Amortizing loans under arbitrary discount functions

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Abstract

A general methodology for loan amortization under arbitrary discount functions is discussed. It is shown that it is always possible to uniquely define a scheme for constructing the loan amortization schedule with an arbitrary assigned discount function. It is also shown that even if the loan amortization is carried out from the sequence of principal payments and the sequence of accrued interest, the underlying discount function can be uniquely determined at the maturities corresponding to the installment payment dates. As a special case of the proposed approach, we derive the amortization method according to the law of simple interest.

Keywords: discount function, amortizing loans, outstanding balance, amortization schedule, compound interest, simple interest.
JEL Code: G31, G32, G33.

1 Introduction

The past decade has seen a renewed interest in Italy for the design of amortizing loans, following an important debate on the consistency of the law of compound interest, also known as the law of exponential capitalization, with the principle, enshrined in Italian law, that interest produced in one period of time cannot produce interest in subsequent periods, a phenomenon called anatocism (for a review, see Annibali et al. (2017)). Two main points are debated. The first is whether anatocism is present when amortizing loans are designed according to the law of compound interest (Fersini and Olivieri, 2015). The second point concerns the possibility of exploring different amortization methods, with a focus on amortization methods consistent with the law of simple interest, also called the law of linear capitalization (Mari and Aretusi, 2018, 2019).

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The problem also has international significance. Several international disputes have shown a general tendency not to accept compound interest (for a comprehensive review see Sinclair (2016)). This is motivated by the fact that the exponential nature of the law of compound interest has an explosive effect in the medium to long term, a factor that greatly affects the risk of default and, therefore, the ability to efficiently plan investments (Cerina, 1993).

In this paper, we will focus on the second point of this debate, namely the possibility of exploring different amortization methods, by providing a general methodology for designing amortizing loans according to arbitrary financial laws, i.e., under arbitrary discount functions. It will be shown that it is always possible to unambiguously define a scheme for constructing the loan amortization schedule with an assigned arbitrary discount function. Moreover, to monitor the interest generation process and understand the interest flow over time an extended amortization schedule is introduced. Like a macro lens to uncover the intimate structure of the amortizing loan, the extended amortization schedule contains all the information needed to fully understand the loan repayment process.

As a consequence of the proposed general methodology, two significant results are presented in this paper.

The first result allows us to design loan amortization using two different but equivalent schemes. In the first scheme, loan amortization is carried out starting from the knowledge of the discount function and the sequence of the loan installments; in the second scheme, loan amortization is performed starting from the sequence of principal payments and the sequence of accrued interest. It will be shown that even if the second scheme is adopted, the underlying discount function can be uniquely determined at the maturities corresponding to the installment payment dates. These findings will be presented more formally in Theorem 1 and Theorem 2.

As a second result, we derive the amortization method under the law of simple interest as a particular case of the proposed methodology. In this method the generation of "interest on interest" is precluded. In fact, we will show that under the law of simple interest, accrued interest is calculated on the present value of the outstanding balance and not on the outstanding balance itself as in the compound interest method of amortization. In this way, the interest component is removed from the outstanding balance and the interest compounding over time is avoided.

This study provides a conceptual framework for designing amortization methods under arbitrary financial laws that appropriately extend the most common way of amortizing a loan, based on the law of compound interest, by including the latter as a special case.

The paper is organized as follows. Section 2 outlines the general methodology for loan design. Section 3 illustrates the standard amortization method. Section 4 presents the extended amortization schedule. In Section 5, Theorem 1 and Theorem 2 are stated and proved. Section 6 presents the loan amortization method under the law of compound interest as a particular case of our methodology. As a further special case, Section 7 provides the loan amortization method under the law of simple interest. The "interest on interest" question is discussed in both Section 6 and Section 7. Some further remarks on loan amortization under a linear capitalization scheme are presented in Section 8.

2 Designing amortizing loans: a general methodology

In this section we provide a general methodology to design amortizing loans. The main goal is to show how to amortize a loan and properly construct amortization schedules under arbitrary discount functions.

2.1 Some basic results

Let us denote by $v(0, T)$ the discount function observed at the current time $t = 0$ (the present). It denotes the value at time $t = 0$ of one unit of money payable at a later time T and can incorporate credit risk (Duffie and Singleton, 1999; Mari and Renò, 2005). By standard no-arbitrage arguments it follows that the discount function must be a strictly positive function (Duffie and Singleton, 1999), i.e.,

$$v(0, T) > 0, \quad T \geq 0, \tag{1}$$

with

$$v(0, 0) = 1. \tag{2}$$

We assume that the discount function is independent of the amount (Richardson, 1946): if x_T denotes a monetary amount payable at time T ($T \geq 0$), its value at time $t = 0$, x_0, is given by

$$x_0 = x_T v(0, T). \tag{3}$$

The monetary amount x_0 is the present value of the amount x_T available at time T (spot evaluation).

