A note of the induced topological pressure for topological systems

Zhitao Xing[1]

1 School of Mathematics and Statistics , Zhaoqing University,

e-mail: xzt-303@163.com

Zhaoqing 526061, Guangdong, P.R.China

Abstract. In this paper, we give an equivalent definition of the induced topological pressure [8]. We also set up a relation for two induced topological pressures with a factor map by using a method which is different from that of [9].

Keywords and phrases: Induced topological pressure, dynamical system, factor map.

1 Introduction and statement of main result

Throughout this paper, a topological dynamical system (for short TDS) means a pair (X, f), where f is a continuous map from a compact metric space (X, d) to itself. For $n \in \mathbb{N}$, the n-th Bowen metric d_n on X is defined by

$$d_n(x, y) = \max\{d(f^i(x), f^i(y)) : i = 0, 1, \ldots, n-1\}.$$

Recall that $C(X, \mathbb{R})$ is the Banach algebra of real-valued continuous functions of X equipped with the supremum norm. For $\varphi \in C(X, \mathbb{R})$, let $(S_n\varphi)(x) := \sum\limits_{i=0}^{n-1} \varphi(f^i x)$.

The notion of the topological entropy plays an important role in topological dynamics and dimension theory [1, 2, 6]. In 1971, Bowen [4] considered a factor map $\pi : (X, f) \to (Y, g)$, and showed that

$$h(f) \le h(g) + \sup_{y \in Y} h(f, \pi^{-1}(y)), \tag{1.1}$$

where $h(f, K)$ denotes the entropy of a compact subset $K \subseteq X$ with respect to f.

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Topological pressure is a generalization to topological entropy for dynamical systems. It was first introduced by Ruelle [5] for expansive dynamical systems, and later by Walters [3, 6] for the general case. Recently, the theory for dynamical systems with different time-scalings has been developed. Jaerisch, Kesseböhmer, and Lamei [7] studied the induced topological pressure of a countable state Markov shift. In [8], the authors defined the induced topological pressure for a topological dynamical system, and established a variational principle for it. In this paper, we give an equivalent definition of the induced topological pressure. We also set up a relation for two induced topological pressures with a factor map by using a method which is different from that of [9].

Let (X, f) be a TDS and $\psi \in C(X, \mathbb{R})$ with $\psi > 0$. For $x \in X, T > 0, \epsilon > 0$, define

$$n(x, T) = \inf\{n \in \mathbb{N} : S_n\psi(x) \geq T\}$$

and

$$B_T(x, \epsilon, f) = \{y \in X : d_{n(x,T)}(x, y) < \epsilon\}; \overline{B}_T(x, \epsilon, f) = \{y \in X : d_{n(x,T)}(x, y) \leq \epsilon\}.$$

Let K be a compact set of X. A subset $F_T \subset X$ is called a (ψ, T, ϵ)-spanning set of K with respect to f, if for any $y \in K$, there exists $x \in F_T$ with $d_{n(x,T)}(x, y) \leq \epsilon$. Let $r_T(f, K, \epsilon)$ denotes the smallest cardinality of any (ψ, T, ϵ)-spanning set of K. Obviously $r_T(f, K, \epsilon) < \infty$. Define

$$r(f, K, \epsilon) = \limsup_{T\to\infty} \frac{1}{T} \log r_T(f, K, \epsilon).$$

Clearly if $0 < \epsilon_1 < \epsilon_2$, then $r_T(f, K, \epsilon_1) \geq r_T(f, K, \epsilon_2)$.

Definition 1.1. *We define the ψ-induced topological entropy of K (with respect to f) by*

$$h_\psi(f, K) = \lim_{\epsilon\to 0} \limsup_{T\to\infty} \frac{1}{T} \log r_T(f, K, \epsilon) \tag{1.2}$$

Remarks.
$h_1(f, X) = h(f)$, where $h(f)$ denotes the topological entropy of f [3, 6].

