A generalized univalent functions with missing coefficients of alternating type

Deborah Olufunmilayo Makinde *
Department of Mathematics, Obafemi Awolowo University,
Ile Ife 220005, Nigeria.
funmideb@yahoo.com; dmakinde@oauife.edu.ng
Fadipe-Joseph O.A
Department of Mathematics, University of Ilorin,
P. M. B. 1515, Ilorin, Nigeria. famelov@unilorin.edu.ng

Abstract

The normalized univalent function of the type

$$f(z) = z + \sum_{n=2}^{\infty}(-1)^{n+k}a_{n+k}z^{n+k}, k \geq 1,$$

has negative coefficient when $n+k$ is odd,and positive coefficient when (n+k) is even. In this paper, the author investigated some properties of univalent functions with negative coefficients of the type $f(z) = z + \sum_{n=2}^{\infty}(-1)^{n+k}a_{n+k}z^{n+k}, k \geq 1$ and obtain some conditions for which the function $f(z)$ belong to the subclass $S^*(\alpha,\beta,k)$, $C^*(\alpha,\beta,k)$.
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*correspondence author.

1 Introduction

Let S denote the class of normalized univalent functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

analytic in the unit disc $U = \{z : |z| < 1\}$

let T denote the subclass of S of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \quad n = 2, 3, ... \tag{1}$$

Let $J(\alpha, \lambda) := \left\{ f(z) \in A : \left| -\alpha z^2 (z|f(z))'' + f'(z)(z|f(z))^2 - 1 \right| \leq \lambda \right\}, \quad z \in E$ where $\lambda > 0$ and $\alpha \in \Re \setminus \left\{ -\frac{1}{2}, 0 \right\}$. Kaur et al [3] established conditions under which functions in the class $J(\alpha, \lambda)$ are starlike of order $\Gamma,\ 0 \leq \Gamma < 1$ while Yi-Ling Cang and Jin-Lin Liu [1] showed certain sufficient conditions for univalency of analytic functions with missing coefficients.

Silverman [4] studied the properties of T in D where $D = \{w : w \text{ is analytic in } U; w(0) = 0, |w(z)| < 1 \text{ in } U\}$. Khairnar and More [2] studied $G(A, B)$, a subclass of analytic function in U, which are of the form $\frac{1+Aw(z)}{1+Bw(z)}$, $-1 \leq A < \beta \leq 1$ where $w(z) \in D$.

Now, we define a subclass H of T to consist of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} (-1)^{n+k} a_{n+k} z^{n+k}, \quad a_{k+1} \geq 0, \quad k \geq 1 \tag{2}$$

And let $G(\alpha, \beta, k)$ denote a subclass of analytic functions in U, which are of the form $\frac{1+\alpha w(z)}{1+\beta w(z)}$, $-1 \leq \alpha < \beta \leq 1$ where $w(z) \in D$.

We shall in this paper investigate a subclass H of T in $G(\alpha, \beta, k)$

We define $S^*(\alpha, \beta, k)$ and $C(\alpha, \beta, k)$ respectively as follows:

$$S^*(\alpha, \beta, k) = \left\{ f : f \in H \text{ and } \frac{zf'}{f} \in G(\alpha, \beta) \right\}$$

$$C(\alpha, \beta, k) = \left\{ f : f \in H \text{ and } \left(\frac{zf'}{f} \right)' \in G(\alpha, \beta) \right\}$$

Lemma 1 [1]: A function

$$f(z) = z + \sum_{n=2}^{\infty} (-1)^{n+1} a_{n+1} z^{n+1}, a_{n+1} \geq 0$$

is in $M(A,B)$ iff

$$\sum_{n=2}^{\infty}\left(\frac{n(B+1)-(A-B)}{A-B}\right)a_{n+1}\leq 1$$

Lemma 2 [1]: A function

$$f(z)=z+\sum_{n=2}^{\infty}(-1)^{n+1}a_{n+1}z^{n+1}, a_{n+1}\geq 0$$

is in $C(A,B)$ iff

$$\sum_{n=2}^{\infty}\left(\frac{[n(B+1)-(A-B)](n+1)}{A-B}\right)a_{n+1}\leq 1$$

$$M(A,B)=\left\{f:f\in Mand\frac{zf'}{f}\in G(A,B)\right\}$$

and

$$C(A,B)=\left\{f:f\in Mand\left(\frac{zf'}{f}\right)'\in G(A,B)\right\}$$

2 Main Results

Theorem 1:

Let a function

$$f(z)=z+\sum_{n=2}^{\infty}(-1)^{n+k}a_{n+k}z^{n+k}, a_{n+k}\geq 0$$

be in $S^*(\alpha,\beta,k)$, then,

$$\sum_{n=2}^{\infty}\left(\frac{n+k-1(\beta+1)-(\alpha-\beta)}{\alpha-\beta}\right)a_{n+k}\leq 1$$

Proof:

Let

$$f(z)=z+\sum_{n=2}^{\infty}(-1)^{n+k}a_{n+k}z^{n+k},\quad k\geq 1$$

Then,

$$f'(z)\geq 0$$

Thus,

$$\frac{zf'(z)}{f(z)} = \frac{z+\sum_{n=2}^{\infty}(-1)^{n+k}(n+k)a_{n+k}z^{n+k}}{z+\sum_{n=2}^{\infty}(-1)^{n+k}a_{n+k}z^{n+k}} = \frac{1+\alpha w(z)}{1+\beta w(z)}$$

$$\Longrightarrow \left|\frac{\sum_{n=2}^{\infty}(-1)^{n+k}(n+k-1)a_{n+k}z^{n+k-1}}{(\alpha-\beta)+\sum_{n=2}^{\infty}(-1)^{n+k}[\alpha-\beta(n+k)]a_{n+k}z^{n+k-1}}\right| = 1$$

for $r \to 1$ we obtain

$$\frac{\sum_{n=2}^{\infty}(n+k-1)a_{n+k}}{\alpha-\beta+\sum_{n=2}^{\infty}(-1)^{n+k}[\alpha-\beta(n+k)]a_{n+k}} \leq 1$$

$$\Longrightarrow \sum_{n=2}^{\infty}(n+k-1)a_{n+k} < (\alpha-\beta)+\sum_{n=2}^{\infty}[\alpha-\beta(n+k)]a_{n+k}$$

$$\Longrightarrow \sum_{n=2}^{\infty}\Big[(n+k-1)-\alpha+\beta(n+k)\Big]a_{n+k} \leq \alpha-\beta$$

$$\Longrightarrow \sum_{n=2}^{\infty}\Big[(n+k-1)-\alpha+\beta((n+k-1)+1)\Big]a_{n+k} \leq \alpha-\beta$$

$$\Longrightarrow \sum_{n=2}^{\infty}\Big[(n+k-1)(\beta+1)+\beta-\alpha\Big]a_{n+k} \leq \alpha-\beta$$

$$\Longrightarrow \sum_{n=2}^{\infty}\left[\frac{(n+k-1)(\beta+1)-(\alpha-\beta)}{\alpha-\beta}\right]a_{n+k} < \sum_{n=2}^{\infty}\left[\frac{(n+k)(\beta+1)-(\alpha-\beta)}{\alpha-\beta}\right]a_{n+k} \leq 1$$

This concludes the proof of Theorem 1.

Theorem 2:

Let a function

$$f(z) = z+\sum_{n=2}^{\infty}(-1)^{n+k}a_{n+k}z^{n+k}, a_{n+k} \geq 0$$

be in $C(\alpha,\beta,k)$ then

$$\sum_{n=2}^{\infty}\left\{\frac{(n+k)(n+k-1)(\beta+1)-(\alpha-\beta)}{\alpha-\beta}\right\}a_{n+k} \leq 1$$

Proof:

Suppose $f(z) \in C(\alpha, \beta, k)$

We have

$$\frac{1+\sum_{n=2}^{\infty}(-1)^{n+k}(n+k)^2 a_{n+k} z^{n+k-1}}{1+\sum_{n=2}^{\infty}(-1)^{n+k}(n+k) a_{n+k} z^{n+k-1}} = \frac{1+\alpha w(z)}{1+\beta w(z)}$$

$$\Longrightarrow \quad \frac{\sum_{n=2}^{\infty}(-1)^{n+k}(n+k-1)(n+k) a_{n+k} z^{n+k-1}}{(\alpha-\beta)+\sum_{n=2}^{\infty}(-1)^{n+k}(n+k)[\alpha-\beta(n+k)] a_{n+k} z^{n+k-1}} = w(z)$$

$$\Longrightarrow \quad \left|\frac{\sum_{n=2}^{\infty}(-1)^{n+k}(n+k-1)(n+k) a_{n+k} z^{n+k-1}}{(\alpha-\beta)+\sum_{n=2}^{\infty}(-1)^{n+k}(n+k)[\alpha-\beta(n+k)] a_{n+k} z^{n+k-1}}\right| \leq 1$$

since $|w(z)| \leq 1$

Let $|z| \longrightarrow 1$, we obtain

$$\frac{(n+k-1)(n+k) a_{n+k}}{(\alpha-\beta)+\sum_{n=2}^{\infty}(n+k)[\alpha-\beta(n+k)] a_{n+k}} \leq 1$$

$$\Longrightarrow \quad \sum_{n=2}^{\infty}(n+k-1)(n+k) a_{n+k} \leq (\alpha-\beta)+\sum_{n=2}^{\infty}(n+k)[\alpha-\beta(n+k)] a_{n+k}$$

$$\Longrightarrow \quad \sum_{n=2}^{\infty}\left\{\frac{(n+k)\Big[(n+k-1)(\beta+1)-(\alpha-\beta)\Big]}{\alpha-\beta}\right\} a_{n+k}$$

$$\leq \quad \sum_{n=2}^{\infty}\left\{\frac{(n+k)\Big[(n+k)(\beta+1)-(\alpha-\beta)\Big]}{\alpha-\beta}\right\} a_{n+k}$$

$$< \quad 1$$

which is the required result.

Theorem 3

Let a function

$$f(z) = z+\sum_{n=2}^{\infty}(-1)^{n+k} a_{n+k} z^{n+k}, a_{n+k} \geq 0$$

be in $S_k^*(\alpha, \beta)$ then $(1-\lambda)\frac{zf'(z)}{f(z)}$ is also in $S^*(\alpha, \beta, k)$ $0 \leq \alpha < 1$

Proof:

Let $f(z) \in S^*(\alpha, \beta, k)$ then,

$$\sum_{n=2}^{\infty}\left(\frac{(n+k-1)(\beta+1)-(\alpha-\beta)}{\alpha-\beta}\right) a_{n+k} \leq 1$$

But,

$$\frac{(1-\lambda)\Big\{z+\sum_{n=2}^{\infty}(-1)^{n+k}(n+k)a_{n+k}z^{n+k}\Big\}}{z+\sum_{n=2}^{\infty}(-1)^{n+k}a_{n+k}z^{n+k}}=\frac{1+\alpha w(z)}{1+\beta w(z)}$$

$$\begin{aligned}
\Rightarrow\quad & z+\sum_{n=2}^{\infty}(-1)^{n+k}(n+k)a_{n+k}z^{n+k}-\left\{\lambda z+\lambda\sum_{n=2}^{\infty}(-1)^{n+k}(n+k)a_{n+k}z^{n+k}\right\}\\
+\quad & z\beta w(z)+\beta w(z)\sum_{n=2}^{\infty}(-1)^{n+k}(n+k)a_{n+k}z^{n+k}\\
-\quad & \left\{\lambda z\beta w(z)+\lambda\beta w(z)\sum_{n=2}^{\infty}(-1)^{n+k}(n+k)a_{n+k}z^{n+k}\right\}\\
=\quad & z+\sum_{n=2}^{\infty}(-1)^{n+k}a_{n+k}z^{n+k}+z\alpha w(z)+\alpha w(z)\sum_{n=2}^{\infty}(-1)^{n+k}a_{n+k}z^{n+k}\\
\Rightarrow\quad & \frac{\sum_{n=2}^{\infty}(-1)^{n+k}\Big[(1-\lambda)(n+k)-1\Big]a_{n+k}z^{n+k-1}-\lambda}{(\alpha-(1-\lambda)\beta)+\sum_{n=2}^{\infty}(-1)^{n+k}\Big[\alpha-(1-\lambda)\beta(n+k)\Big]a_{n+k}z^{n+k-1}}=w(z)\\
\Rightarrow\quad & \left|\frac{\sum_{n=2}^{\infty}(-1)^{n+k}\Big[(1-\lambda)(n+k)-1\Big]a_{n+k}z^{n+k-1}-\lambda}{(\alpha-(1-\lambda)\beta)+\sum_{n=2}^{\infty}(-1)^{n+k}\Big[\alpha-(1-\lambda)\beta(n+k)\Big]a_{n+k}z^{n+k-1}}\right|=|w(z)|\leq 1
\end{aligned}$$

