulation and a solution algorithm for the DUO problem. Wie (2002) developed an algorithm to solve the user equilibrium route choice problem. Jang, Ran and Choi (2005) proposed a route-based discrete variational inequality model of ideal dynamic user optimal (DUO) route choice. They presented a projection-based approach with column generation to solve the model. Bellei, Gentile and Papola (2005) formulated within-day dynamic traffic assignment as a fixed-point problem. The fixed-point problem is solved through the Bather’s method. In the solution process, an implicit path enumeration network loading procedure is used as an extension of Dial’s algorithm.

Bliemer and Bovy (2003) proposed a multiple-user-class macroscopic dynamic traffic assignment model. The model is specified as a quasi-variational inequality problem. A nested modified projection method is proposed to solve the assignment problem. The solution algorithm requires path enumeration. Ran, Lee, and Shin (2002) proposed a link-based variational inequality model of dynamic traffic assignment with the extended capability of performing rolling horizon implementation. The model can be solved to convergence by a relaxation /diagonalization algorithm. Akamatsu（2001）presents an algorithm for solving nonlinear complementarity formulation of the dynamic user equilibrium (DUE) traffic assignment for a one-to-many origin-destination network. Kim and Jayakrishnan (2006) studied dynamic traffic assignment based on arrival time-based OD demand. Mahut, Florian and Tremblay (2008) formulated dynamic traffic assignment model as a time discrete variational inequality problem and use the MSA and a gradient-like method to solve the model. Ramadurai et al. (2008) developed a linear complementarity formulation for the single bottleneck model. Liu et al. (2003) proposed a fuzzy dynamic traffic assignment model. A fuzzy shortest path algorithm is used to find the fuzzy shortest paths and assign traffic to each of them by using the C-logit method. Ran et al. (2002) presented a new algorithm for solving the dynamic route choice problems without time-space network expansion. Lo and Szeto (2002) developed a cell-based nonlinear complementarity formulation of dynamic traffic assignment (DTA). HAMDOUCH et al. (2004) proposed a VI model of dynamic traffic assignment where strategic choices are an integral part of user behaviour. Han and Lo (2004) developed a descent direction of the merit function for co-coercive variational inequality (VI) problems and implemented the solution method for traffic assignment problems. Rong et al. (2018) proposed a continuum dynamic model for autonomous vehicles in a polycentric urban city by considering the environment impact of traffic emission. Mao et al. (2019) established a multi-objective dynamic traffic assignment model with the objectives of not only minimum travel time but minimum decline of present serviceability index for pavements. ZHANG et al. (2019) proposed a dynamic traffic assignment method based on the connected transportation system to express the time-varied traffic flows caused by uncertain traffic demand and supply in the real traffic network accurately.

**3. LINK-BASED VARIATIONAL INEQUALITY (VI) DUO MODEL**

**3.1 Some Definitions and notations**

Some definitions for are given as follows:

**Departure Horizon**: The time period in which there are vehicles departing from an origin and entering the network. Denote it as. All departing flow rate from any origins is zero after.

**Assigning Horizon**: The time period from the beginning to the time point at which the last vehicle entering the network reaches its destination. Denote it as. is the whole analysis time period.

**Time Increment**: The length of the time interval used to partition and. Denote it as. Each time increment is a unit of time. The time interval is.

Let, where is the set of natural number. Similarly, let.

**Time-Space Network**: The network with time dimension, showing the network state at each time interval.

Fig. 2 shows an example of time-space network with 4 time interval for the 3-link network in Fig. 1.  is the number of vehicles on link at interval .



Fig. 1. A 3-link network.



Fig. 2. Time-space network with 4 time intervals for the 3-link network

Notations used in this paper are given as follows:

: number of vehicles on link  at beginning of interval 

: inflow into link  during interval 

: exit flow from link  during interval 

**:** inflow into link **** from origin  to destination at time 

**:** outflow from link **** from origin  to destination at time 

: departure flow from origin toward destination  during interval 

: actual travel time over link **** for flows entering link  at time 

: actual travel time for route  between O-D pair  for flows departing origin at time 

: actual travel time for route  between origin and node  for flows departing origin  at time 

: minimal actual route travel time between O-D pair  for flows departing origin  at time 

The above notations are for discrete case. The time interval is taken as time point when the notations are used in continuous case. Other notations will be defined when needed.

**3.2 Link-based VI Formulation of DUO**

Assume the network is empty at, and only travel demands departing within the departure horizon are considered. The link-based DUO continuous VI model can be expressed as

 (1a)

where , , , , and are the cardinalities of the set nodes, links and O-D pairs, etc. ,

or in expanded form as

 (1b)

where

,  (1c)

This formulation is equivalent to the following link-based DUO route choice conditions:

  (2a)

  (2b)

 (2c)

The above formulation and conditions comes from Ran and Boyce (1996) with some modification. In Ran and Boyce (1996), the link cost term is defined as

 (3)

which is different from (1c).