Definition 1.2. *Let (X, f) be a TDS, and let K be a compact set of X, $\varphi, \psi \in C(X, \mathbb{R})$ with $\psi > 0$. For $T > 0, \epsilon > 0$, put*

$$Q_{\psi,T}(f, K, \varphi, \epsilon) = \inf\{\sum_{x\in F_T} \exp(S_{n(x,T)}\varphi)(x) : F_T \text{ is a } (\psi, T, \epsilon)\text{-spanning set of } K\}.$$

We define the ψ-induced topological pressure of φ (with respect to f and K) by

$$P_\psi(f, K, \varphi) = \lim_{\epsilon\to 0} \limsup_{T\to\infty} \frac{1}{T} \log Q_{\psi,T}(f, K, \varphi, \epsilon) \tag{1.3}$$

Remarks.
(i) If $0 < \epsilon_1 < \epsilon_2$, then $Q_{\psi,T}(f, K, \varphi, \epsilon_1) \geq Q_{\psi,T}(f, K, \varphi, \epsilon_2)$, which implies the existence of the $P_\psi(f, K, \varphi)$ in (1.3).
(ii) $P_1(f, X, \varphi) = P(\varphi)$, where $P(\varphi)$ denotes the topological pressure of φ [3, 6].
(iii) It is easy to see $P_\psi(f, X, \varphi) = P_\psi(\varphi)$, where $P_\psi(\varphi)$ denotes the ψ-induced topological pressure of φ [8].

By using a method which is different from that of [9], we obtain the result of this paper, as follows.

Theorem 1.1. *Let* $(X, d), (Y, \rho)$ *be compact metric spaces, and let* $f : X \to X, g : Y \to Y$ *be continuous maps,* $\pi : X \to Y$ *a factor map, i.e., a continuous surjective map with* $\pi \circ f = g \circ \pi$, $\varphi, \psi \in C(Y, \mathbb{R})$ *with* $\psi > 0$. *Then*

$$P_{\psi\circ\pi}(\varphi \circ \pi) \leq P_\psi(\varphi) + \sup_{y\in Y} h_{\psi\circ\pi}(f, \pi^{-1}(y)). \tag{1.4}$$

2 Some lemmas

In this section, we give some lemmas, which will be needed for the proof of Theorem 1.1.

Lemma 2.1. *Let* (Y, g) *be a TDS, and let* ρ *be a compatible metric on* Y, $\psi \in C(Y, \mathbb{R})$ *with* $\psi > 0, m = \min\{\psi(x) : x \in Y\}$. *For each* $y \in \overline{B}_T(x, \delta, g)$, *we have*

$$|n(x, T) - n(y, T)| \leq \frac{T + m}{m^2} var(\psi, \delta) + \frac{\|\psi\|}{m}.$$

where $var(\psi, \delta) := \sup\{|\psi(x) - \psi(y)| : \rho(x, y) \leq \delta\}$.

Proof. Clearly $n(x, T) \leq \frac{T}{m} + 1$ for any $x \in Y$. Notice for each $y \in \overline{B}_T(x, \delta, g)$,

$$m|n(x, T) - n(y, T)| - n(x, T)var(\psi, \delta) \leq |S_{n(x,T)}\psi(x) - S_{n(y,T)}\psi(y)| \leq \|\psi\|.$$

Then

$$|n(x, T) - n(y, T)| \leq \frac{n(x, T)}{m} var(\psi, \delta) + \frac{\|\psi\|}{m} \leq \frac{T + m}{m^2} var(\psi, \delta) + \frac{\|\psi\|}{m}.$$

Lemma 2.2. *Let* (Y, g) *be a TDS, and let* ρ *be a compatible metric on* Y, $\varphi, \psi \in C(Y, \mathbb{R})$ *with* $\psi > 0, m = \min\{\psi(x) : x \in Y\}$. *For each* $y \in \overline{B}_T(x, \delta, g)$, *we have*

$$\exp S_{n(y,T)}\varphi(y) \leq e^{(\frac{T}{m}+1)var(\varphi,\delta)+\frac{T+m}{m^2}\|\varphi\|var(\psi,\delta)+\frac{\|\psi\|\|\varphi\|}{m}} \exp S_{n(x,T)}\varphi(x),$$

where $var(\psi, \delta) := \sup\{|\psi(x) - \psi(y)| : \rho(x, y) \leq \delta\}$.

