**A COMPARATIVE STUDY OF A CLASS OF IMPLICIT MULTI-DERIVATIVE METHODS FOR THE NUMERICAL SOLUTION OF SECOND -ORDER ORDINARY DIFFERENTIAL EQUATIONS**

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**Abstract**

In this paper, we considered the development, analysis and implementation of a class of multi-derivative multistep methods for solving second order ordinary differential equations.

The methods incorporated more analytical properties of the differential equations into the conventional implicit linear multistep method. The step size and the order of the derivatives of the methods have been varied to ensure accuracy and efficiency in the method. The basic properties of these methods were analysed the results showed that the methods are accurate, consistent, convergent and zero-stable. A comparative study of the new methods was carried out using some second order ODEs, to determine the effect of increasing the step-size (k) and the order of the derivative (l)

The result shows that the methods are more efficient and accurate when the step-size (k) and the order of derivative are increased.

**1.0 Introduction.**

The formulation of physical phenomenon in engineering, sciences and economics often to lead to differential equations. Though these problems exists in theory and principles, their mathematical formulation leads to differential equations According to Ross(1989), Auzinger et’al, (1990),Courant(2007); the objects involves obeys certain physical and chemical laws that has to do with rate of change. Awoyemi (1992) posited that differential equations occur in connection with mathematical description of some problems that arise in various branches of science and social science such as Mechanics, Chemistry, Biology and Economics .Differential equations constitute a large and important aspect of today’s mathematics. Ordinary differential equation involves the derivative of the dependent variable with respect to a single independent variable (Boyce 2001). Only a few of these differential equations have analytical solution hence the resort to numerical method or approximation.

A differential equation together with an initial condition is called initial value problem (IVP)

The general first order initial value problem is of the form:

,,  (1)

**1.1 Linear Multistep Method LMM**

According to Lambert (1973), the general linear k-step Multistep Method (LMM) for first order ordinary differential equation is given as

 (2)

Where 

where and are the parameters to be determined and and

When, the method is explicit and implicit if.

In our work, we shall consider the development of methods for the solution general second order ordinary differential equations when k = 2, 3, 4.

The LMM for the integration of second order ordinary differential equation is of the form

 (3)

We shall investigate the inclusion of more analytical properties of the differential equation by way of considering more derivative properties of the differential equation. The study also attempts to determine the effect of increasing the order of the derivatives as well as varying the step-size of the Linear Multistep Method, This we did by reformulating the method (3) in the form:

  (4)

Where l is the order of derivatives,

The LMM (3) involves more derivatives properties of the differential equations .The aim of the study is to compare the accuracy and stability of some implicit multiderivative multistep methods

**2.0 Methodology**

The local truncation error of the method (3) is of the form

 (5)

Where l is the order of the derivatives of.

Adopting Taylor series expansion of the variables   and  given as  (6)

Expanding equation (4) and combining terms in equal powers of  , we have:

where

 (7)

The method (3) may be written in the form

 (8)

Then we have

  (9)

If =0 method is explicit and implicit if 

**2.1 Derivation of Specific methods**

**Two step second derivative method ()**

Eqn (8) becomes

  (10)

 (11)

Adopting Taylor’s expansion in (9) and substituting the results and collecting in equal powers of h yield the following system of equations given as

 

 ,  

 (12)

Solving the above set of equations yields

   , 

Substituting the above values into equation (2.3) and simplifying we obtained the two step second derivative method of the form

 (13)

**Two- step third derivative method** )

From (2.2) we have,

 (14)

By adopting Taylor’s series expansion of  in (2.6) and combining terms in equal powers of  to obtain the following set of equations

  









 (14)

Solving the above set of equations gives

 ,  

Substituting these values into (2.6) yields the scheme

 (15)

**Two-step fourth-derivative method** 

Using (2.2) ,we have

  (16)

Again, by adopting Taylor’s series expansion on (16) and combining terms in equal powers of h , we obtained the following set of equations

 













 (17)