(2a) states that if time-space link  is on the minimal actual route (dynamic shortest path) from origin to destination at time,; otherwise, . (2b) states that if time-space link  is on the minimal actual route from origin to destination at time, or if,; otherwise, or if , . (2c) is nonnegative condition for inflow.

Below we prove traffic status satisfying (1) is in a DUO status or equivalent to (2a), (2b), (2c).

**Proof:**

1. **Necessity.** By (2a) and (2c),, , this implies . By

(2b), . Thus,  holds. Integrating it over , we have (1).

1. **Sufficiency.** (2a) and (2c) hold by definition. Let the optimal solution of

(1) be. To prove (2b) holds for, we first find a feasible solution such that (2b) holds, or. Suppose (2b) does not hold for, we have. We further has, or . This contradicts (1). Thus (2b) holds for.

**4. SOLUTION ALGORITHMS FOR LINK-BASED VI DUO MODEL**

**4.1Discrete Link-based VI DUO Model**

To solve the DUO problem, the continuous VI formulation is discretized with each time interval being time increment. The estimated actual travel time on each time-space link is a multiple of the time increment and is fixed at each time increment, i.e.,

 (4)

whereis an integer and , is time increment.. This round-off method is used only in the flow propagation constraints. The round-off error can be made as small as desired by making the time increment smaller (Ran and Boyce, 1996).

The link-based DUO discrete-time VI formulation is

 (5a)

or in expanded form as

 (5b)

where, , and

, (6)

is the feasible region defined by the following constraint.

Path flow conservation constraint:

 (7)

Link inflow conservation constraint:

 (8)

Link outflow conservation constraint:

 (9)

Node flow conservation constraint:

 (10)

where  is the set of links whose tail node is  (after) and  is the set of links whose head node is  (before).

Link flow propagation constraint:

 (11)

The link state equation:

 (12a)

or

 (12b)

(12a) is forward formula, (12b) is backward formula.

Path-link flow incidence constraint:

 (13)

whereis defined as:

 (14)

Nonnegative constraint:

 (15)

With flow propagation constrain (11), outflow  and link volume  can be expressed by inflow  as follows (Ran, 2002; Chen, 1998):

 (16)

where

 (17)

and

 (18)

where

 (19)

**4.2 Relaxation**

At each relaxation, we temporarily fix (Ran and Boyce, 1996; Ran, 2002):

1. Actual travel time in the link flow propagation constraints as  and corresponding actual route travel time  as ;
2. Actual travel time  in the VI cost term  as  and
3. Minimal travel times as, as  and  as for each link and each origin and destination.

At each relaxation, a time-space network is implicitly formed with fixed link flow propagation constraints and fixed actual route travel time. Via relaxation, the VI cost term becomes

 (20)

**4.3 Optimization Problem**

An optimization problem which is equivalent to the discrete VI under relaxation can thus be formulated, as follows:

 (21)

The gradient of (21) is shown to be

 (22)

(21) is equivalent to the cost term of discrete VI (5b) under relaxation. This indicates the above optimization program is equivalent to the discrete VI (5).

By using (18), we have

 (23a)

where

Letting  and, (23a) can be expressed as

 (23b)

(23b) can be rewritten as

 (23c)

Letting 

(23c) can be rewritten as

 (23d)

Substitute (23d) into (21), we have

 (24)

Since all cross effects are fixed in each relaxation,  is the only variable for each summation term of (21) and (24). At each relaxation, the VI formulation of DUO problem was transformed into a series of static user equilibrium traffic assignment problems over the time-space network of the relaxation, which can be solved by Frank-Wolfe algorithm. Call the relaxation as outer iteration and solving static user equilibrium traffic assignment problems over the time-space network of the relaxation as inner iteration.

At the iteration of the inner iteration (Frank-Wolfe algorithm), the descending direction of nonlinear programming (21) can be found by solving the following linear program:

 (25)

in.

whereis sub-problem variable, is gradient of with respect to evaluated at .

(25) is equivalent to:

 (26)

in.

where

 (27)

(27) can be decomposed by origin-destination pair. The resulting sub-problem for O-D pair  is:

 (28)

in.

(28) can be further decomposed by each O-D flow,. The resulting sub-problem for O-D flow  is:

 (29)

in.

(29) can be viewed as a shortest path problem over the time-space network of the relaxation. The minimum of (29) is found by assigning to the actual minimum cost route (dynamic shortest path) of O-D pair  at time interval. The cost of each time-space link is defined as (27). The shortest path for (27) can be found on the original network, with the time interval for each link recorded on the original network to track the shortest path on time-space network. As an example, Fig. 3 shows how to record time interval on the original network for demand.



Fig. 3. An example of recording time intervals on original network.