Proof. For each $y \in \overline{B}_T(x,\delta,g)$, it follows from Lemma 2.1 that

$$\begin{aligned}
\exp S_{n(y,T)}\varphi(y) &= \exp(S_{n(y,T)}\varphi(y) - S_{n(x,T)}\varphi(x) + S_{n(x,T)}\varphi(x)) \\
&\le \exp|S_{n(y,T)}\varphi(y) - S_{n(x,T)}\varphi(x)| \exp S_{n(x,T)}\varphi(x) \\
&\le e^{n(x,T)var(\varphi,\delta)+|n(x,T)-n(y,T)|\|\varphi\|} \exp S_{n(x,T)}\varphi(x) \\
&\le e^{(\frac{T}{m}+1)var(\varphi,\delta)+\frac{T+m}{m^2}\|\varphi\|var(\psi,\delta)+\frac{\|\psi\|\|\varphi\|}{m}} \exp S_{n(x,T)}\varphi(x).
\end{aligned}$$

3 The proof of Theorem 1.1

Now we give the proof of Theorem 1.1. Let $m = \min\{\psi(x) : x \in Y\}$. To show the inequality, for any $\epsilon > 0$, we choose $\delta_1 > 0$ small enough so that

$$d(u,v) < 4\delta_1 \Rightarrow d_{2+[\frac{\|\psi\|}{m}]}(u,v) \le \epsilon, \tag{3.5}$$

where $[\frac{\|\psi\|}{m}]$ denotes the integer part of $\frac{\|\psi\|}{m}$. Clearly, we may assume

$$a := \sup_{y\in Y} h_{\psi\circ\pi}(f, \pi^{-1}(y)) < \infty.$$

Fix $\delta_1 > 0$ and $\tau > 0$. For any $y \in Y$, we choose $T_y > 0$ such that there exist a $(\psi\circ\pi, T_y, \delta_1)$-spanning set E_y of $\pi^{-1}(y)$ with minimal cardinality such that $|E_y| = r_{T_y}(f, \pi^{-1}(y), \delta_1)$ and

$$\log r_{T_y}(f, \pi^{-1}(y), \delta_1) \le (h_{\psi\circ\pi}(f, \pi^{-1}(y)) + \tau)T_y \le (a+\tau)T_y.$$

Denote $U_y = \{u \in X : \exists z \in E_y \text{ s.t } d_{n(z,T_y)}(u,z) < 2\delta_1\}$, then U_y is an open neighborhood of $\pi^{-1}(y)$ and

$$(X\setminus U_y) \cap \bigcap_{\gamma>0} \pi^{-1}(\overline{B_\gamma(y)}) = \emptyset,$$

where $B_\gamma(y) = \{z \in Y : \rho(y,z) < \gamma\}$. By the finite intersection property of compact sets, there is a $W_y = B_{\gamma_y}(y), (\gamma_y > 0)$ for which $\pi^{-1}(W_y) \subset U_y$. Since Y is compact, there exists $W_{y_1}, W_{y_2} \dots W_{y_r}$ cover Y. Let $\delta_2 > 0$ be a Lebesgue number for Y for this open cover. For $T > 0$, we choose $0 < \delta < \frac{1}{2}\delta_2$ so that $\frac{T+m}{m^2}var(\psi,\delta) + \frac{\|\psi\|}{m} \le 2 + [\frac{\|\psi\|}{m}]$. Let F_T be a (ψ, T, δ)-spanning set of Y. For each $y \in F_T, 0 \le j < n(y,T)$, pick $\triangle_y(j) \in \{y_1, y_2 \dots y_r\}$ such that $\overline{B_\delta(g^j(y))} \subset W_{\triangle_y(j)}$. Define recursively

$$\begin{aligned}
t_0(y) &= 0; \\
t_1(y; z_0) &= n(z_0, T_{\triangle_y(0)}), z_0 \in E_{\triangle_y(0)}; \\
t_2(y; z_0, z_1) &= t_1(y;z_0) + n(z_1, T_{\triangle_y(t_1(y;z_0))}), z_1 \in E_{\triangle_y(t_1(y;z_0))}; \\
&\dots \\
t_{s+1}(y; z_0, z_1, \dots, z_s) &= t_s(y; z_0, z_1, \dots, z_{s-1}) + n(z_s, T_{\triangle_y}(t_s(y; z_0, z_1, \dots, z_{s-1}))),
\end{aligned}$$