The knowledge of the discount function allows us to define an equivalence relationship between monetary amounts due at different future times (forward evaluation). Indeed, let us denote by x_{T_1} a sum of money due at time T_1 ($T_1 \geq 0$) and by x_{T_2} a sum of money due at time T_2 ($T_2 \geq 0$), they are assumed to be financially equivalent if and only if they have the same present value (Richardson, 1946), i.e.,

$$x_{T_2} v(0, T_2) = x_{T_1} v(0, T_1). \tag{4}$$

The rationale is that if x_{T_1} and x_{T_2} satisfy Equation (4), they can be transformed into each other. Stated in a different way, x_{T_1} and x_{T_2} are financially equivalent if and only if they differ only in the interest component. Indeed, Equation (4) is equivalent to the following relationship

$$x_{T_2} = x_{T_1} \frac{v(0, T_1)}{v(0, T_2)}, \tag{5}$$

showing that x_{T_2}, the financially equivalent amount of x_{T_1}, can be determined by first discounting x_{T_1} from time T_1 to current time $t = 0$, thus eliminating the interest component, and then imputing accrued interest in the time interval $[0, T_2]$ by capitalizing the obtained value from time $t = 0$ to time T_2. It is straightforward to prove that the binary relation defined by Equation (5) is an equivalence relation. Indeed, it is trivially reflexive and symmetric. It is also a transitive relation because if $x_{T_1} v(0, T_1) = x_{T_2} v(0, T_2)$ and $x_{T_2} v(0, T_2) = x_{T_3} v(0, T_3)$, it follows that $x_{T_1} v(0, T_1) = x_{T_3} v(0, T_3)$, regardless of the temporal ordering of T_1, T_2, and T_3. The binary relation defined by Equation (5), being reflexive, symmetric and transitive, provides an equivalence relation between amounts of money due at different times.

The extension of the definition of financial equivalence to cash flows is straightforward. Indeed, let us consider the cash flow,

$$\mathbf{x} = \{x_{t_1}, x_{t_2}, \cdots, x_{t_n}\}, \tag{6}$$

where $0 \leq t_1 < t_2 < \cdots < t_n$. The amount S_T at time $T \geq 0$ is financially equivalent to the cash flow $\mathbf{x}$ if and only if the present value of S_T is equal to the present value of $\mathbf{x}$, that is, if and only if the following relationship holds (Richardson, 1946; Broverman, 2017),

$$S_T v(0, T) = \sum_{k=1}^{n} x_{t_k} v(0, t_k), \tag{7}$$

from which we get

$$S_T = \frac{1}{v(0,T)} \sum_{k=1}^{n} x_{t_k} v(0,t_k). \tag{8}$$

Again, the rationale is that if Equation (8) is satisfied, the amount S_T can be transformed into the cash flow $\mathbf{x}$ and vice versa, because each term in the r.h.s. of Equation (8), i.e. $x_{t_k} v(0,t_k)/v(0,T)$, has only one financially equivalent amount x_{t_k} at time t_k. Equation (8) has a very interesting financial interpretation: each term x_{t_k} is first discounted from time t_k to time $t = 0$ to eliminate the interest component, then it is capitalized from time $t = 0$ to time T to include the interest accrued in the time interval $[0, T]$. In the case $T = 0$, Equation (8) becomes

$$S_0 = \sum_{k=1}^{n} x_{t_k} v(0,t_k). \tag{9}$$

It should be emphasized that Equation (5) and Equation (8) can also be derived from no-arbitrage arguments (Brigo and Mercurio, 2006).

Finally, we close this section by pointing out that the equivalence relationship is established at time $t = 0$ on the basis of the information contained in the discount function at time $t = 0$ and that it is not necessarily preserved over time. Due to the unpredictability of the time evolution of the discount function, monetary amounts that are financially equivalent at time $t = 0$ may no longer be financially equivalent at a later time.

2.2 Designing amortizing loans

The methodology outlined in the previous section can be employed to design amortizing loans under arbitrary discount functions. To show this, let us consider at time $t = 0$ a loan with a principal amount S_0 that will be repaid with a series of nonnegative future installments,

$$\mathbf{r} = \{R_1, R_2, \cdots, R_n\}, \tag{10}$$

scheduled at regular time intervals $1, 2, \cdots, n$, with $R_n > 0$. If we denote by $v(0, T)$ the discount function at time $t = 0$, the following relationship must hold, as a consequence of Equation (9),

$$S_0 = \sum_{k=1}^{n} R_k v(0,k). \tag{11}$$

Let us denote by M_k the outstanding balance after the payment of the k-th installment. By definition, M_k, $k = 1, 2, \cdots, n-1$, is the monetary amount due at time k that is financially equivalent to receiving the stream of future installments $\mathbf{r_k} = \{R_{k+1}, R_{k+2}, \cdots, R_n\}$. It can be computed from Equation (8), thus obtaining

$$M_k = \frac{1}{v(0,k)} \sum_{j=k+1}^{n} R_j v(0,j), \quad k = 1, 2, \cdots, n-1. \tag{12}$$