Cost term (27) contains the fixed actual travel time  and  at each relaxation for every link,. They are dynamic shortest path on time-space network. Section (2.3) describes an efficient algorithm to find dynamic shortest paths on the original network based on time-space link travel times.

Notice the difference between cost term (27) and. If  does not contribute to, the shortest paths based on (27) and are the same. To see this, let  be the set of the actual minimum cost route of O-D pair  at time interval at the iteration of the inner iteration, where  is the number of the actual minimum cost route of. Consider the path cost of any, , with , where  are sequential links on route , is the number of links on route . The path cost of  (denote it as) is the sum of all the cost of time-space links on the path, or

 (30a)

If is the same path as the minimum route (with path cost) under the relaxation, then we have

 (30b)

(16a) reduces to

(31)

Since and is fixed at each relaxation, equation (31) implies is also the minimum cost route if cost term  is used. However, if contributes to, the shortest paths based on (27) and are not necessarily the same. To see this, now let  be the actual minimum cost route of O-D pair  at time interval at the iteration of the inner iteration based on cost term. For any with,, its path cost based on cost term is



Its path cost based on cost term (27) is



Because  may not be the same path as the minimum route (with path cost) under the relaxation, (30b) do not necessarily hold, and the shortest paths based on (27) and are not necessarily the same.

The step size along the descending direction can be decided by solving the following one-dimensional search problem:

 (32)

After the optimal step size is found, the solution at the inner iteration can be updated as

  (33)

**4.4 Solution Algorithm**

The algorithm for solving the ideal DUO route choice model (5) is summarized as follows.

**Step 0: Outer Initialization**.

Compute, where is the static minimum travel time of O-D.

Set. Set,  . Find an initial

feasible solution . Set outer iteration counter. Set an outer iteration

convergence criterion.

**Step 1: Relaxation**.

**Step 1.0:** Find a new estimation of actual link travel times:, find

  , where \* denotes the solution obtained from the

most recent inner iteration or from outer initialization. Findand.

**Step 1.1:** Find,, and  by using dynamic shortest

path algorithm, ,.

**Step 2: Inner Iteration**

**Step 2.0: Inner Initialization**. reset the inner initial feasible solution to be consistent with the flow propagation constrain at the current relaxation. Set an inner iteration counter( or a convergence criterion).

**Step 2.1:** **Update**. Compute. Update by equation (27).

**Step 2.2: Direction Finding**. Based on, search for shortest routes for all OD

pairs over the physical network without time-space expansions. Perform an all-or-

nothing assignment following the link flow propagation constrain, yielding

sub-problem solution .

**Step 2.3: Line Search**. Solve the one-dimensional search problem (32) using a line

search procedure such as the bisection method and find the optimal step size .

**Step 2. 4: Move**. Find a new solution  by (33).

**Step 2. 5: Convergence Test for Inner Iteration**.

If  >, set, go to Step

2.1.; otherwise, set,, go to Step 3.

**Step 3: Convergence Test for Outer Iteration**. If, stop. The

current Solution , ,  is in a near optimal state; otherwise, set

 and go to Step 1.

In the above algorithm,  are solutions at outer iteration. is the estimation of link travel time at outer iteration . is the floored link travel time.  and are solutions at inner iteration . is the estimation of link travel time based on them. The number of inner iterations at each relaxation can also be pre-specified.

All inflow of the time-space link is zero and the corresponding link travel time is free flow travel time unless the link is assigned flow. The initial feasible solution in outer initialization can be found by performing all-or-nothing assignment on the dynamic shortest path based on free flow link travel time for all OD pairs.

At each relaxation, a time-space network is implicitly formed. The algorithm then performs FW iteration on the time-space network. The, and  at the  relaxation are calculated using the solution at the relaxation. Notice the solution  at the outer iteration cannot be used as the initial solution in the inner iteration of the  relaxation unless andat the two relaxations are exactly the same (which indicates the implicit time-space networks of the two relaxations are the same). If andat the two relaxations are different, the solution  at the outer iteration is not a feasible solution in the inner iteration of the  relaxation. A procedure to reset the initial feasible solution for the inner iteration at each relaxation is needed to make the initial feasible solution consistent with the current flow propagation. Ran’s (1996) original relaxation with FW algorithm lacks this critical step, which leads to wrong solution that is not consistent with the definition and constraint of DUO.

In inner iterations, the shortest paths with link cost term can be found by dynamic shortest path algorithm. Or they can be found by static shortest path algorithm with arrival time interval for each link recorded on the original network as shown in Fig. 3. When performing all-or-nothing assignment for,, the assigned value should be  instead of . As an example, Fig. 4 shows how  should be assigned on the time-space network. The time-space links on the dynamic shortest paths are highlighted as thick black. The assigned volumes resulting from  are ,, and .