$$z_s \in E_{\triangle_y(t_s(y;z_0,z_1,\dots,z_{s-1}))} \tag{3.6}$$

until one gets a $t_{q+1}(y; z_0, z_1 \dots z_q) \geq n(y, T)$. Clearly the number of q depends on the choice of $z_0, z_1 \dots z_{q-1}$. Set $q(y; z_0, z_1 \dots z_{q-1}) = q$, we yet denote $q(y; z_0, z_1 \dots z_{q-1})$ by q for convenience. For $y \in F_T$ and $z_0 \in E_{\triangle_y(0)}, z_1 \in E_{\triangle_y(t_1(y;z_0))}, \dots, z_q \in E_{\triangle_y(t_q(y;z_0,z_1,\dots,z_{q-1}))}$, define

$$
\begin{aligned}
V(y; z_0, z_1, \dots, z_q) = \{u \in X : d(f^{t+t_s(y;z_0,z_1,\dots,z_{s-1})}(u), f^t(z_s)) < 2\delta_1 \\
\text{for all } 0 \leq t < n(z_s, T_{\triangle_y}(t_s(y; z_0, z_1, \dots, z_{s-1}))), 0 \leq s \leq q\}.
\end{aligned}
$$

It is not hard to see that

$$
\bigcup_{z_0 \in E_{\triangle_y(0)}, z_1 \in E_{\triangle_y(t_1(y;z_0))}, \dots, z_q \in E_{\triangle_y(t_q(y;z_0,z_1,\dots,z_{q-1}))}} V(y; z_0, z_1, \dots, z_q) \supset \pi^{-1}(\overline{B}_T(y, \delta, g)). \tag{3.7}
$$

In fact, for any $u \in \pi^{-1}(\overline{B}_T(y, \delta, g))$, we have

$$
\rho(g^j(y), g^j(\pi u)) \leq \delta, \quad \forall 0 \leq j < n(y, T).
$$

Then

$$
\pi(f^j(u)) = g^j(\pi u) \in \overline{B_\delta(g^j(y))} \subset W_{\triangle_y(j)}, \quad \forall 0 \leq j < n(y, T).
$$

This implies that

$$
f^j(u) \in \pi^{-1}(W_{\triangle_y(j)}) \subset U_{\triangle_y(j)}, \quad \forall 0 \leq j < n(y, T), \tag{3.8}
$$

and hence, there exists $\widetilde{z_0} \in E_{\triangle_y(0)}$ with $d_{n(\widetilde{z_0}, T_{\triangle_y(0)})}(\widetilde{z_0}, u) < 2\delta_1$. If $n(\widetilde{z_0}, T_{\triangle_y(0)}) \geq n(y, T)$, let $t_1(y; \widetilde{z_0}) = n(\widetilde{z_0}, T_{\triangle_y(0)})$, we have $u \in V(y; \widetilde{z_0})$ and finish the proof. Otherwise, it follows from (3.8) that there exists $\widetilde{z_1} \in E_{\triangle_y(t_1(y;\widetilde{z_0}))}$ such that

$$
d_{n(\widetilde{z_1}, T_{\triangle_y(t_1(y;\widetilde{z_0}))})}(\widetilde{z_1}, f^{n(\widetilde{z_0}, T_{\triangle_y(0)})}(u)) < 2\delta_1.
$$

By this means, we get the minimal $q(y; \widetilde{z_0}, \widetilde{z_1} \dots \widetilde{z_{q-1}})$ with $t_{q+1}(y; \widetilde{z_0}, \widetilde{z_1} \dots \widetilde{z_q}) \geq n(y, T)$. This implies that $u \in V(y; \widetilde{z_0}, \widetilde{z_1}, \dots, \widetilde{z_q})$. Since u is arbitrary, this shows (3.7).

Notice for each $x \in X$,

$$
\begin{aligned}
n(x, T) &= \inf\{n : (S_n\psi \circ \pi)(x) \geq T\} \\
&= \inf\{n : \sum_{i=0}^{n-1} \psi \circ \pi(f^i(x) \geq T\} \\
&= \inf\{n : \sum_{i=0}^{n-1} \psi(g^i\pi(x)) \geq T\} \\
&= n(\pi(x), T).
\end{aligned} \tag{3.9}
$$