Equation (12) clearly shows that the values of the outstanding balance, M_k, $k = 1, 2, \cdots, n-1$, are strictly positive. Of course it must be $M_n = 0$ because after the last payment at time n the outstanding balance is zero. Moreover, since at time $t = 0$ the outstanding balance coincides with the principal amount, we pose $M_0 = S_0$. We note that each term R_j in Equation (12) is first discounted at time $t = 0$ to eliminate the interest component, then it is capitalized from time 0 to time k to include the interest accrued in the time interval $[0, k]$. Equation (12) provides the so-called *prospective* method for computing the outstanding balance. In addition, since from Equation (11) we get

$$\sum_{j=k+1}^{n} R_j v(0,j) = S_0 - \sum_{j=1}^{k} R_j v(0,j), \quad k = 1, 2, \cdots, n-1, \tag{13}$$

we can recast Equation (12) in the following useful form

$$M_k = \frac{1}{v(0,k)} \left(S_0 - \sum_{j=1}^{k} R_j v(0,j) \right), \quad k = 1, 2, \cdots, n-1, \tag{14}$$

that provides the so-called *retrospective* method for computing the outstanding balance.

The dynamics of the outstanding balance can be also determined recursively by comparing the outstanding balance at time $k-1$ with the outstanding balance at time k, thus obtaining

$$M_{k-1} = \frac{v(0,k)}{v(0,k-1)} (M_k + R_k), \quad k = 1, 2, \cdots, n. \tag{15}$$

It should be noted that Equation (15) could have been obtained directly as a consequence of the financial equivalence between the outstanding balance M_{k-1} at time $k-1$ and the amount $M_k + R_k$ at time k, which is the sum of the outstanding balance at time k and the k-th installment. Finally, it is worth pointing out that Equation (12) can be recovered as the only solution of Equation (15) under the terminal condition $M_n = 0$, thus proving the equivalence of the above representations of the outstanding balance.

3 The standard amortization schedule

In the following, we use the notation $v(0, T, T')$ to denote the implied forward discount function at time $t = 0$ defined by the ratio

$$v(0, T, T') = \frac{v(0, T')}{v(0, T)}, \quad 0 \leq T \leq T'. \tag{16}$$

We note that for $T = 0$ the implied forward discount function coincides with the discount function. Using the implied forward discount function, Equation (15) can be cast in a more expressive form[1],

$$M_k = M_{k-1} + i(0, k-1, k)M_{k-1} - R_k, \tag{17}$$

where

$$i(0, k-1, k) = \frac{1}{v(0, k-1, k)} - 1, \tag{18}$$

is the forward rate and quantifies the interest accrued in the time interval $[k-1, k]$. In this regard, we note that the dynamics of the outstanding balance has a simple structure driven by two components, namely accrued interest and loan repayments. If we recast Equation (17) in the following form

$$R_k = M_{k-1} - M_k + i(0, k-1, k)M_{k-1}, \tag{19}$$

we can see that each installment R_k can be decomposed into two components, namely

$$R_k = C_k + I_k, \tag{20}$$

where

$$C_k = M_{k-1} - M_k, \tag{21}$$

and

$$I_k = i(0, k-1, k)M_{k-1}. \tag{22}$$

Equation (21) shows that C_k quantifies the change in the outstanding balance over the time interval $[k-1, k]$ and Equation (22) shows that I_k is the interest accrued over the same time interval. Finally, it is straightforward to show that the outstanding balance, M_k, can also be expressed as

$$M_k = S_0 - \sum_{j=1}^{k} C_j, \tag{23}$$

[1] Unless otherwise stated, the index k takes values from 1 to n.

and that the following relationship holds,

$$\sum_{k=1}^{n} C_k = S_0. \tag{24}$$

For this reason, in the literature the numbers C_k $(k = 1, 2, \cdots, n)$ are called principal payments.

The standard amortization schedule is a table that shows all the financial information of the loan mentioned above (Pressacco et al., 2022; Broverman, 2017). In particular, the amortization schedule exhibits for each k the vector

$$\phi_k = \{k, R_k, C_k, I_k, M_k\}, \tag{25}$$

starting from the initial vector $\phi_0 = \{0, 0, 0, 0, S_0\}$ which is reported in the first row of the table. All the financial quantities contained in ϕ_k can be easily computed in the proposed approach. For example (but this is not the only way), under an assigned discount function, the amortization schedule can be constructed iteratively as follows: starting from the principal amount $M_0 = S_0$ and the loan repayment plan, R_k, obtained as a solution of Equation (11) with $R_k \geq 0$ and $R_n > 0$, accrued interest I_k can be calculated by using Equation (22); then C_k can be obtained from Equation (20) by taking the difference

$$C_k = R_k - I_k, \tag{26}$$

and, finally, M_k can be computed from Equation (21),

$$M_k = M_{k-1} - C_k. \tag{27}$$

3.1 A numerical example

To illustrate the standard amortization method, consider a loan with principal amount $S_0 = 100$ repaid with an annuity consisting of $n = 5$ equal installments due at regular intervals $k = 1, 2, \cdots, 5$. The values of the discount function at time $t = 0$ are reported in Table 1.

k	1	2	3	4	5
$v(0,k)$	0.9346	0.8573	0.7513	0.7084	0.6560

Table 1: The discount function.