Fig. 4 Assigned volumes on the time-space expansion network.

Since any route on the time-space network corresponds to a unique route on the original physical network, the assignment of any time-dependent demand  can also be performed on the original network if arrival time interval for the link is recorded. Fig. 5 shows how  should be assigned on the original network for the 3-link network. The same method is used to assign all time dependent demand,  on the original network.



Fig. 5. Assigned volumes on the original network.

The departure horizon is the same for all relaxations. The assignment horizon and the time-space network is fixed at each relaxation but may change from relaxation to relaxation. The assignment horizon and time-space network will finally tend to be

fixed. A necessary condition of the convergence of the algorithm is that the time-space

network remains the same at successive relaxations. As explained above, the

algorithm does not need time-space network. The introduction of time-space network

is for better explaining and understanding the solution process.

The actual assignment horizon at the end of the solution is,

where. When FIFO condition holds, departure horizon

 and assignment horizon  have the following relationship under DUO

status: , where =,  is the minimal actual

route travel time from origin to destination  at time .

**5. DYNAMIC SHORTEST-PATH ALGORITHM**

Let *G =* (*V, A*) be a directed network with node set *V* and arc set *A*. Any link *a* is indexed by (), or *a=*,where  and  are the ‘from node’ and ‘to node’. Denote link *a=*at time interval asor, node  at time interval as, the travel time on link**at time interval as ,

.is its floored value. Denote by  the minmum travel time

to destination departing node at time . The optimality condition of

minimum travel times are defined by the following functional form (Cooke and

Halsey, 1966; Ziliaskopoulos and Mahmassani, 1993; ISMAIL CHABINI, 1998)



When the FIFO condition is valid, the label-correcting algorithm can be generalized

to solve the time-dependent minimum paths (dynamic shortest paths) problem with

the same time complexity as the static shortest paths problem (Dreyfus,1969;

Kaufman and Smith,1993; ISMAIL CHABINI, 1998). Below we introduce an

algorithm to find the dynamic shortest path without time-space network. The

algorithm is a generalization of Moore algorithm to find the time-dependent minimum

paths problem. Readers may refer to Sheffi (1985) for the detailed description of

Moore algorithm. In order to describe our algorithm for dynamic shortest path, we

introduce the following denotations.

Denote =, where, and

=

Further denote =,, where

=

Denote 

Let, or 

Our algorithm to find the dynamic shortest path between any node *r* and *s* at timeis described as follows:

**Step 0**: Initialization.

Set =0, , =, =,=0,,.

Set and =.

**Step 1**: Set =, choose ** such that

=min.

Label, ,.

Set= and =. If =or =

or, stop; otherwise, go to Step 2.

**Step 2**:

**Step 2.1**: Search among and choose such that 



Label,,.

**Step 2.2**:Set =. Set =. If =or = or, stop; otherwise, go to Step 2.1

The above shortest path algorithm is the forward label-correcting method. It finds the dynamic shortest path from a given origin  at time to any other nodes in the network. A travel cost  is associated with each link *a=*at. Each

node  has three labels: , and .  is the minimum cost from the origin

node to node  along the shortest path at .  is the time interval when one

departing node  at  and traveling along the shortest path reaches node . 

is the node just preceding node  along the shortest path. A sequence list is used to

help keep track of the nodes. The list includes all the nodes that have yet to be

examined as well as the nodes requiring further examination.

In initialization, the algorithm sets allandto infinity and all to zero. And place the origin nodeon the sequence list with label =0, . Each iteration starts with the selection of a node  from the sequence list for examination. All nodes,, that can be reach from  by traversing only a single link are tested in the examination process. If the minimum path to  through  at is shorter than the previous path to , then and are updated. In other words, if + <, then the current shortest path form the origin node to  can be improved by going through node . To reflect this change, the label list is updated by setting :=+, :=+, the predecessor list is updated by setting : =, and the sequence list is updated by adding to it. Once all the nodes  (that can be reached from) are tested, the examination of node  is complete and it is deleted from the sequence list. The algorithm terminates when the sequence list is empty. The dynamic shortest path from the origin at to any other node can be found by tracing the predecessor list back to the origin node. The corresponding time interval for each node on the shortest path is given by .

**6. A NUMERICAL EXAMPLE**

An example is presented below to show the application of the above algorithms.

The configuration of the network is shown in Fig.6. In the network, each link is

assumed as a one-lane street with a length of 0.5 mi. The free flow speed is assumed

to be 25 miles / hour. The following linear travel time function is used to enforce

FIFO condition:, where is the length of link,is free

flow speed, is link travel time on link at time , is number of vehicles

on link  at time . Four O-D pairs are considered. Five 20 s departure time

intervals are specified. The OD flows are 10 vehicle units per time interval. The O-D

pairs and the time-dependent O-D demand are shown in Table 1. In this example, the

departure horizon is 5 time increments, and the time increment is 20 seconds.