If $V(y; z_0, z_1, \dots, z_q) \cap \pi^{-1}(\overline{B}_T(y, \delta, g)) \neq \emptyset$, pick any $v(y; z_0, z_1, \dots, z_q) \in V(y; z_0, z_1, \dots, z_q) \cap \pi^{-1}(\overline{B}_T(y, \delta, g)) \neq \emptyset$, we have

$$
\overline{B}_T(v(y; z_0, z_1, \dots, z_q), \epsilon, f) \supset V(y; z_0, z_1, \dots, z_q). \tag{3.10}
$$

In fact, for any $v \in V(y; z_0, z_1, \dots, z_q)$, we have for all $0 \le t < n(z_s, T_{\triangle_y}(t_s(y; z_0, z_1, \dots, z_{s-1})))$ and $0 \le s \le q$,

$$d(f^{t+t_s(y;z_0,z_1,\dots,z_{s-1})}(v), f^t(z_s)) < 2\delta_1.$$

Since $v(y; z_0, z_1, \dots, z_q) \in V(y; z_0, z_1, \dots, z_q)$, we get

$$d(f^{t+t_s(y;z_0,z_1,\dots,z_{s-1})}(v(y; z_0, z_1, \dots, z_q)), f^t(z_s)) < 2\delta_1.$$

Hence

$$d(f^j(v(y; z_0, z_1, \dots, z_q)), f^j(v)) < 4\delta_1, \quad 0 \le j \le t_{q+1}(y; z_0, z_1, \dots, z_q).$$

By Lemma 2.1, we have

$$\begin{aligned} n(v(y; z_0, z_1, \dots, z_q), T) &= n(\pi(v(y; z_0, z_1, \dots, z_q), T) \\ &\le n(y, T) + \frac{T+m}{m^2} var(\psi, \delta) + \frac{\|\psi\|}{m} \\ &\le n(y, T) + 2 + [\frac{\|\psi\|}{m}]. \end{aligned}$$

Now that $n(y, T) \le t_{q+1}(y; z_0, z_1, \dots, z_q)$, it follows from (3.5) that

$$d_{n(y,T)+2+[\frac{\|\psi\|}{m}]}(v(y; z_0, z_1, \dots, z_q), v) \le \epsilon.$$

Therefore

$$d_{n(v(y;z_0,z_1,\dots,z_q),T)}(v(y; z_0, z_1, \dots, z_q), v) \le \epsilon.$$

That is, we show (3.10). Combing (3.7) and (3.10), we obtain

$$\bigcup_{z_0 \in E_{\triangle_y(0)}, z_1 \in E_{\triangle_y(t_1(y;z_0))}, \dots, z_q \in E_{\triangle_y(tq(y;z_0,z_1,\dots,z_{q-1}))}} \overline{B}_T(v(y; z_0, z_1, \dots, z_q), \epsilon, f) \supset \pi^{-1}(\overline{B}_T(y, \delta, g)).$$

Let

$$E_T = \{v(y; z_0, z_1, \dots, z_q) : y \in F_T, z_0 \in E_{\triangle_y(0)}, z_1 \in E_{\triangle_y(t_1(y;z_0))}, \dots, z_q \in E_{\triangle_y(t_q(y;z_0,z_1,\dots,z_{q-1}))}\}.$$

Clearly E_T is a $(\psi \circ \pi, T, \epsilon)$-spanning set of X. For $y \in F_T$, there exists a permissible $(z_0^{'}, z_1^{'}, \dots, z_q^{'})$ such that the number of permissible $(z_0, z_1, \dots, z_q)$ is at most

$$N_y = \prod_{s=0}^{q} r_{T_{\triangle_y}(t_s(y;z_0^{'},z_1^{'},\dots,z_{s-1}^{'}))}(f, \pi^{-1}(\triangle_y(t_s(y; z_0^{'}, z_1^{'}, \dots, z_{s-1}^{'}))), \delta_1), \tag{3.11}$$

where $t_0(y; z_{-1}^{'}) = 0$.