The amount of each payment can be computed by using Equation (11), thus getting

$$R = \frac{S_0}{\sum_{k=1}^{n} v(0,k)}. \tag{28}$$

The standard amortization schedule, obtained by following the iterative procedure discussed above, is given in Table 2.

k	R_k	C_k	I_k	M_k
0	0	0	0	100
1	25.59	18.59	7.00	81.41
2	25.59	18.25	7.34	63.16
3	25.59	16.68	8.91	46.47
4	25.59	22.78	2.81	23.70
5	25.59	23.70	1.89	0

Table 2: The standard amortization schedule.

4 The extended amortization schedule

Before proceeding further, it is necessary to explore one aspect that is definitely relevant to our analysis. Is it correct to identify accrued interest with paid interest? Looking at Equation (11), we can see that each term R_k is discounted at time $t = 0$. Discounting removes the interest component from R_k, thus providing the portion of the principal that is actually repaid with the k-th installment (in concordance also with the decomposition of a loan into single-payment loans). In this picture, the interest content of each installment is then given by the difference $R_k - R_k v(0,k)$. Let us pose, therefore,

$$S_{0,k} = R_k v(0,k), \tag{29}$$

and

$$J_k = R_k\big[1 - v(0,k)\big], \tag{30}$$

to indicate, respectively, the portion of principal and the portion of interest actually paid with the k-th installment. In addition to the representation provided by the Equation (20), R_k also admits, therefore, the following decomposition

$$R_k = S_{0,k} + J_k. \tag{31}$$

Of course, $S_{0,k} \neq C_k$ and $J_k \neq I_k$, however, the following equalities hold,

$$\sum_{k=1}^{n} S_{0,k} = \sum_{k=1}^{n} C_k = S_0, \tag{32}$$

$$\sum_{k=1}^{n} J_k = \sum_{k=1}^{n} I_k. \tag{33}$$

as a consequence of Equation (11) and Equation (20). Since $S_{0,k}$ is the present value of R_k, it contains no interest and is, therefore, pure capital. For this reason, we will refer to the amounts $S_{0,k}$ $(k = 1, 2, \cdots, n)$ as principal "bare" payments.

The financial quantities we have just introduced, namely $S_{0,k}$ and J_k, allow for a meaningful representation of outstanding balance. In fact, by substituting Equation (31) into Equation (17) we get

$$M_k = M_{k-1} - S_{0,k} + I_k - J_k. \tag{34}$$

Since I_k is the interest accrued in the time interval $[k-1, k]$ and J_k is the amount of interest actually paid with the k-th installment, it follows that whenever $J_k < I_k$, the interest component of M_k increases by the amount $I_k - J_k$; if $J_k > I_k$, the interest component of M_k decreases by the amount $J_k - I_k$. Furthermore, since $C_k = M_{k-1} - M_k$, Equation (34) also provides the relationship between C_k and $S_{0,k}$, namely

$$C_k = S_{0,k} + J_k - I_k, \tag{35}$$

showing that C_k, despite being called principal payment, contains a well-defined interest component. Moreover, let us denote by $D_{0,k}$ the value of the principal not yet actually repaid with the first k installments, i.e, the difference between S_0 and the sum of the first k principal bare payments,

$$D_{0,k} = S_0 - \sum_{j=1}^{k} S_{0,j} = \sum_{j=k+1}^{n} S_{0,j}, \quad k = 1, 2, \cdots, n-1. \tag{36}$$

By substituting Equation (29) into Equation (12) we obtain a very expressive relationship between M_k and $D_{0,k}$, namely

$$M_k = \frac{D_{0,k}}{v(0,k)}, \tag{37}$$

or, equivalently,

$$D_{0,k} = M_k v(0,k), \tag{38}$$

showing that $D_{0,k}$ is the present value of the outstanding balance M_k. Of course, it is $D_{0,n} = 0$ and $D_{0,0} = S_0$. Since $D_{0,k}$ is the present value of M_k, it contains no interest and is, therefore, pure capital[2]. As a consequence, the difference $M_k - D_{0,k}$ quantifies the interest component in the outstanding balance. It is given by

$$M_k - D_{0,k} = \sum_{j=1}^{k} (I_j - J_j), \tag{39}$$

as it is straightforward to prove by recursively applying Equation (34).

Finally, since $C_k = M_{k-1} - M_k$ and $S_{0,k} = D_{0,k-1} - D_{0,k}$, we also obtain the following interesting picture: C_k is given by the difference between the outstanding balance at time $k-1$ and the outstanding balance at time k; $S_{0,k}$ is given by the difference between the present value of the outstanding balance at time $k-1$ and the present value of the outstanding balance at time k.

In the extended amortization schedule, we will provide synoptically all relevant financial information about the loan, showing explicitly for each k the vector

$$\phi_k^{\text{ext}} = \{k, R_k, C_k, I_k, M_k, S_{0,k}, J_k, D_{0,k}\}, \tag{40}$$

starting from the initial vector $\phi_0^{\text{ext}} = \{0, 0, 0, 0, S_0, 0, 0, S_0\}$ reported in the first row of the table. In the extended amortization schedule, the traditional schedule is shown to the left of the vertical bar. On the right-hand side some additional information is given concerning, for each epoch k, the financial quantities $S_{0,k}$, J_k and $D_{0,k}$. Like a macro lens to uncover the intimate structure of the amortizing loan, the part to the right of the vertical bar contains all the information needed to monitor the interest generation process and understand the interest flow over time.