for $|z|=r\to 1$

$$\begin{aligned}
\Rightarrow\quad & \frac{\sum_{n=2}^{\infty}\Big[(1-\lambda)(n+k)-1\Big]a_{n+k}-\lambda}{(\alpha-(1-\lambda)\beta)+\sum_{n=2}^{\infty}\Big[\alpha-(1-\lambda)\beta(n+k)\Big]a_{n+k}}\leq 1\\
\Rightarrow\quad & \sum_{n=2}^{\infty}\Big[(1-\lambda)(n+k)-1\Big]a_{n+k}-\lambda\leq\alpha-(1-\lambda)\beta+\sum_{n=2}^{\infty}\Big[\alpha-(1-\lambda)\beta(n+k)\Big]a_{n+k}\\
\Rightarrow\quad & \sum_{n=2}^{\infty}\Big[(1-\lambda)(n+k)-1\Big]a_{n+k}-\sum_{n=2}^{\infty}\Big[\alpha-(1-\lambda)\beta(n+k)\Big]a_{n+k}\leq\alpha-(1-\lambda)\beta\\
\Rightarrow\quad & \sum_{n=2}^{\infty}\left[\Big((1-\lambda)(n+k)-1\Big)-\alpha+(1-\lambda)\beta(n+k)\right]a_{n+k}\leq\alpha-(1-\lambda)\beta\\
\Rightarrow\quad & \sum_{n=2}^{\infty}\left[\Big((1-\lambda)(n+k)-1\Big)-\alpha+\beta\Big\{(1-\lambda)((n+k)-1)+1\Big\}\right]a_{n+k}\leq\alpha-(1-\lambda)\beta\\
\Rightarrow\quad & \sum_{n=2}^{\infty}\left[\Big((1-\lambda)(n+k)-1\Big)(\beta+1)-\alpha+(1-\lambda)\beta\right]a_{n+k}\leq\alpha-(1-\lambda)\beta
\end{aligned}$$

But,

$$\Rightarrow \quad \frac{\sum_{n=2}^{\infty}\left[\Big((1-\lambda)(n+k)-1\Big)(\beta+1)-(\alpha-(1-\lambda)\beta)\right]a_{n+k}}{\alpha-(1-\lambda)\beta}$$

$$\leq \quad \frac{\sum_{n=2}^{\infty}(n+k-1)(\beta+1)-(\alpha-\beta)}{\alpha-\beta}$$

This shows that $(1-\lambda)\frac{zf'(z)}{f(z)}$ belongs to $S_k^*(\alpha,\beta)$.

Theorem 4

Let

$$f(z) = z + \sum_{n=2}^{\infty}(-1)^{n+k}a_{n+k}z^{n+k}, a_{n+k} \geq 0.$$

belong to the class $C(\alpha,\beta,k)$ then $(1-\lambda)(\frac{zf'(z)}{f(z)})'$ also belong to the class $C(\alpha,\beta,k)$

Proof:

$$\frac{(1-\lambda)\{1+\sum_{n=2}^{\infty}(-1)^{n+k}(n+k)^2a_{n+k}z^{n+k-1}\}}{1+\sum_{n=2}^{\infty}(-1)^{n+k}(n+k)a_{n+k}z^{n+k-1}} = \frac{1+\alpha w(z)}{1+\beta w(z)}$$

and following the proof of Theorem 3, we obtain the result.

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