To show (3.11), we give some notions which will be needed in next proof. Following (3.6), we suppose $q(y; z_0, z_1 \dots z_{q-1}) \ge 1$. For each $1 \le s \le q(y; z_0, z_1 \dots z_{q-1})$, if $z_{s-1} \in E_{\triangle_y(t_{s-1}(y;z_0,z_1,\dots,z_{s-2}))}$, we call z_{s-1} *directs* $E_{\triangle_y(t_s(y;z_0,z_1,\dots,z_{s-1}))}$ and $E_{\triangle_y(t_{s-1}(y;z_0,z_1,\dots,z_{s-2}))}$

is a *corresponding set* of $E_{\triangle_y(t_s(y;z_0,z_1,\ldots,z_{s-1}))}$. We say a permissible $(z_0, z_1, \ldots, z_q)$ is a $q+1$-*string*, and z_q is a *terminal point* of the permissible $(z_0, z_1, \ldots, z_q)$. For each $z \in E_{\triangle_y(t_q(y;z_0,z_1,\ldots,z_{q-1}))}$, if z is a terminal point of a $q+1$-string, we also say $E_{\triangle_y(t_q(y;z_0,z_1,\ldots,z_{q-1}))}$ is a *terminal set of* $q+1$-*step.*

Now we show (3.11). Let

$$p = \max\{q(y; z_0, z_1, \ldots, z_{q-1}) : z_0 \in E_{\triangle_y(0)}, z_1 \in E_{\triangle_y(t_1(y;z_0))}, \ldots, z_q \in E_{\triangle_y(t_q(y;z_0,z_1,\ldots,z_{q-1}))}\},$$

and

$$\mathcal{E} := \{E_{y_1}, E_{y_2}, \ldots, E_{y_r}\}.$$

If $p = 0$, it is clear that (3.11) holds.

If $p = 1$, there exists terminal sets of 2-step. We assume $E_{01}, \ldots, E_{0p_1} \in \mathcal{E}, (1 \leq p_1 \leq |E_{\triangle_y(0)}|)$ are all terminal sets of 2-step and $|E_{01}| = \max\{|E_{0i}| : 1 \leq i \leq p_1\}$. Then the sum of the number of all 1-strings and the number of all 2-strings is at most $|E_{\triangle_y(0)}||E_{01}|$. Let $z_0^{'} \in E_{\triangle_y(0)}$ directs E_{01}. Then the permissible $(z_0^{'})$ such that (3.11) holds.

If $p = 2$, there exists terminal sets of 3-step. We assume

$$E_{0l_11}, \ldots, E_{0l_1s_1}; E_{0l_21}, \ldots, E_{0l_2s_2}; \ldots; E_{0l_t1} \ldots E_{0l_ts_t}, (1 \leq t \leq p_1)$$

are all terminal sets of 3-step and satisfy the following:

(i) For each $1 \leq i \leq t, 1 \leq j \leq s_i$, E_{0l_i} is a corresponding set of E_{0l_ij}, where $1 \leq s_i \leq |E_{0l_i}|$.

(ii) For each $1 \leq i \leq t$, $|E_{0l_i1}| = \max\{|E_{0l_ij}| : 1 \leq j \leq s_i\}$.

There exists $1 \leq k \leq t$ with

$$|E_{0l_k}||E_{0l_k1}| = \max\{|E_{0l_i}||E_{0l_i1}| : 1 \leq i \leq t\}.$$

Considering that the possibility of the existent terminal set of 2-step, if $|E_{0l_k}||E_{0l_k1}| \geq |E_{01}|$, we obtain that the number of permissible $(z_0, z_1, \ldots, z_q)$ is at most

$$|E_{\triangle_y(0)}||E_{0l_k}||E_{0l_k1}|.$$

Choose $z_0^{'} \in E_{\triangle_y(0)}$ with $z_0^{'}$ directs E_{0l_k}, $z_1^{'} \in E_{0l_k}$ with $z_1^{'}$ directs E_{0l_k1}. Then the permissible $(z_0^{'}, z_1^{'})$ such that (3.11) holds. If $|E_{01}| \geq |E_{0l_k}||E_{k1}|$, we have the number of permissible $(z_0, z_1, \ldots, z_q)$ is at most $|E_{\triangle_y(0)}||E_{01}|$ and permissible $(z_0^{'})$ with $z_0^{'}$ directs E_{01} such that (3.11) holds.