4.1 A numerical example

Referring to the numerical example discussed in the previous section, the extended amortization schedule is shown in Table 3.

[2]We remark that, in the case of early repayment at time k, M_k (and not $D_{0,k}$, which is its present value) is the amount the borrower is required to repay to the lender.

k	R_k	C_k	I_k	M_k	$S_{0,k}$	J_k	$D_{0,k}$
0	0	0	0	100	0	0	100
1	25.59	18.59	7.00	81.41	23.92	1.67	76.08
2	25.59	18.25	7.34	63.16	21.94	3.65	54.14
3	25.59	16.68	8.91	46.47	19.23	6.36	34.92
4	25.59	22.78	2.81	23.70	18.13	7.46	16.79
5	25.59	23.70	1.89	0	16.79	8.80	0

Table 3: The extended amortization schedule.

5 Uncovering the financial law behind an amortizing loan

In this section, we discuss a loan amortization technique that can be configured as a second well-defined amortization scheme (Pressacco et al., 2022). With no apparent reference to an underlying discount function, in this scheme the input is given by the principal amount, S_0, the sequence of principal payments, C_k, and the sequence of accrued interest, I_k. To simplify the notation, let us pose

$$B_k = S_0 - \sum_{j=1}^{k} C_j, \quad B_0 = S_0. \tag{41}$$

We assume that the sequences of numbers C_k and I_k satisfy the following conditions:

(G1) $B_n = 0$;

(G2) $I_k = f(k)B_{k-1} \ \big(f(k) > -1\big), \quad k = 1, 2, \cdots, n$;

(G3) $C_k + I_k \geq 0, \quad k = 1, 2, \cdots, n-1, \quad C_n + I_n > 0$.

From this figure, the loan installments and outstanding balance are calculated as follows,

$$R_k = C_k + I_k, \tag{42}$$

and

$$M_k = M_{k-1} - C_k, \quad M_0 = S_0. \tag{43}$$

Condition *(G1)* ensures that $M_n = 0$, i.e.,

$$\sum_{k=1}^{n} C_k = S_0, \tag{44}$$

and that $M_k = B_k$; condition *(G2)* also allows for negative rates to be taken into account; condition *(G3)* ensures that the installments, R_k, are nonnegative with $R_n > 0$. Moreover, conditions *(G1)*-*(G3)* imply that

$$M_k > 0, \quad k = 1, 2, \cdots, n-1. \tag{45}$$

Indeed, if there is $\bar{k}$ such that $M_{\bar{k}} \leq 0$, $\bar{k} = 1, 2, \cdots, n-1$, it follows that $M_{\bar{k}+1} = (1 + f(\bar{k}+1))M_{\bar{k}} - R_{\bar{k}} \leq 0$ and so on until time n where $M_n < 0$ since $R_n > 0$.

We will show that even if this second scheme is adopted, the underlying discount function can be uniquely determined at the maturities corresponding to the installment payment dates. In addition, we will show that this second amortization schemes is equivalent to the scheme discussed in Section 3. These results are more formally described by the following Theorem 1 and Theorem 2. In particular, Theorem 1 summarizes the findings obtained in the Section 3.

Theorem 1 *Let S_0 a strictly positive number and consider for $k = 1, 2, \cdots, n$: (i) a sequence of strictly positive numbers $v(0, k)$; (ii) a sequence of nonnegative numbers R_k, with $R_n > 0$, such that*

$$S_0 = \sum_{k=1}^{n} R_k v(0, k). \tag{46}$$

If M_k is computed according to

$$M_k = \frac{1}{v(0, k)} \sum_{j=k+1}^{n} R_j v(0, j), \quad k = 1, 2, \cdots, n-1, \tag{47}$$

and $M_n = 0$, then there exist a unique sequence of numbers C_k and a unique sequence of numbers I_k, $k = 1, 2, \cdots, n$, satisfying conditions (G1)-(G3), such that the amortizing schedule can be computed according to the following rules,

$$R_k = C_k + I_k, \tag{48}$$

and

$$M_k = M_{k-1} - C_k, \quad M_0 = S_0. \tag{49}$$

The proof of Theorem 1 is provided in Appendix A.

The converse is also true. Indeed, we will prove that the following proposition holds.

Theorem 2 *Let S_0 a strictly positive number and consider for $k = 1, 2, \cdots, n$: (i) a sequence of numbers C_k and (ii) a sequence of numbers I_k satisfying conditions (G1)-(G3). If the amortizing schedule is computed according to the following rules,*

$$R_k = C_k + I_k, \tag{50}$$

and

$$M_k = M_{k-1} - C_k, \quad M_0 = S_0, \tag{51}$$

there exists a unique sequence of numbers,

$$v(0,k) = \prod_{j=1}^{k} \frac{1}{1+f(j)} \quad k = 1, 2, \cdots, n, \tag{52}$$

such that the following relationships hold,

$$S_0 = \sum_{k=1}^{n} R_k v(0,k), \tag{53}$$

$$M_k = \frac{1}{v(0,k)} \sum_{j=k+1}^{n} R_j v(0,j), \quad k = 1, 2, \cdots, n-1. \tag{54}$$

Moreover, the number $v(0,k)$, $k = 1, 2, \cdots, n$, are strictly positive.