Solving the above set of equations gives

 , ,  ,  ,  , 

 ,  , ,  , 

On substituting the above values into ( ) and simplifying we obtain

 (18)

**Three-step second-derivative method** 

Again (9) , we have

 

 (19)

By adopting Taylor’s series expansion on (19) and combining terms in equal powers of h, we obtained the following set of equations:

 ,  , 

 ,





 (20)

Solving the above set of equations gives



Substituting the above value in (19) and re-arranging, we have the method

 (21)

**Four-step second-derivative method** 

Applying equation (9)

 (22)  (23)

 ,  , 

 ,











Solving the above set of equations gives

Substituting the values above into equation (22) yields

 (24)

**3.0** **Basic Properties of the Methods**

The local truncation error of our method in (2.2) is given as

 ,  and  (25)

Adopting Taylor series expansion of (25) and collect terms in equal powers of , we have

 (26)

where and

 (27)

**Two-step second derivative methods (k=2,l=2)**

From the lte

 (28)

where  ,  , 

 , 

, 

By imposing an accuracy of order 4, we have

 and  hence the method ( 12) is of order 4

**Two-step third derivative method (k=2.l=3)**

Let the lte be given by



Where (29)

 ,  , 









 (30)

By imposing an accuracy of order 6 , we have that

 ,  .Hence the scheme is order 6 and error constant 

**Two-step fourth derivative method (k=2, l=4)**

The lte for the method is given by

 (31)

By imposing an accuracy of order 10, we have that

 

Hence the scheme is of order 10 and error constant 

**Three-step second-derivative method** 

The lte for the method (21) is given by

 (31)

By imposing an accuracy of order 5, we have that



Hence, the method (21) has order p = 5, error constant 

**Four-step second-derivative method** 

Using similar approach, the order of the method (24) is p = 6 and the error constant is 

Table 0: Coefficients and order of the methods

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| k |  |  |  |  |  |  |  |  |  |  |  |  |  |  | p |
| 2 | 2 | 1 | -2 | 1 |  |  |  |  |  |  |  |  |  |  | 5 |
| 2 | 3 | 1 | -2 | 1 |  |  |  |  |  |  |  |  | 0 |  | 6 |
| 2 | 4 | 1 | -2 | 1 |  |  |  |  |  |  |  |  | 0 |  | 10 |
| 3 | 2 | -1 | 3 | -3 | 1 |  |  |  |  |  |  |  |  |  | 5 |
| 4 | 2 | 1 |  |  |  | 1 |  |  |  |  |  |  |  |  | 6 |

**Consistency**

According to Lambert (1973), Awoyemi (1999, 2001), a linear multistep method of the type (3) for the solution of nth order ordinary differential equation is consistent if and only if the following conditions are satisfied

1. 
2. 
3. 

**Two-step second derivative method**

1. Since in method (12 ) ,then the first condition is satisfied
2.  .thus the second condition is satisfied
3. 

LHS = 

RHS =. Thus satisfying the third condition. Hence the method is consistent.

**Two-step third derivative method.**

1. The implicit scheme (15) is of order 6, then the first condition is satisfied
2. . Hence the second condition is satisfied
3. LHS:  , RHS=

LHS = RHS=1.Hence the third condition is also satisfied. The method is therefore consistent.

**Two-step fourth derivative method**

1. , hence the first condition is satisfied
2. . Again the second condition is satisfied
3. LHS= 
4. RHS = 

LHS = RHS =1. The third condition is also satisfied .Hence the method is consistent.

**Zero stability**

Given the linear k-step method (2), we consider its first and second characteristics polynomial as

  (2.20)

Where, as before  and 

**Theorem (Root condition)** A linear multistep method is zero – stable for any initial value problem of the form (1) where satisfies the hypothesis of Picard’s Theorem, if and only if, all the roots of the first characteristics polynomial of the method are inside the closed unit disc in the complex plane, with any which lie on the unit circle being simple (Endre and Mayers 2003) .

**Definition (Jain and Iyengar2008):** The linear multistep method (3) is said to satisfy the root condition if the root of the equation  lie inside the unit circle in the complex plane, and are simple if it lie on the unit circle.