Proceeding in this way, if $p > 2$, for each $1 \leq i \leq t$, we assume there exists a permissible $(z_1^{(i)}, \ldots, z_q^{(i)})$ such that the number of permissible $(z_1, \ldots, z_q)$ with $z_1 \in E_{0l_i}$ is at most

$$\prod_{s=1}^{q} r_{T_{\triangle_y(t_s(y;z_0^{(i)},z_1^{(i)},\ldots,z_{s-1}^{(i)}))}}(f, \pi^{-1}(\triangle_y(t_s(y; z_0^{(i)}, z_1^{(i)}, \ldots, z_{s-1}^{(i)}))), \delta_1),$$

where $z_1^{(i)} \in E_{0l_i}$ and $z_0^{(i)}$ directs $E_{0l_i}, q := q(z_0^{(i)}, z_1^{(i)}, \ldots, z_{q-1}^{(i)})$.
There exists $1 \leq k \leq t$ with

$$\begin{aligned} &\prod_{s=1}^{q} r_{T_{\triangle_y}(t_s(y;z_0^{(k)},z_1^{(k)},\ldots,z_{s-1}^{(k)}))}(f, \pi^{-1}(\triangle_y(t_s(y; z_0^{(k)}, z_1^{(k)}, \ldots, z_{s-1}^{(k)}))), \delta_1) \\ = \quad &\max\{\prod_{s=1}^{q} r_{T_{\triangle_y}(t_s(y;z_0^{(i)},z_1^{(i)},\ldots,z_{s-1}^{(i)}))}(f, \pi^{-1}(\triangle_y(t_s(y; z_0^{(i)}, z_1^{(i)}, \ldots, z_{s-1}^{(i)}))), \delta_1) : 1 \leq i \leq t\}. \end{aligned}$$

If

$$\prod_{s=1}^{q} r_{T_{\triangle_y}(t_s(y;z_0^{(k)},z_1^{(k)},\ldots,z_{s-1}^{(k)}))}(f, \pi^{-1}(\triangle_y(t_s(y; z_0^{(k)}, z_1^{(k)}, \ldots, z_{s-1}^{(k)}))), \delta_1) \geq |E_{01}|,$$

then the number of permissible $(z_0, z_1, \ldots, z_q)$ is at most

$$\prod_{s=0}^{q} r_{T_{\triangle_y}(t_s(y;z_0^{(i)},z_1^{(i)},\ldots,z_{s-1}^{(i)}))}(f, \pi^{-1}(\triangle_y(t_s(y; z_0^{(i)}, z_1^{(i)}, \ldots, z_{s-1}^{(i)}))), \delta_1).$$

This implies $(z_0^{(k)}, z_1^{(k)}, \ldots, z_q^{(k)})$ such that (3.11) holds.
If

$$\prod_{s=1}^{q} r_{T_{\triangle_y}(t_s(y;z_0^{(k)},z_1^{(k)},\ldots,z_{s-1}^{(k)}))}(f, \pi^{-1}(\triangle_y(t_s(y; z_0^{(k)}, z_1^{(k)}, \ldots, z_{s-1}^{(k)}))), \delta_1) \leq |E_{01}|,$$

we have the number of permissible $(z_0, z_1, \ldots, z_q)$ is at most $|E_{\triangle_y(0)}||E_{01}|$ and permissible $(z_0^{'})$ with $z_0^{'}$ directs E_{01} such that (3.11)holds and finish the proof of (3.11).

Let $v \in V(y; z_0^{'}, z_1^{'}, \ldots, z_q^{'}), N = \max\{n(z, T_{y_i}) : z \in E_{y_i}, i = 1, 2 \ldots r\}$. Then

$$\begin{aligned} \log N_y &= \sum_{s=0}^{q} \log r_{T_{\triangle_y}(t_s(y;z_0^{'},z_1^{'},\ldots,z_{s-1}^{'}))}(f, \pi^{-1}(\triangle_y(t_s(y; z_0^{'}, z_1^{'}, \ldots, z_{s-1}^{'}))), \delta_1) \\ &\leq (a+\tau)(T_{\triangle_y(0)} + T_{\triangle_y(t_1(y;z_0^{'}))} + \ldots + T_{\triangle_y(t_q(y;z_0^{'},z_1^{'},\ldots,z_q^{'}))}) \\ &\leq (a+\tau)(S_{n(z_0^{'},T_{\triangle_y(0)})}\psi \circ \pi(z_0^{'}) + \ldots + S_{n(z_q^{'},T_{\triangle_y(tq(y;z_0^{'},z_1^{'},\ldots,z_q^{'})})}\psi \circ \pi(z_q^{'})) \\ &\leq (a+\tau)[(n(y,T) + N)Var(\psi \circ \pi, 2\delta_1) + S_{n(y,T)+N}\psi \circ \pi(v)]. \end{aligned} \tag{3.12}$$