The proof of Theorem 2 is provided in Appendix A.

As Theorem 2 clearly shows, the rule for calculating interest, expressed by condition *(G2)*, plays a crucial role in identifying the discount function, allowing it to be uniquely determined. Moreover, we note that Equation (52) can be cast in the following recursive form,

$$v(0,k) = \frac{v(0,k-1)}{1+f(k)}, \quad k = 1, 2, \cdots, n, \tag{55}$$

with $v(0,0) = 1$.

As an example, it is easy to verify that the discount function represented in Table 1 can be easily discovered from the amortization schedule shown in Table 2 by using Equation (52) or, equivalently, Equation (55).

6 Amortizing loans under the law of compound interest

As a special case of the general approach proposed in this paper, we derive the amortization method according to the law of compound interest, which is the most common way of amortizing loans. In such a case the discount function at time $t = 0$ is expressed as follows,

$$v(0,T) = \frac{1}{(1+i)^T}, \tag{56}$$

where i denotes the interest rate level at time $t = 0$. Within this framework, the forward discount function is given by

$$v(0,T,T') = \frac{1}{(1+i)^{T'-T}}, \tag{57}$$

and the forward rate reads

$$i(0,T,T') = (1+i)^{T'-T} - 1. \tag{58}$$

6.1 The amortization method

Let us consider at time $t = 0$ a loan with a principal amount S_0 which will be repaid with a series of future nonnegative installments,

$$\mathbf{r} = \{R_1, R_2, \cdots, R_n\}, \tag{59}$$

scheduled at regular time intervals $1, 2, \cdots, n$, with $R_n > 0$. From Equation (11), the following relationship must hold,

$$S_0 = \sum_{k=1}^{n} \frac{R_k}{(1+i)^k}. \tag{60}$$

According to the law of compound interest, the dynamics of the outstanding balance, described by Equation (17), becomes

$$M_k = M_{k-1} + iM_{k-1} - R_k, \tag{61}$$

so that each installment can be decomposed in the following form,

$$R_k = C_k + I_k, \tag{62}$$

where

$$C_k = M_{k-1} - M_k, \tag{63}$$

quantifies the change of the outstanding balance in the time interval $[k-1, k]$, and

$$I_k = iM_{k-1}, \tag{64}$$

is the interest accrued over the same time interval. Then, the amortization method is uniquely defined according to the schemes provided by Theorem 1 or Theorem 2.

6.1.1 Numerical examples

To illustrate the amortization method, consider a loan with a principal amount of $S_0 = 100$ repaid with an annuity consisting of $n = 5$ equal installments due at regular intervals $k = 1, 2, \cdots, 5$. We assume that the interest rate level is $i = 10\%$. The amount of each installment is computed according to Equation (60),

$$R = \frac{S_0}{\sum_{k=1}^{n} v(0,k)}, \tag{65}$$

where

$$v(0,k) = \frac{1}{(1+i)^k}. \tag{66}$$

The extended amortization schedule is depicted in Table 4.

k	R_k	C_k	I_k	M_k	$S_{0,k}$	J_k	$D_{0,k}$
0	0	0	0	100	0	0	100
1	26.38	16.38	10.00	83.62	23.98	2.40	76.02
2	26.38	18.02	8.36	65.60	21.80	4.58	54.22
3	26.38	19.82	6.56	45.78	19.82	6.56	34.40
4	26.38	21.80	4.58	23.98	18.02	8.36	16.38
5	26.38	23.98	2.40	0	16.38	10.00	0

Table 4: Constant installments.

Looking at Table 4, we note the correspondence $I_k = J_{n-k+1}$ (and $C_k = S_{0,n-k+1}$). However, such a relationship is accidental. In fact, if we consider the loan described in the previous example but with constant principal payments, $C_k = S_0/n$, this correspondence disappears, as the amortization schedule presented in Table 5 clearly shows.

6.2 The "interest on interest" phenomenon

Under the law of compound interest, the interest accrued in the time interval $[k-1, k]$ is computed on the outstanding balance at time $k-1$ according to

$$I_k = iM_{k-1}. \tag{67}$$

We recall that M_{k-1} is related to $D_{0,k-1}$ by Equation (37) which, in the law of compound interest, becomes

$$M_{k-1} = (1+i)^{k-1} D_{0,k-1}. \tag{68}$$

k	R_k	C_k	I_k	M_k	$S_{0,k}$	J_k	$D_{0,k}$
0	0	0	0	100	0	0	100
1	30.00	20.00	10.00	80.00	27.27	2.73	72.73
2	28.00	20.00	8.00	60.00	23.14	4.86	49.59
3	26.00	20.00	6.00	40.00	19.53	6.47	30.05
4	24.00	20.00	4.00	20.00	16.39	7.61	13.66
5	22.00	20.00	2.00	0	13.66	8.34	0

Table 5: Constant principal payments.