The main consequence of zero stability is to control the propagation of the errors as the integration progresses Fatunla (1988)

**Definition (Jain and Iyengar 2008)**

A linear multistep method of the type (3) is said to be zero stable if the root of the first characteristics polynomial  lie inside the unit circle in the complex plane, and are of multiplicity not exceeding at most two if they lie in unit circle

Therefore for the methods (13), (15), (18), the first characteristic polynomials are all of the forms

.

Solving this gives. Hence the roots are within unit circle, the methods are therefore zero-stable

For the methods (21) and (24), all the roots lie on the unit circle and none of the roots has modulus greater than one. Hence they all zero stable.

**Region of Absolute Stability of Method**

**Two –step second derivative method.**

The method is given as



The first and second characteristics polynomial is:  ,

Using the boundary locus method

 where 

. Therefore setting imaginary 

And , With 



Hence the region of absolute stability of the Two-step second derivative method is (-6,0)

**Two-step third derivative method**

The first and second characteristics polynomial are :

, .

Again by applying the boundary locus method

 , where 

Therefore,

.

Considering only the real part: , and with 



**Two –step fourth derivative methods**

The first and second characteristics polynomial of the method are :

, .

Using the boundary locus method

 ,where 

Therefore,

.

Considering only the real part: , and with 



**Numerical Examples**

**Problem 1**

 ,  , 

Exact solution: 

**.Problem 2**

A highly oscillatory problem

 ,  ,  

Exact solution: 

**Problem 3**

 ,  ,  , 

Exact solution: 

**Table 1: Results of problem 1**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  |  | Computed Results for: | | | | |
| X | Exact  Solution | K=2,L=2 | K=2,L=3 | K=2,L=4 | K=3, L=2 | K=4, L=2 |
| 0.1 | 1.005004168 | 1.0050041667 | 1.0050041680 | 1.0050041681 | 1.005004176 | 1.005004161 |
| 0.2 | 1.020066756 | 1.0200666667 | 1.0200667556 | 1.0200667556 | 1.020066756 | 1.020066755 |
| 0.3 | 1.045338514 | 1.0453383403 | 1.0453385141 | 1.0453385141 | 1.045338513 | 1.045338513 |
| 0.4 | 1.081072372 | 1.0810721042 | 1.0810723718 | 1.0810723728 | 1.081072356 | 1.081072378 |
| 0.5 | 1.127625965 | 1,1276255417 | 1,1276259649 | 1,1276259652 | 1.127625868 | 1.127625926 |
| 0.6 | 1.185465218 | 1.1854644028 | 1.1854652166 | 1.1854652182 | 1.185464800 | 1.185465432 |
| 0.7 | 1.255169006 | 1.2551671042 | 1.2551689978 | 1.2551690056 | 1.255167568 | 1.255167749 |
| 0.8 | 1.337434946 | 1.3374302292 | 1.3374349165 | 1.3374349463 | 1.337430756 | 1.337442102 |
| 0.9 | 1.433086385 | 1.4330750278 | 1.4330862887 | 1.4330863854 | 1.433075613 | 1.433044487 |
| 1.0 | 1.543080635 | 1.5430549167 | 1.5430803571 | 1.5430806348 | 1.543058556 | 1.543318962 |