It follows from (3.9) and Lemma 2.1 that

$$n(y,T) \leq n(v,T) + 2 + [\frac{\|\psi\|}{m}]$$

and

$$\begin{aligned} (3.12) &\leq (a+\tau)[(n(y,T) + N)Var(\psi \circ \pi, 2\delta_1) + S_{n(v,T)}\psi \circ \pi(v) + (2 + [\frac{\|\psi\|}{m}] + N)\|\psi\|] \\ &\leq (a+\tau)[(\frac{T}{m} + 1 + N)Var(\psi \circ \pi, 2\delta_1) + T + \|\psi\| + (2 + [\frac{\|\psi\|}{m}] + N)\|\psi\|], \end{aligned} \tag{3.13}$$

where $Var(\psi \circ \pi, \delta) = \sup\{|\psi \circ \pi(x) - \psi \circ \pi(y)| : d(x, y) < \delta, x, y \in X\}$. Let $v := v(y; z_0, z_1, \ldots, z_q)$. Combining (3.13) and Lemma 2.2, we have

$$\begin{aligned}
&\sum_{v \in E_T} \exp(S_{n(v,T)}\varphi \circ \pi(v)) \\
&\leq \sum_{y \in F_T} \sum_{v \in V(y; z_0, z_1, \ldots, z_q) \cap \pi^{-1}(\overline{B}(y,T,\delta))} \exp(S_{n(v,T)}\varphi \circ \pi(v)) \\
&\leq \sum_{y \in F_T} \sum_{v \in V(y; z_0, z_1, \ldots, z_q) \cap \pi^{-1}(\overline{B}(y,T,\delta))} \exp(|S_{n(v,T)}\varphi \circ \pi(v) - S_{n(y,T)}\varphi(y)| + S_{n(y,T)}\varphi(y)) \\
&\leq \sum_{y \in F_T} \exp S_{n(y,T)}\varphi(y) \sum_{v \in V(y; z_0, z_1, \ldots, z_q) \cap \pi^{-1}(\overline{B}(y,T,\delta))} \exp[n(y,T)var(\varphi, \delta) + |n(v,T) - n(y,T)| \|\varphi\|] \\
&\leq \exp\{(a+\tau)[(\frac{T}{m} + 1 + N)Var(\psi \circ \pi, 2\delta_1) + T + \|\psi\| + (2 + [\frac{\|\psi\|}{m}] + N)\|\psi\|]\} \\
&\quad \exp[(\frac{T}{m} + 1)var(\varphi, \delta)] \exp[(2 + [\frac{\|\psi\|}{m})]\|\varphi\|] \sum_{y \in F_T} \exp S_{n(y,T)}\varphi(y) \qquad (3.14)
\end{aligned}$$

Now that $\delta \to 0$ as $T \to \infty$, it is easy to see

$$\limsup_{T \to \infty} \frac{1}{T} \log Q_{\psi \circ \pi, T}(f, \varphi \circ \pi, \epsilon) \leq (a+\tau)(\frac{1}{m} Var(\psi \circ \pi, 2\delta_1) + 1) + P_\psi(\varphi),$$

where

$$Q_{\psi \circ \pi, T}(f, \varphi \circ \pi, \epsilon) = \inf\{\sum_{v \in E_T} \exp(S_{n(v,T)}\varphi)(v) : E_T \textit{ is a } (\psi \circ \pi, T, \epsilon)\text{-spanning set of } X\}.$$

Notice $Var(\psi \circ \pi, 2\delta_1) \to 0$ as $\delta_1 \to 0$. Since $\delta_1 \to 0$ as $\epsilon \to 0$, we have

$$P_{\psi \circ \pi}(\varphi \circ \pi) \leq P_\psi(\varphi) + a + \tau.$$

As $\tau \to 0$, we obtain

$$P_{\psi \circ \pi}(\varphi \circ \pi) \leq P_\psi(\varphi) + \sup_{y \in Y} h_{\psi \circ \pi}(f, \pi^{-1}(y)).$$

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