Fig.6. Simulation network for sample problem 1

Table 1 O-D pairs and time-dependent O-D demand for example 1

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| O-D | Departure time interval *k* | | | | |
| 1 | 2 | 3 | 4 | 5 |
| 1-9 | 10 | 10 | 10 | 10 | 10 |
| 9-1 | 10 | 10 | 10 | 10 | 10 |
| 3-7 | 10 | 10 | 10 | 10 | 10 |
| 7-3 | 10 | 10 | 10 | 10 | 10 |

The program of the algorithm was run on a computer with 1.5 GHz frequency processor. The inner iteration (FW algorithm) convergence test method was set as a prespecified number. The outer iteration (Relaxation) convergence test method was set as 

where is the actual travel time difference of link *a* at time *k*

between successive relaxations. The operation of the program is shown in Table 2.

Table 5.2. Convergence criterion and computation time for sample problem 5.1

|  |  |  |  |
| --- | --- | --- | --- |
| Inner iteration convergence criterion | Outer iteration convergence criterion | Total relaxations  (Outer iterations) | Total computation time (minute) |
| *n*=4 | 0.002 | 8 | 25.8 |

The assignment horizon *K* is found to be 21 time increments. Table 4a shows the

output of . Table 4 shows the output of, , links on each path and

the arrival time interval for each link on a path.

We take the following examples to verify that the solution satisfies the constraints and the dynamic user-optimal conditions.

Path flow conservation constraint (7):

**=**

**=**3.4424+1.865+3.1786+1.1275+0.2891+0.0974

=10

Link inflow conservation constraint (8):

+=2.1353+2.1353=4.2706=

Link outflow conservation constraint (9):

+=2.1353+2.1353=4.2706=

Node flow conservation constraint (10):

=4.7216+0+5.2784=10

=3.3945+3.1047+3.5008=10

Link flow propagation constraint (11):

===2.1353

===2.1353

Where =1.2428 minutes. For a time increment of 20 seconds, =4.

The link state equation (12b):

=+= 4.3082 + = 8.5788

The actual travel times on the used paths from origin 1 toward destination 9 departing at time increment 1 are as follows:

**=**

**=** 1.2232+1.2171+1.2171+1.2242

**=** 4.8816 minutes

Similarly, we have 4.8888 minutes, 4.8841 minutes, 4.878 minutes, 4.8871 minutes, 4.8798 minutes. They are nearly equal, which is consistent with the DUO route choice condition.

As can be checked in the same way, all the solution output satisfies the constraints and the ideal dynamic user optimal conditions. This verifies the rationale of the above model and solution algorithm.

**7. CONCLUSIONS**

In this paper, we have presented a new relaxation algorithm to solve the link-based variational inequality model of ideal dynamic user optimal route choice problem based on previous studies. It is a modified version of Ran’s algorithm and has corrected some critical mistakes in the original algorithm. The algorithm does not need time-space network expansion and is efficient in solving real problem.

Further research can be conducted in several directions: 1) consider the stochastic factors in link travel times and develop stochastic DUO models and solution algorithms; 2) combine other stages of travel choice and develop combined dynamic transportation network models and solution algorithms; 2) adopting accelerating techniques to accelerate the convergence of the algorithm.

**Competing interests**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

**Availability of Data and Materials**

The datasets generated and/or analyzed during the current study are not publicly available due to moderate confidentiality but are available from the corresponding author on reasonable request.

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**Authors' contributions**

All authors wrote the main text and reviewed the manuscript. Tianze Xu made the figures and tables.

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Table 4. The resultant path flowand path travel time  for sample problem 1.