The interest accrued in the time interval $[k-1,k]$ can be, therefore, expressed as

$$I_k = i(1+i)^{k-1} D_{0,k-1}. \tag{69}$$

Since $D_{0,k-1}$ is pure capital, Equation (69) shows that the phenomenon of generating "interest on interest" is implicit in the law of compound interest and arises as a consequence of calculating accrued interest according to Equation (67). As discussed below, amortizing loans designed under the law of simple interest are not affected by this mechanism of interest compounding over time.

Finally, we show that the law of compound interest is the only financial law characterized by the property that accrued interest in each time interval is calculated as a given percentage, say i, of the outstanding balance at the beginning of the time interval, as described by Equation (67). This result is a consequence of Theorem 2, with $f(k) = i$. In fact, from Equation (52) we get

$$v(0,k) = \frac{1}{(1+i)^k}. \tag{70}$$

7 Amortizing loans under the law of simple interest

As a special case of the general approach proposed in this paper, we derive the amortization method under the law of simple interest. In this case, the discount function is given by

$$v(0,T) = \frac{1}{1+iT}, \tag{71}$$

where i denotes the interest rate level at time $t = 0$. Within this framework, the forward discount function is given by

$$v(0, T, T') = \frac{1 + iT}{1 + iT'}, \tag{72}$$

and the forward rate reads

$$i(0, T, T') = \frac{i(T' - T)}{1 + iT}. \tag{73}$$

7.1 The amortization method

Let us consider at time $t = 0$ a loan with a principal amount S_0 which will be repaid with a series of future nonnegative installments,

$$\mathbf{r} = \{R_1, R_2, \cdots, R_n\}, \tag{74}$$

scheduled at regular time intervals $1, 2, \cdots, n$, with $R_n > 0$. From Equation (11), the following relationship must hold,

$$S_0 = \sum_{k=1}^{n} \frac{R_k}{1 + ik}. \tag{75}$$

Under the law of simple interest, the dynamics of the outstanding balance, described by Equation (17), becomes

$$M_k = M_{k-1} + \frac{iM_{k-1}}{1 + i(k-1)} - R_k, \tag{76}$$

so that each installment can be decomposed in the following form

$$R_k = C_k + I_k, \tag{77}$$

where

$$C_k = M_{k-1} - M_k, \tag{78}$$

quantifies the change of the outstanding balance in the time interval $[k-1, k]$, and

$$I_k = \frac{iM_{k-1}}{1 + i(k-1)}, \tag{79}$$

is the interest accrued over the same time interval. Then, the amortization method is uniquely defined according to the schemes provided by Theorem 1 or Theorem 2. It is worth noting a very important difference from the amortization method based on the law of compound interest. Indeed, looking at Equation (79), we observe that under the law of simple interest, the interest accrued in the time interval $[k-1, k]$ is calculated multiplying by i the present value of the outstanding balance at time $k-1$. In this way, the interest component of the outstanding balance is removed, thus preventing interest compounding over time.

As a final remark, consider a single-payment loan, i.e., a loan with a principal amount of S_0 repaid with a single strictly positive installment $R_n = S_0(1+in)$ at time n. In this case, the dynamics of the outstanding balance can be determined by applying Equation (12) thus obtaining

$$M_k = (1+ik)S_0, \quad k = 1, 2, \cdots, n-1, \tag{80}$$

and $M_n = 0$. Therefore, the outstanding balance grows linearly over time until time n and then equals 0 due to the payment of the n-th installment. By applying Equation (79), we see that accrued interest is constant over each time interval, namely

$$I_k = iS_0, \tag{81}$$

just as required by the law of simple interest. The significant implications of Equation (79) will be further discussed below.

7.1.1 Numerical examples

To illustrate the amortization method with simple interest, consider a loan with a principal amount of $S_0 = 100$ repaid with an annuity consisting of $n = 5$ equal installments due at regular time intervals $k = 1, 2, \cdots, 5$. We assume that the interest rate level is $i = 10\%$. The amount of each installment is computed by using Equation (75), thus getting

$$R = \frac{S_0}{\sum_{k=1}^{n} v(0,k)}, \tag{82}$$

with

$$v(0,k) = \frac{1}{1+ik}. \tag{83}$$

The extended amortization schedule is depicted in Table 6. In the case of constant principal payments the amortization schedule is shown in Table 7.

7.2 The absence of the "interest on interest" phenomenon

Under the law of simple interest, the interest accrued in the time interval $[k-1, k]$ is computed on the present value of the outstanding balance M_{k-1}, as expressed by

k	R_k	C_k	I_k	M_k	$S_{0,k}$	J_k	$D_{0,k}$
0	0	0	0	100	0	0	100
1	25.69	15.69	10.00	84.31	23.35	2.34	76.65
2	25.69	18.03	7.66	66.29	21.41	4.28	55.24
3	25.69	20.17	5.52	46.12	19.76	5.93	35.48
4	25.69	22.14	3.55	23.98	18.35	7.34	17.13
5	25.69	23.98	1.71	0	17.13	8.56	0

Table 6: Constant installments.

k	R_k	C_k	I_k	M_k	$S_{0,k}$	J_k	$D_{0,k}$
0	0	0	0	100	0	0	100
1	30.00	20.00	10.00	80.00	27.27	2.73	72.73
2	27.27	20.00	7.27	60.00	22.73	4.55	50.00
3	25.00	20.00	5.00	40.00	19.23	5.77	30.77
4	23.08	20.00	3.08	20.00	16.48	6.59	14.29
5	21.43	20.00	1.43	0	14.29	7.14	0

Table 7: Constant principal payments.