**Table2: Errors in the computed Results for Problem 1**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | Errors in the Computed Results for: | | | | |
| X | K=2,L=2 | K=2,L=3 | K=2,L=4 | K=3, L=2 | K=4, L=2 |
| 0.1 | 1.3891370276E-009 | 2.4824586831E-013 | 2.2204460493E-016 | 2.482458683E-013 | 2.482458683E-013 |
| 0.2 | 8.8952409216E-008 | 1.4837020501E-012 | 8.8817841970E-016 | 6.352030013E-011 | 6.352030013E-011 |
| 0.3 | 1.7385108242E-007 | 5.2498005942E-012 | 1.1102230246E-015 | 1.628859714E-009 | 1.628860379E-009 |
| 0.4 | 2.6767178785E-007 | 3.5677683030E-011 | 1.5543122345E-015 | 1.628289725E-008 | 5.750255161E-009 |
| 0.5 | 4.2353971352E-007 | 2.8045521461E-010 | 2.2204460493E-015 | 9.715082094E-008 | 3.882476273E-008 |
| 0.6 | 8.1546448927E-007 | 1.6867125474E-009 | 9.9920072216E-015 | 4.182422608E-007 | 2.135768764E-007 |
| 0.7 | 1.9014642758E-006 | 7.8351103383E-009 | 7.3496764230E-014 | 1.437575377E-006 | 1.256609340E-006 |
| 0.8 | 4.7171381774E-006 | 2.9762252884E-008 | 4.7228887468E-013 | 4.190749275E-006 | 7.155541700E-006 |
| 0.9 | 1.1351670996E-005 | 9.6715442099E-008 | 2.4824586831E-012 | 1.077294876E-005 | 4.150958687E-005 |
| 1.0 | 2.5718148576E-005 | 2.7771802280E-007 | 1.0969891662E-011 | 2.507925966E-005 | 2.383271489E-004 |

**Table 2: Results of problem 2**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| X-  value | Exact  Solution | K=2,L=2 | K=2,L=3 | K=2,L=4 | K=3 ,L=2 | K=4, L=2 |
| 0.01 | 0.10197987E+01 | 0.101979E+01 | 0.101979E+01 | 0.101979E+01 | 0.101979E+01 | 0.101979 E+01 |
| 0.02 | 0.10391894E+01 | 0.103919E+01 | 0.103918E+01 | 0.103919E+01 | 0.103919E+01 | 0.103919 E+01 |
| 0.03 | 0.10581646E+01 | 0.105816E+01 | 0.105817E+01 | 0.105817E+01 | 0.105817E+01 | 0.105816 E+01 |
| 0.04 | 0.10767164E+01 | 0.107672E+01 | 0.107672E+01 | 0.107672E+01 | 0.107672E+01 | 0.107671 E+01 |
| 0.05 | 0.10948376E+01 | 0.109484E+01 | 0.109484E+01 | 0.109484E+01 | 0.109484E+01 | 0.109484 E+01 |
| 0.06 | 0.11125208E+01 | 0.111252E+01 | 0.111252E+01 | 0.111252E+01 | 0.111252E+01 | 0.111252 E+01 |
| 0.07 | 0.11297591E+01 | 0.112976E+01 | 0.112976E+01 | 0.112976E+01 | 0.112976E+01 | 0.112976 E+01 |
| 0.08 | 0.11465455E+01 | 0.114655E+01 | 0.114655E+01 | 0.114655E+01 | 0.114655E+01 | 0.114654 E+01 |
| 0.09 | 0.11628733E+01 | 0.116287E+01 | 0.116287E+01 | 0.116287E+01 | 0.116287E+01 | 0.116287 E+01 |
| 0.1 | 0.11787359E+01 | 0.117874E+01 | 0.117874E+01 | 0.117874E+01 | 0.117874E+01 | 0.117873 E+01 |

**Table 3: Comparison of Errors in the results problem 2**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | Errors in the Computed results for: | | | |  |
|  | K=2,L=2 | K=2,L=3 | K=2,L=4 | K=3, L=2 | K=4, L=2 |
| 0.01 | 2.6577E-11 | 4.4408E-16 | 2.2204E-16 | 8.9262E-14 | 4.44089210E-16 |
| 0.02 | 5.3390E-11 | 8.8818E-16 | 6.6611E-16 | 5.7212E-12 | 3.26405569E-14 |
| 0.03 | 7.9982E-11 | 1.3323E-15 | 1.3322E-15 | 4.3430E-12 | 5.51558799E-13 |
| 0.04 | 1.0379E-10 | 1.3322E-15 | 1.7763E-15 | 8.4632E-12 | 1.84985360E-12 |
| 0.05 | 1.1596E-10 | 8.8818E-16 | 2.4424E-15 | 1.9333E-11 | 1.25917055E-11 |
| 0.06 | 9.3451E-11 | 1.2656E-14 | 3.1086E-15 | 3.6497E-11 | 7.00373093E-11 |
| 0.07 | 1.3794E-11 | 5.7288E-14 | 3.5527E-15 | 5.8766E-11 | 4.06566336E-10 |
| 0.08 | 3.0168E-10 | 1.9385E-13 | 4.9968E-15 | 8.3445E-11 | 2.33935693E-09 |
| 0.09 | 9.3806E-10 | 5.5822E-13 | 4.2188E-15 | 1,0513E-10 | 1.34819704E-08 |
| 0.1 | 2.1970E-09 | 1.4308E-12 | 4.6629E-15 | 1.1373E-10 | 7.76816826E-08 |