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| |  |  |  |  |  |  |  |  |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | | Path number | *r* | *s* | *k* |  |  | Links on the path | | | | Arrival time for each link  on the path | | | | | 1 | 1 | 9 | 1 | 3.4424 | 4.8816 | 5 | 15 | 23 | 24 | 1 | 5 | 9 | 13 | | 2 | 1 | 9 | 1 | 1.865 | 4.8888 | 3 | 7 | 14 | 19 | 1 | 5 | 9 | 13 | | 3 | 1 | 9 | 1 | 3.1786 | 4.8841 | 3 | 4 | 9 | 19 | 1 | 5 | 9 | 13 | | 4 | 1 | 9 | 1 | 1.1275 | 4.878 | 5 | 13 | 17 | 24 | 1 | 5 | 9 | 13 | | 5 | 1 | 9 | 1 | 0.2891 | 4.8871 | 3 | 7 | 17 | 24 | 1 | 5 | 9 | 13 | | 6 | 1 | 9 | 1 | 0.0974 | 4.8798 | 5 | 13 | 14 | 19 | 1 | 5 | 9 | 13 | | 7 | 1 | 9 | 2 | 3.4424 | 4.9728 | 5 | 15 | 23 | 24 | 2 | 6 | 10 | 14 | | 8 | 1 | 9 | 2 | 1.865 | 4.9737 | 3 | 7 | 14 | 19 | 2 | 6 | 10 | 14 | | 9 | 1 | 9 | 2 | 2.3457 | 4.9516 | 3 | 4 | 9 | 19 | 2 | 6 | 10 | 14 | | 10 | 1 | 9 | 2 | 1.9793 | 4.9777 | 5 | 13 | 17 | 24 | 2 | 6 | 10 | 14 | | 11 | 1 | 9 | 2 | 0.2703 | 4.9827 | 3 | 7 | 17 | 24 | 2 | 6 | 10 | 14 | | 12 | 1 | 9 | 2 | 0.0974 | 4.9686 | 5 | 13 | 14 | 19 | 2 | 6 | 10 | 14 | | 13 | 1 | 9 | 3 | 3.0312 | 5.0523 | 5 | 15 | 23 | 24 | 3 | 7 | 11 | 15 | | 14 | 1 | 9 | 3 | 1.865 | 5.0622 | 3 | 7 | 14 | 19 | 3 | 7 | 11 | 15 | | 15 | 1 | 9 | 3 | 3.1398 | 5.0339 | 3 | 4 | 9 | 19 | 3 | 7 | 11 | 15 | | 16 | 1 | 9 | 3 | 1.9249 | 5.0657 | 5 | 13 | 17 | 24 | 3 | 7 | 11 | 15 | | 17 | 1 | 9 | 3 | 0.039 | 5.0583 | 5 | 13 | 14 | 19 | 3 | 7 | 11 | 15 | | 18 | 1 | 9 | 4 | 3.5008 | 5.1337 | 5 | 15 | 23 | 24 | 4 | 8 | 12 | 16 | | 19 | 1 | 9 | 4 | 2.1353 | 5.1539 | 3 | 7 | 14 | 19 | 4 | 8 | 12 | 16 | | 20 | 1 | 9 | 4 | 3.1431 | 5.119 | 3 | 4 | 9 | 19 | 4 | 8 | 12 | 16 | | 21 | 1 | 9 | 4 | 1.163 | 5.1395 | 5 | 13 | 17 | 24 | 4 | 8 | 12 | 16 | | 22 | 1 | 9 | 4 | 0.0578 | 5.1392 | 5 | 13 | 14 | 19 | 4 | 8 | 12 | 16 | | 23 | 1 | 9 | 5 | 3.5398 | 5.1328 | 5 | 15 | 23 | 24 | 5 | 9 | 13 | 17 | | 24 | 1 | 9 | 5 | 1.865 | 5.1522 | 3 | 7 | 14 | 19 | 5 | 9 | 13 | 17 | | 25 | 1 | 9 | 5 | 3.3945 | 5.122 | 3 | 4 | 9 | 19 | 5 | 9 | 13 | 17 | | 26 | 1 | 9 | 5 | 1.1818 | 5.1352 | 5 | 13 | 17 | 24 | 5 | 9 | 13 | 17 | | 27 | 1 | 9 | 5 | 0.0188 | 5.14 | 5 | 13 | 14 | 19 | 5 | 9 | 13 | 17 | | 28 | 9 | 1 | 1 | 3.5398 | 4.8805 | 22 | 21 | 16 | 6 | 1 | 5 | 9 | 13 | | 29 | 9 | 1 | 1 | 2.1541 | 4.8917 | 20 | 12 | 8 | 1 | 1 | 5 | 9 | 13 | | 30 | 9 | 1 | 1 | 3.1786 | 4.8855 | 20 | 10 | 2 | 1 | 1 | 5 | 9 | 13 | | 31 | 9 | 1 | 1 | 1.1275 | 4.8722 | 22 | 