Equation (79), namely

$$I_k = \frac{iM_{k-1}}{1 + i(k-1)}, \tag{84}$$

and not on M_{k-1} as required by the law of compound interest, i.e., $I_k = iM_{k-1}$. In this way, the interest compounding over time, i.e. the generation of "interest on interest" is precluded. Indeed, we recall that M_{k-1} is related to $D_{0,k-1}$ by Equation (37) which, in the law of simple interest, becomes

$$M_{k-1} = (1 + i(k-1))D_{0,k-1}. \tag{85}$$

The accrued interest in the time interval $[k-1, k]$ is, therefore, given by

$$I_k = iD_{0,k-1}. \tag{86}$$

Since $D_{0,k-1}$ is pure capital and, therefore, contains no interest, capitalization of interest over time is avoided.

Finally, we will show that the law of simple interest is the only financial law in which interest accrued in each time interval is calculated as a given percentage, say i, of the present value of the outstanding balance at the beginning of the time interval, namely

$$I_k = iv(0, k-1)M_{k-1}. \tag{87}$$

This result is a consequence of Theorem 2, with

$$f(k) = iv(0, k-1). \tag{88}$$

In fact, by substituting Equation (88) into Equation (52) we get

$$v(0, k) = \frac{1}{1+ik}. \tag{89}$$

8 Concluding remarks

In this paper we have provided a general methodology for designing amortizing loans with arbitrary discount functions. Although we have discussed loans with installment payments at regular time intervals, the extension to the case of time intervals of variable amplitude is straightforward. Moreover, as a special case of the proposed methodology, we have illustrated the amortization method based on the law of simple interest and shown that in this case the phenomenon of generating "interest on interest" is precluded.

Some authors proposed a different method for designing amortizing loans under a linear capitalization scheme (Annibali et al., 2017). To illustrate their procedure, let us consider at time $t = 0$ a loan with a principal amount S_0 that will be repaid with a series of future nonnegative installments,

$$\mathbf{r} = \{R_1, R_2, \cdots, R_n\}, \tag{90}$$

scheduled at regular time intervals $1, 2, \cdots, n$, with $R_n > 0$. The starting point of their analysis is that the loan principal and each installment are linearly capitalized at loan maturity n, using the interest rate i observed at time $t = 0$, thus obtaining

$$S_0(1 + in) = \sum_{k=1}^{n} R_k \big[1 + i(n-k)\big]. \tag{91}$$

We point out that this approach can be considered as a special case of the methodology proposed in this study with the following discount function

$$v(0, k) = \frac{1 + i(n-k)}{1 + in}. \tag{92}$$

However, it should be noted this procedure produces spurious results that are not consistent with the law of simple interest. Consider, for example, a loan with a principal amount S_0 at time $t = 0$ that will be repaid with a single strictly positive installment $R_n = S_0(1 + in)$ at time n. Following this approach, the dynamics of the outstanding balance is given by

$$M_k = \frac{1 + in}{1 + i(n - k)} S_0, \tag{93}$$

showing that the outstanding balance does not follow a linear behavior, as it should be according to the law of simple interest and as obtained from Equation (80). In a different but equivalent way, interest does not accrue linearly over time. For these reasons, we believe that there is only one method for designing loans with amortization according to the law of simple interest, the one described in this study.

A

A.1 Proof of Theorem 1

Under the assumptions of Theorem 1, the sequences of numbers C_k and I_k are given by Equation (21) and Equation (22), respectively. Then, it is straightforward to verify that conditions *(G1)*-*(G3)* hold with $f(k) = i(0, k - 1, k)$.

A.2 Proof of Theorem 2

Preliminarily we note that from condition *(G2)* the numbers $v(0, k)$ defined by Equation (52) are strictly positive since $f(k) > -1$. By substituting Equation (50) into Equation (51) we obtain,

$$M_k = M_{k-1} + f(k)M_{k-1} - R_k, \tag{94}$$

where condition *(G2)* has been used. Solving with respect to M_{k-1}, we get,

$$M_{k-1} = \frac{R_k + M_k}{1 + f(k)}. \tag{95}$$

By using Equation (52), we can rewrite Equation (95) in the following recursive form,

$$M_{k-1} = \frac{v(0, k)}{v(0, k - 1)}(R_k + M_k), \quad k = 1, 2, \cdots, n, \tag{96}$$

with $v(0, 0) = 1$. Equation (53) and Equation (54) can be then recovered by backward induction starting from $M_n = 0$ and recalling that $M_0 = S_0$. To prove the uniqueness, we observe that the system of n linear equations in the n unknowns $v(0,k)$, $k=1,2,\cdots,n$, described by Equation (96), admits one and only one solution.

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