**Table 4: Results of problem 3**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| X-  Value | Exact  Solution | K=2,L=2 | K=2,L=3 | K=2,L=4 | K=3 ,L=2 |
| 0.1 | 1.050041730 | 1.050041667 | 1.050041701 | 1.050041766 | 1.050041666 |
| 0.2 | 1.100335349 | 1.100333333 | 1.100335265 | 1.100335333 | 1.100333335 |
| 0.3 | 1.151140438 | 1.151125002 | 1.151140800 | 1.151140999 | 1.151125002 |
| 0.4 | 1.202732557 | 1.202666670 | 1.202732110 | 1.202732670 | 1.202666670 |
| 0.5 | 1.255412816 | 1.255208330 | 1.255413807 | 1.255414336 | 1.255208337 |
| 0.6 | 1.309519609 | 1.309000000 | 1.309513000 | 1.309516990 | 1.309000005 |
| 0.7 | 1.365443760 | 1.364291673 | 1.365439683 | 1.365449833 | 1.364291673 |
| 0.8 | 1.423648937 | 1.421333333 | 1.423639335 | 1.423639876 | 1.421333340 |
| 0.9 | 1.484700287 | 1.480375008 | 1.484688898 | 1.484712008 | 1.480375008 |
| 1.0 | 1.549306154 | 1.541666676 | 1.549166676 | 1.549356676 | 1.541666676 |

**Table 5: Comparison of errors in the results of problem 3**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| X-  Value | Errors in | | | |
| K=2,L=2 | K=2,L=3 | K=2,L=4 | K=3,L=2 |
| 0.1 | 6.26118E-08 | 2.90000E-08 | 1.00000E-09 | 6.2612E-08 |
| 0.2 | 2.01440E-07 | 5.80220E-08 | 1.60000E-08 | 2.0144E-06 |
| 0.3 | 1.54459E-05 | 3.27100E-07 | 3.90000E-08 | 1.5436E-05 |
| 0.4 | 6.58874E-05 | 4.47013E-07 | 1.13000E-07 | 6.5887E-05 |
| 0.5 | 2.04479E-04 | 9.91200E-07 | 1.52000E-06 | 2.0441E-04 |
| 0.6 | 5.19604E-04 | 3.60912E-06 | 2.61000E-06 | 5.1960E-04 |
| 0.7 | 1.15209E-03 | 4.07700E-06 | 6.07310E-06 | 1.1521E-03 |
| 0.8 | 2.31561E-03 | 9.60231E-06 | 9.06100E-06 | 2.3156E-03 |
| 0.9 | 4.32529E-03 | 1.13890E-05 | 1.17210E-05 | 4.3253E-03 |
| 1.0 | 7.63945E-03 | 1.39478E-04 | 5.05220E-05 | 7.6395E-03 |

**Conclusion**

In this study, we have developed a class of implicit multi-derivative linear multistep methods for the numerical solution of general second order ordinary differential equations. Analysis of the basic properties showed that the methods are consistent, zero-stable, convergent and absolutely stable. The results displayed in tables 1-3 shows that there is a remarkable improvement in accuracy if the order of the derivatives (l) is increased rather than the steps. Also it was observed that the order of the methods increases when the derivative is increased. This can be seen in table 0 above.

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