18 | 11 | 6 | 1 | 5 | 9 | 13 | | 32 | 9 | 1 | 2 | 3.4424 | 4.97 | 22 | 21 | 16 | 6 | 2 | 6 | 10 | 14 | | 33 | 9 | 1 | 2 | 2.1353 | 4.9793 | 20 | 12 | 8 | 1 | 2 | 6 | 10 | 14 | | 34 | 9 | 1 | 2 | 2.3457 | 4.9545 | 20 | 10 | 2 | 1 | 2 | 6 | 10 | 14 | | 35 | 9 | 1 | 2 | 2.0766 | 4.9664 | 22 | 18 | 11 | 6 | 2 | 6 | 10 | 14 | | 36 | 9 | 1 | 3 | 3.0703 | 5.0495 | 22 | 21 | 16 | 6 | 3 | 7 | 11 | 15 | | 37 | 9 | 1 | 3 | 1.865 | 5.0678 | 20 | 12 | 8 | 1 | 3 | 7 | 11 | 15 | | 38 | 9 | 1 | 3 | 3.1398 | 5.0367 | 20 | 10 | 2 | 1 | 3 | 7 | 11 | 15 | | 39 | 9 | 1 | 3 | 1.9249 | 5.0544 | 22 | 18 | 11 | 6 | 3 | 7 | 11 | 15 | | 40 | 9 | 1 | 4 | 3.4424 | 5.1308 | 22 | 21 | 16 | 6 | 4 | 8 | 12 | 16 | | 41 | 9 | 1 | 4 | 1.8839 | 5.1559 | 20 | 12 | 8 | 1 | 4 | 8 | 12 | 16 | | 42 | 9 | 1 | 4 | 3.3945 | 5.1244 | 20 | 10 | 2 | 1 | 4 | 8 | 12 | 16 | | 43 | 9 | 1 | 4 | 1.2604 | 5.1297 | 22 | 18 | 11 | 6 | 4 | 8 | 12 | 16 | | 44 | 9 | 1 | 4 | 0.0188 | 5.1438 | 22 | 18 | 8 | 1 | 4 | 8 | 12 | 16 | | 45 | 9 | 1 | 5 | 3.4814 | 5.1315 | 22 | 21 | 16 | 6 | 5 | 9 | 13 | 17 | | 46 | 9 | 1 | 5 | 1.865 | 5.1515 | 20 | 12 | 8 | 1 | 5 | 9 | 13 | 17 | | 47 | 9 | 1 | 5 | 3.3945 | 5.1259 | 20 | 10 | 2 | 1 | 5 | 9 | 13 | 17 | | 48 | 9 | 1 | 5 | 1.2402 | 5.1314 | 22 | 18 | 11 | 6 | 5 | 9 | 13 | 17 | | 49 | 9 | 1 | 5 | 0.0188 | 5.1419 | 22 | 18 | 8 | 1 | 5 | 9 | 13 | 17 | | 50 | 3 | 7 | 1 | 3.4424 | 4.8819 | 9 | 19 | 22 | 21 | 1 | 5 | 9 | 13 | | 51 | 3 | 7 | 1 | 1.865 | 4.8888 | 2 | 7 | 11 | 15 | 1 | 5 | 9 | 13 | | 52 | 3 | 7 | 1 | 3.1786 | 4.8841 | 2 | 1 | 5 | 15 | 1 | 5 | 9 | 13 | | 53 | 3 | 7 | 1 | 1.1275 | 4.878 | 9 | 12 | 17 | 21 | 1 | 5 | 9 | 13 | | 54 | 3 | 7 | 1 | 0.2891 | 4.8871 | 2 | 7 | 17 | 21 | 1 | 5 | 9 | 13 | | 55 | 3 | 7 | 1 | 0.0974 | 4.8798 | 9 | 12 | 11 | 15 | 1 | 5 | 9 | 13 | | 56 | 3 | 7 | 2 | 3.4424 | 4.9728 | 9 | 19 | 22 | 21 | 2 | 6 | 10 | 14 | | 57 | 3 | 7 | 2 | 1.865 | 4.9737 | 2 | 7 | 11 | 15 | 2 | 6 | 10 | 14 | | 58 | 3 | 7 | 2 | 2.3457 | 4.9516 | 2 | 1 | 5 | 15 | 2 | 6 | 10 | 14 | | 59 | 3 | 7 | 2 | 1.9793 | 4.9777 | 9 | 12 | 17 | 21 | 2 | 6 | 10 | 14 | | 60 | 3 | 7 | 2 | 0.2703 | 4.9827 | 2 | 7 | 17 | 21 | 2 | 6 | 10 | 14 | | 61 | 3 | 7 | 2 | 0.0974 | 4.9686 | 9 | 12 | 11 | 15 | 2 | 6 | 10 | 14 | | 62 | 3 | 7 | 3 | 3.0312 | 5.0523 | 9 | 19 | 22 | 21 | 3 | 7 | 11 | 15 | | 63 | 3 | 7 | 3 | 1.865 | 5.0622 | 2 | 7 | 11 | 15 | 3 | 7 | 11 | 15 | | 64 | 3 | 7 | 3 | 3.1398 | 5.0339 | 2 | 1 | 5 | 15 | 3 | 7 | 11 | 15 | | 65 | 3 | 7 | 3 | 1.9249 | 5.0657 | 9 | 12 | 17 | 21 | 3 | 7 | 11 | 15 | | 66 | 3 | 7 | 3 | 0.039 | 5.0583 | 9 | 12 | 11 | 15 | 3 | 7 | 11 | 15 | | 67 | 3 | 7 | 4 | 3.5008 | 5.1337 | 9 | 19 | 22 | 21 | 4 | 8 | 12 | 16 | | 68 | 3 | 7 | 4 | 2.1353 | 5.1539 | 2 | 7 | 11 | 15 | 4 | 8 | 12 | 16 | | 69 | 3 | 7 | 4 | 3.1431 | 5.119 | 2 | 1 | 5 | 15 | 4 | 8 | 12 | 16 | | 70 | 3 | 7 | 4 | 1.163 | 5.1395 | 9 | 12 | 17 | 21 | 4 | 8 | 12 | 16 | | 71 | 3 | 7 | 4 | 0.0578 | 5.1392 | 9 | 12 | 11 | 15 | 4 | 8 | 12 | 16 | | 72 | 3 | 7 | 5 | 3.5398 | 5.1328 | 9 | 19 | 22 | 21 | 5 | 9 | 13 | 17 | | 73 | 3 | 7 | 5 | 1.865 | 5.1522 | 2 | 7 | 11 | 15 | 5 | 9 | 13 | 17 | | 74 | 3 | 7 | 5 | 3.3945 | 5.122 | 2 | 1 | 5 | 15 | 5 | 9 | 13 | 17 | | 75 | 3 | 7 | 5 | 1.1818 | 5.1352 | 9 | 12 | 17 | 21 | 5 | 9 | 13 | 17 | | 76 | 3 | 7 | 5 | 0.0188 | 5.14 | 9 | 12 | 11 | 15 | 5 | 9 | 13 | 17 | | 77 | 7 | 3 | 1 | 3.5398 | 4.8805 | 23 | 24 | 20 | 10 | 1 | 5 | 9 | 13 | | 78 | 7 | 3 | 1 | 2.1541 | 4.8917 | 16 | 13 | 8 | 4 | 1 | 5 | 9 | 13 | | 79 | 7 | 3 | 1 | 3.1786 | 4.8855 | 16 | 6 | 3 | 4 | 1 | 5 | 9 | 13 | | 80 | 7 | 3 | 1 | 1.1275 | 4.8722 | 23 | 18 | 14 | 10 | 1 | 5 | 9 | 13 | | 81 | 7 | 3 | 2 | 3.4424 | 4.97 | 23 | 24 | 20 | 10 | 2 | 6 | 10 | 14 | | 82 | 7 | 3 | 2 | 2.1353 | 4.9793 | 16 | 13 | 8 | 4 | 2 | 6 | 10 | 14 | | 83 | 7 | 3 | 2 | 2.3457 | 4.9545 | 16 | 6 | 3 | 4 | 2 | 6 | 10 | 14 | | 84 | 7 | 3 | 2 | 2.0766 | 4.9664 | 23 | 18 | 14 | 10 | 2 | 6 | 10 | 14 | | 85 | 7 | 3 | 3 | 3.0703 | 5.0495 | 23 | 24 | 20 | 10 | 3 | 7 | 11 | 15 | | 86 | 7 | 3 | 3 | 1.865 | 5.0678 | 16 | 13 | 8 | 4 | 3 | 7 | 11 | 15 | | 87 | 7 | 3 | 3 | 3.1398 | 5.0367 | 16 | 6 | 3 | 4 | 3 | 7 | 11 | 15 | | 88 | 7 | 3 | 3 | 1.9249 | 5.0544 | 23 | 18 | 14 | 10 | 3 | 7 | 11 | 15 | | 89 | 7 | 3 | 4 | 3.4424 | 5.1308 | 23 | 24 | 20 | 10 | 4 | 8 | 12 | 16 | | 90 | 7 | 3 | 4 | 1.8839 | 5.1559 | 16 | 13 | 8 | 4 | 4 | 8 | 12 | 16 | | 91 | 7 | 3 | 4 | 3.3945 | 5.1244 | 16 | 6 | 3 | 4 | 4 | 8 | 12 | 16 | | 92 | 7 | 3 | 4 | 1.2604 | 5.1297 | 23 | 18 | 14 | 10 | 4 | 8 | 12 | 16 | | 93 | 7 | 3 | 4 | 0.0188 | 5.1438 | 23 | 18 | 8 | 4 | 4 | 8 | 12 | 16 | | 94 | 7 | 3 | 5 | 3.4814 | 5.1315 | 23 | 24 | 20 | 10 | 5 | 9 | 13 | 17 | | 95 | 7 | 3 | 5 | 1.865 | 5.1515 | 16 | 13 | 8 | 4 | 5 | 9 | 13 | 17 | | 96 | 7 | 3 | 5 | 3.3945 | 5.1259 | 16 | 6 | 3 | 4 | 5 | 9 | 13 | 17 | | 97 | 7 | 3 | 5 | 1.2402 | 5.1314 | 23 | 18 | 14 | 10 | 5 | 9 | 13 | 17 | | 98 | 7 | 3 | 5 | 0.0188 | 5.1419 | 23 | 18 | 8 | 4 | 5 | 9 | 13 | 17 | |